Approximation of Harmonic Maps and Wave Maps

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Partial differential equations with a nonlinear pointwise constraint defined through a manifold N occur in a variety of applications: The magnetization of a ferromagnet can be described by a unit length vector field $m : \Omega \to S^2$ and the orientation of the rod-like molecules that constitute a liquid crystal is often modeled by a vector field that attains its values in the real projective plane $\mathbb{R}P^2$ thus respecting the head-to-tail symmetry of the molecules. Other applications arise in geometric modeling, quantum mechanics, and general relativity. Simple examples reveal that it is impossible to satisfy pointwise constraints exactly by lowest order finite elements. For two model problems we discuss the practical realization of the constraint and the efficient solution of the resulting nonlinear systems of equations.

Let $T > 0$, $\Omega \subset \mathbb{R}^m$, $m = 2, 3$, a bounded Lipschitz domain, and $N \subset \mathbb{R}^{n+1}$ a convex hypersurface, i.e., $N = \partial\mathcal{C}$ for a convex set $\mathcal{C}$. Let either $\widehat{X} = \Omega$ and g denote the standard Euclidean metric on $\mathbb{R}^m$ or $\widehat{X} = (0,T) \times \Omega$ and g denote the Lorentzian metric on $\mathbb{R}^{m+1}$. We then consider critical points of the functional

$$E_{g,\widehat{X}}(v) = \int_{\widehat{X}} \big|Dv\big|_g^2 \,\mathrm{d}\widehat{x}$$

among mappings $v : \widehat{X} \to N$ and subject to certain boundary condtions. If $\widehat{X} = \Omega$ then critical points $u : \Omega \to N$ are called *harmonic maps into* N and satisfy

$$-\Delta u \perp T_u N, \qquad u|_\Gamma = u_{\mathrm{D}}, \tag{1}$$

where $\Gamma = \partial\Omega$. If $\widehat{X} = (0,T) \times \Omega$ we look for critical points $u : (0,T) \times \Omega \to N$ called *wave maps into* N that solve the initial boundary value problem

$$\partial_t^2 u - \Delta u \perp T_u N, \quad \partial_n u = 0, \quad u(0,\cdot) = u_0, \quad \partial_t u(0,\cdot) = v_0. \tag{2}$$

To approximate harmonic maps or wave maps we consider a regular triangulation $\mathcal{T}_h$ of Ω into triangles or tetrahedra and nodes (vertices of elements) contained in the set $\mathcal{N}_h$. We assume that $\mathcal{T}_h$ is weakly acute, i.e., $\int_\Omega \nabla\varphi_z \cdot \nabla\varphi_y \,\mathrm{d}x \le 0$ for distinct $z, y \in \mathcal{N}_h$ and the nodal basis $\big(\varphi_z : z \in \mathcal{N}_h\big)$ of the lowest order finite element space $\mathbb{V}_h$ subordinate to $\mathcal{T}_h$. We let $\mathcal{I}_h : C(\overline{\Omega}) \to \mathbb{V}_h$ denote the nodal interpolation operator. According to [2, 4] the triangulation $\mathcal{T}_h$ is weakly acute if and only if

$$\big\|\nabla\mathcal{I}_h\big[P \circ v_h\big]\big\| \le |P|_{W^{1,\infty}}\big\|\nabla v_h\big\| \qquad \forall v_h \in \mathbb{V}_h \ \forall P \in W^{1,\infty}(\mathbb{R}). \tag{3}$$

This fact implies the discrete maximum principle for the Dirichlet problem: If $u_h \in \mathbb{V}_h$ is minimal for $v_h \mapsto \|\nabla v_h\|^2$ among all $v_h \in \mathbb{V}_h$ subject to $v_h|_\Gamma = u_{\mathrm{D},h}$ then $\widetilde{u}_h := \mathcal{I}_h\big[P \circ u_h\big]$, for $P(s) := \min\{s, \overline{u}_{\mathrm{D},h}\}$ and $\overline{u}_{\mathrm{D},h} := \max u_{\mathrm{D},h}$, satisfies $\big\|\nabla\widetilde{u}_h\big\| \le \big\|\nabla u_h\big\|$. Thus $\widetilde{u}_h = u_h$ and $u_h \le \overline{u}_{\mathrm{D},h}$.

Motivated by work in [1, 2] we propose an iterative approximation of harmonic maps into N by a successive minimization of the Dirichlet energy subject to the linearized constraint about the current iterate and a subsequent projection of the update:

Alg. 1. *Let* $u_h^0 \in \mathbb{V}_h^{n+1}$ *such that* $u_h^0(z) \in N$ *for all* $z \in \mathcal{N}_h$ *and* $u_h^0|_\Gamma = u_{\mathrm{D},h}$*:*

(1) *Compute* $v_h^{j+1} \in \mathbb{V}_{h,0}^{n+1}$ *such that* $v_h^{j+1}(z) \in T_{u_h^j(z)}N$ *for all* $z \in \mathcal{N}_h$ *and*

$$\big(\nabla\big[u_h^j + v_h^{j+1}\big], \nabla w_h\big) = 0$$

for all $w_h \in \mathbb{V}_{h,0}^{n+1}$ *such that* $w_h(z) \in T_{u_h^j(z)}N$ *for all* $z \in \mathcal{N}_h$*.*

(2) *Set*

$$u_h^{j+1} := \mathcal{I}_h\big[\pi_\mathcal{C} \circ \big(u_h^j + v_h^{j+1}\big)\big].$$

Here, T_pN denotes the tangent space of N at $p \in N$ and $\pi_\mathcal{C}$ is the orthogonal projection onto the convex set $\mathcal{C}$. Well posedness of the iteration is a consequence of the Lax-Milgram lemma and the fact that $\pi_\mathcal{C}$ globally well defined. Stability follows from choosing $w_h = v_h^{j+1}$, i.e.,

$$\big\|\nabla\big[u_h^j + v_h^{j+1}\big]\big\|^2 - \big\|\nabla u_h^j\big\|^2 + \big\|\nabla v_h^{j+1}\big\|^2 = 2\big(\nabla\big[u_h^j + v_h^{j+1}\big], \nabla v_h^j\big) = 0$$

and the fact that owing to (3) we have $\big\|\nabla u_h^{j+1}\big\| \le \big\|\nabla\big[u_h^j + v_h^{j+1}\big]\big\|$.

The observation that $\partial_t u \perp T_u N$ holds for wave maps into N motivates a similar iteration for their approximation and has first been employed in [5]. We let $\tau > 0$ denote a time-step size and d_t the corresponding backward difference operator.

Alg. 2. *Let* $u_h^0, v_h^0 \in \mathbb{V}_h^{n+1}$ *such that* $u_h^0(z) \in N$ *for all* $z \in \mathcal{N}_h$*:*

(1) *Compute* $v_h^{j+1} \in \mathbb{V}_h^{n+1}$ *such that* $v_h^{j+1}(z) \in T_{u_h^j(z)}N$ *for all* $z \in \mathcal{N}_h$ *and*

$$\big(d_t v_h^{j+1}, w_h\big) + \big(\nabla\big[u_h^j + \tau v_h^{j+1}\big], \nabla w_h\big) = 0$$

for all $w_h \in \mathbb{V}_h^{n+1}$ *such that* $v_h^j(z) \in T_{u_h^j(z)}N$ *for all* $z \in \mathcal{N}_h$*.*

(2) *Set*

$$u_h^{j+1} := \mathcal{I}_h\big[\pi_\mathcal{C} \circ \big(u_h^j + \tau v_h^{j+1}\big)\big].$$

Upon choosing $w_h = v_h^{j+1}$ we deduce that

$$d_t\big\|v_h^{j+1}\big\|^2 + \tau\big\|d_t v_h^{j+1}\big\|^2 + \big\|\nabla\big[u_h^j + v_h^{j+1}\big]\big\|^2 - \big\|\nabla u_h^j\big\|^2 + \tau\big\|\nabla v_h^{j+1}\big\|^2 = 0.$$

Again the fact that $\big\|\nabla u_h^{j+1}\big\| \le \big\|\nabla\big[u_h^j + v_h^{j+1}\big]\big\|$ implies unconditional stability.

Convergence of the approximations to harmonic maps and wave maps as $h \to 0$ and $(h,\tau) \to 0$, respectively, can be established via weak compactness results. Necessary modifications of the algorithms for the case that N is nonconvex are discussed in [3, 4]. Nonstandard finite element methods are assumed to improve the performance of the algorithms and are currently under investigation.

References

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