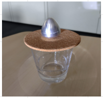
Babuška's paradox in linear and nonlinear bending theories

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ICERM Program on Numerical PDEs

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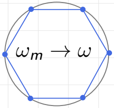
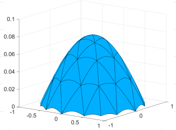
Solids SPP 2256 Variational Methods


$\omega_m \to \omega$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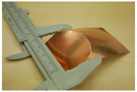
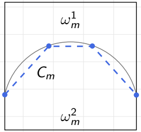
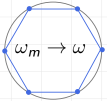



ω_m^1
C_m
ω_m^2

The paradox

Simply supported plates: Linear Kirchhoff model with $u|_{\partial\omega} = 0$

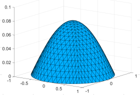
$$I(u) = \frac{\sigma}{2}\int_\omega |\Delta u|^2 \,\mathrm{d}x + \frac{1-\sigma}{2}\int_\omega |D^2 u|^2 \,\mathrm{d}x - \int_\omega fu \,\mathrm{d}x$$



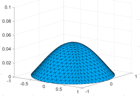
Babuška '61: Incorrect convergence $u_m \to u_\infty \neq u$
for solutions u_m on polygons $\omega_m \subset \omega$

Euler–Lagrange eq's: If $\partial\omega$ piecewise $C^{2,1}$

$$\Delta^2 u = f \quad \text{in } \omega$$
$$u = 0 \quad \text{on } \partial\omega$$
$$\Delta u + (1-\sigma)\kappa\partial_n u = 0 \quad \text{on } \partial\omega$$



- Pointwise clamped condition in corners $\nabla u(c_i) = 0$
- For polygonal domains ω_m term involving κ disappears



Limit $u_\infty = \lim_{m\to\infty}$ solves

$$\Delta^2 u_\infty = f \quad \text{in } \omega, \qquad u_\infty = 0 \quad \text{on } \partial\omega, \qquad \Delta u_\infty = 0 \quad \text{on } \partial\omega$$

Polygonal approximation: Failure of compactness for $\kappa\partial_n u$

Regularity: Role of corners in Euler–Lagrange equations ?

Clamped BC: No paradox due to density of $C_c^\infty(\omega)$ in $H_0^2(\omega)$

Operator splitting: Two standard Poisson problems if ω convex & polygonal

Isoparametric methods: No paradox for quadratic boundary approximations

[Ciarlet & Raviart '72, Zlámal '72, Brenner, Neilan & Sung '13, Bonito, Guignard, Nochetto, Yang '23, ...]

Nonconforming/dG methods: Correct convergence on simplicial meshes

[Arnold & Walker '20, Wissel '23, ...]

Selected references:

- [Babuška '61] Domain perturbations
- [Scott '76] Ideas for avoding the paradox
- [Rannacher '79] Special treatment of BC
- [Utku & Carey '83] Penalty approaches
- [Maz'ya & Nazarov '86] Other plate paradoxes
- [Babuška & Pitkäranta '90] Hard and soft simple support
- [Davini '02] Exterior approximations
- [De Coster, Nicaise & Sweers '19] Variational re-formulations

Modeling: Paradox due to limitations of linear Kirchhoff model?

- No paradox for Reissner–Mindlin (asymptotically, fixed $t > 0$)
- Free support $u|_{\partial\omega} \geq 0$ different due to non-positivity of Green's functions [Nazarov, Sweers & Stylianou '11]

Curvature quantities: Elementary calculations yield

$$|D^2 u|^2 - |\Delta u|^2 = -2 \det D^2 u$$

Null Lagrangian: Express determinant as divergence ($Jv = v^\perp$)

$$2 \det D^2 u = \operatorname{div}(J D^2 u J \nabla u)$$

Simple support: Condition $u = 0$ on $\partial\omega$ yields $\partial_\tau u = 0$ and

$$\partial_\tau^2 u = -\kappa \partial_n u$$

Representation: Using density of $H^3 \cap H_0^1$ functions ($f = 0$) for p/w $C^{2,1}$ bdy

$$I(u) = \frac{\sigma}{2} \int_\omega |\Delta u|^2 \,\mathrm{d}x + \frac{1-\sigma}{2} \int_\omega |D^2 u|^2 \,\mathrm{d}x \qquad I_m(u) = \frac{1}{2} \int_{\omega_m} |\Delta u|^2 \,\mathrm{d}x + 0$$

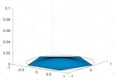
$$= \frac{1}{2} \int_\omega |\Delta u|^2 \,\mathrm{d}x + \frac{1-\sigma}{2} \int_{\partial\omega} \kappa (\partial_n u)^2 \,\mathrm{d}s$$

Consequence: Failure of Γ-convergence $I_m \to I$ for functionals I_m using polygonal domain approximations ω_m if $\partial\omega$ has curved parts

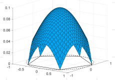
Ways to avoid it

Reduced BC: Impose simple support in corners of ω_m only

Full BC:



Reduced BC:



Approximations: $I_m = I|_{\omega_m}$ for

$$v \in \widetilde{V}_m = \{v \in H^2(\omega_m) : \; v(c_i) = 0, \; i = 0, \dots, m\}$$

Justification: Γ-convergence w.r.t. strong convergence in $L^2(\omega)$ using trivial extensions of functions and derivatives

- *Stability:* If $I_m(v_m) \le c$ then $\mathcal{I}_m v_m \in H^1_0$ and $D^2 v_m \rightharpoonup D^2 v$ for $v \in H^2 \cap H^1_0$

$$\liminf_{m \to \infty} I_m(v_m) \ge I(v)$$

- *Consistency:* If $v \in H^3 \cap H^1_0$ then restrictions $v_m = v|_{\omega_m}$ admissible in V_m and $D^2 v_m \to D^2 v$, hence $I_m(v_m) \to I(v)$
- *Equicoercivity:* Perturbed Poincaré inequality $\|\nabla \mathcal{I}_m v_m\| \le \widetilde{c}_P \|D^2 v_m\|$

Other topologies: Strong convergence of $\mathcal{I}_m v_m$ in H^1_0

Rates: Bound $\|D^2(u - u_m)\|_{L^2(\omega_m)} \le c(u,f)|\omega \setminus \omega_m|^{1/2} = O(h_m)$

Necessary: Need to introduce nonconformity in BCs

Goal: Abstract Γ-convergence result $I_h \to I$ for approximations

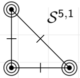
$$I_h(u_h) = \frac{\sigma}{2}\int_{\omega_h} |\Delta_h u_h|^2 \,\mathrm{d}x + \frac{1-\sigma}{2}\int_{\omega_h} |D_h^2 u_h|^2 \,\mathrm{d}x, \quad u_h \in V_h$$

- ω_h domain triangulated by simplicial mesh $\mathcal{T}_h$
- $V_h \subset L^2(\omega_h)$ finite element space including BCs
- D_h^2 approximation of D^2

Assumptions: $\omega_h \subset \omega$ convex, boundary nodes of $\mathcal{T}_h$ belong to $\partial\omega$

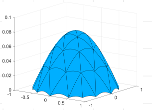
- *Equicoercivity:* $\mathcal{J}_h v_h \in H_0^1$ with $\|\nabla \mathcal{J}_h v_h\|^2 \lesssim I_h(v_h)$
- *Stability of D_h^2:* If $v_h \rightharpoonup v$ then $D_h^2 v_h \rightharpoonup D^2 v$
- *Interpolation in V_h:* $\mathcal{I}_h v \in V_h$ and $D_h^2 \mathcal{I}_h v \to D^2 v$ for $v \in H^3 \cap H_0^1$

Theorem (Correct convergence) [B. & Tscherner '24+]
If conditions are satisfied then $I_h \to I$ w.r.t. strong convergence in L^2.



Argyris element: H^2-conforming FE using quintic polynomials

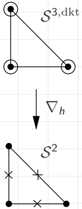
- *Stability of D_h^2:* Trivial as $D_h^2 = D^2$
- *Equicoercivity:* Integration by parts and interpolation



$$\|\nabla \mathcal{I}_h^{\mathrm{p1}} v_h\|^2 = \int_{\omega_h} \mathcal{I}_h^{\mathrm{p1}} v_h(-\Delta v_h) + \nabla \mathcal{I}_h^{\mathrm{p1}} v_h \cdot \nabla(\mathcal{I}_h^{\mathrm{p1}} v_h - v_h)\,\mathrm{d}x$$

$$\leq \big(c_P\|\Delta v_h\| + c_{\mathrm{p1}} h\|D^2 v_h\|\big)\|\nabla \mathcal{I}_h^{\mathrm{p1}} v_h\|$$

- *Interpolation:* Use averaging in highest order derivatives for quasiinterpolation of $v \in H^3(\omega)$



Discrete Kirchhoff element: Discrete gradient $\nabla_h : V_h \to W_h$ and $D_h^2 = \nabla\nabla_h$

- *Stability of D_h^2:* Approximation properties of ∇_h
- *Equicoercivity:* As above with Δ_h and $D_{\mathcal{T}}^2$
- *Interpolation:* Use canonical interpolation operator $\mathcal{I}_h^{\mathrm{dkt}}$

BC's: No canonical way to impose conditions for Hermite elements

Discrete operator: Elementwise integration by parts in strong form

$$\begin{aligned} a_h(v_h, w_h) = (D_h^2 v_h, D_h^2 w_h) \\ &+ (\{\partial_n \nabla_h v_h\}, [\![\nabla_h w_h]\!])_{\cup \mathcal{S}_h \setminus \partial\omega_h} + (\{\partial_n \nabla_h w_h\}, [\![\nabla_h v_h]\!])_{\cup \mathcal{S}_h \setminus \partial\omega_h} \\ &- (\{\partial_n \Delta_h v_h\}, [\![w_h]\!])_{\cup \mathcal{S}_h} - (\{\partial_n \Delta_h w_h\}, [\![v_h]\!])_{\cup \mathcal{S}_h} \end{aligned}$$

Stabilization: With suitable parameters $\gamma_0, \gamma_1 > 0$

$$s_h(v_h, w_h) = \gamma_0 (h_{\mathcal{S}}^{-3} [\![v_h]\!], [\![w_h]\!])_{\cup \mathcal{S}_h} + \gamma_1 (h_{\mathcal{S}}^{-1} [\![\nabla_h v_h]\!], [\![\nabla_h w_h]\!])_{\cup \mathcal{S}_h \setminus \partial\omega_h}$$

Discrete energy: SIPG formulation for $u_h \in \mathcal{L}^\ell(\mathcal{T}_h)$

$$I_h(u_h) = \frac{1}{2} a_h(u_h, u_h) + \frac{1}{2} s_h(u_h, u_h)$$

BC via penalty: Since $\operatorname{dist}(\partial\omega, \partial\omega_h) \leq ch^2$, for $v \in H^3 \cap H_0^1$

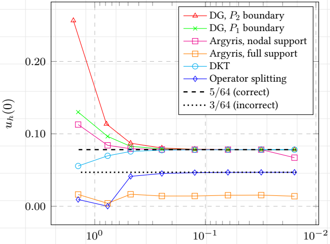
$$\varepsilon^{-1} \int_{\partial\omega_h} |v|^2 \, \mathrm{d}s \lesssim \varepsilon^{-1} h^4 \|v\|_{H^3(\omega)} \to 0$$

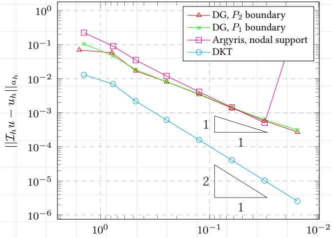
- *Stability of D_h^2:* Apply lifting $H_h(v_h) \in L^2(\omega)^{2\times 2}$ [Bonito et al. '23]
- *Equicoercivity:* As above with node averaging
- *Interpolation:* Use quadratic Lagrange interpolant

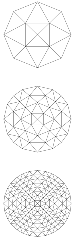
Babuška's example: For $\omega = B_1(0)$ and $f = 1$ obtain

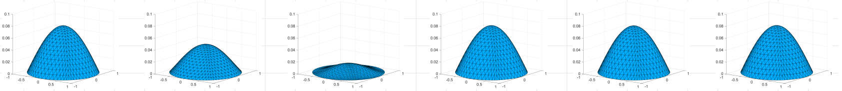
$$u(x) = \frac{(5+\sigma) - (6+2\sigma)|x|^2 + (1+\sigma)|x|^4}{64(1+\sigma)}, \qquad u_\infty(x) = \frac{3}{64} - \frac{1}{16}|x|^2 + \frac{1}{64}|x|^4$$

Experiment: Midpoint- and H^2-errors for $\sigma = 0$







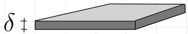


u $\quad u_\infty \quad u_h^{arg,0} \quad u_h^{arg,red} \quad u_h^{dkt} \quad u_h^{dg}$

Thin sheet folding

3D hyperelasticity: Isotropic & objective material

$$I^{3d}(y) = \int_\Omega W(\nabla y)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega f \cdot y\,\mathrm{d}x\,\mathrm{d}t$$



Bending: $\Omega_\delta = \omega \times (-\delta/2, \delta/2)$ and

$$\min_{y \in \mathcal{A}} I^{3d}(y) \sim \delta^3$$

Rigidity: [Friesecke, James, Müller '02] $\min_{R \in SO(3)} \|\nabla y - R\| \leq C \big\|\mathrm{dist}(\nabla y, SO(3))\big\|$

Theorem [Kirchhoff 1850, FJM '02]. Functionals $\delta^{-3} I^{3d}$ Γ-converge to

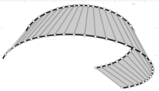
$$I^{2d}(y) = \frac{1}{2}\int_\omega |II|^2\,\mathrm{d}x - \int_\omega \widetilde{f} \cdot y\,\mathrm{d}x$$

for isometries $y : \omega \to \mathbb{R}^3$, i.e., $I = \mathbb{I}_{2\times 2}$, with fundamental forms

$$I = \nabla y^\mathsf{T} \nabla y, \qquad II = \nu^\mathsf{T} D^2 y, \qquad \nu = \partial_1 y \times \partial_2 y$$

Isometry condition implies $\partial_j^2 y \cdot \partial_k y = 0$ and

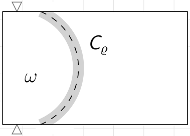
$$|II|^2 = 4H^2 = |\Delta y|^2 = |D^2 y|^2, \quad K = 0$$



Prepared material: Inhomogeneous material and deformation $y : \Omega \to \mathbb{R}^3$

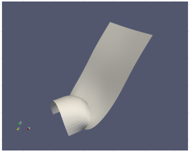
$$I^{3d}(y) = \int_\Omega W(x, \nabla y)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega f \cdot y\,\mathrm{d}x\,\mathrm{d}t$$

Material softer (damaged) along arc $C \subset \overline{\omega}$



Model reduction: $\Omega = \omega \times (-\delta/2, \delta/2)$

- narrow region C_ϱ, soft material $\mathbb{C}_\varepsilon$
- appropriate scaling relations $\varepsilon, \varrho, \delta$
- ∇y discontinuous across C



For isometry $y \in H^2(\omega \setminus C; \mathbb{R}^3) \cap W^{1,\infty}(\omega; \mathbb{R}^3)$

$$I_{\mathrm{fold}}(y) = \frac{1}{2}\int_{\omega\setminus C} |D^2 y|^2\,\mathrm{d}x - \int_\omega \widetilde{f} \cdot y\,\mathrm{d}x$$

Proof: [B., Bonito & Hornung '22] following [Friesecke, James & Müller '02]

Related: [Conti & Dolzmann '09, Santilli & Schmidt '23]

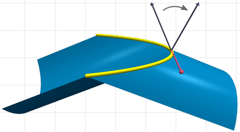
Isometries: Piecewise C^1 isometry [Kirchheim '01, Müller & Pakzad '05] $y : \omega \to \mathbb{R}^3$

$$(\nabla y)^{\mathsf{T}} \nabla y = \mathbb{I}_{2\times 2}$$

Folding arcs: Folding curve $b : I \to \overline{\omega}$ maintains geodesic curvature κ under isometric deformation

Darboux frames: Normals n^ℓ define Darboux frames

$$r^\ell = [\gamma', n^\ell, \gamma' \times n^\ell] \in SO(3), \quad \gamma = y \circ b$$



Folding angle: Since frames share tangent γ'

$$r^2 = R(\theta, \gamma') r^1$$

Curvatures: *Geodesic* $\kappa = (r_1^\ell)' \cdot r_2^\ell$ and *normal* $\mu^\ell = (r_1^\ell)' \cdot r_3^\ell$ curvatures and *torsion* $\tau^\ell = (r_2^\ell)' \cdot r_3^\ell$ related via, unless $\theta \in 2\pi\mathbb{Z}$,

$$\kappa \sin\left(\frac{\theta}{2}\right) = \pm \mu^\ell \cos\left(\frac{\theta}{2}\right), \quad \tau^2 = \tau^1 + \theta'$$

Related: Simpler version in [Duncan & Duncan '82]

Implications: For deformed plate along crease C

- if $\kappa = 0$ then either unfolded or folded back or θ constant and $\mu^\ell = 0$
- if $\kappa \neq 0$ then either unfolded or $\mu^\ell \neq 0$ and θ uniquely defined

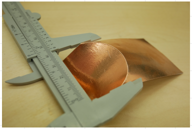
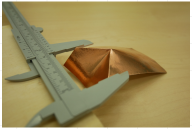
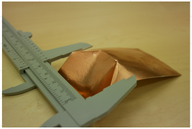
Related: Periodic kirigami structures (e.g., maps and deployable structures)

- [Liu, Choi, Mahadevan '21] 17 patterns define periodic tilings of the plane
- [James & Liu '22+] Origami structures with curved tiles between creases

Polygonal creases

No paradox? $|D^2 y|^2 = |\Delta y|^2$ for isometries; K enters via iso constraint

Experiment: Singularities at corner points, i.e., $y_m \notin H^2(\omega \setminus C_m)$?

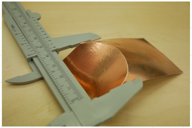
Theorem [B., Bonito & Hornung '24+]
There are no nontrivially folded isometries $y_m \in H^2(\omega \setminus C_m)$ for polygonal crease lines C_m which are C^1 in the closure of a subdomain.

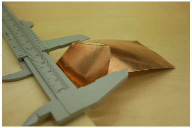
Idea of proof:

- y_m folded $\implies$ flat ($\mu = 0$, θ constant) or folded back (μ arbitrary, $\theta = \pi$)
- If $y_m \in C^1(\overline{\omega}_1)$ then $\nabla y(x_c^{\pm}) = Q_{\pm} \nabla y(x_c)$ with $Q_+ \neq Q_-$
- Obtain jumps of ∇y_m in corner x_c contradicting H^2 property □

Slit segments

Idea: Use slits along segments

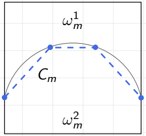


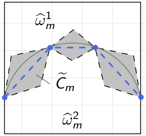


Approximation:

$$\mathcal{A}_m = \{y \in H^2(\omega \setminus C_m) \cap W^{1,\infty}(\omega) : \\ y \text{ iso \& continuous in corners}\}$$

$$I_m(y) = \frac{1}{2} \int_{\omega \setminus C_m} |D^2 y|^2 \, \mathrm{d}x, \quad y \in \mathcal{A}_m$$





Theorem [B., Bonito & Hornung '24+] Γ-convergence $I_m \to I$.

Proof: (i) $D^2 y_m$ p/w bounded in L^2 gives weak limit H; linear interpolation of y_m gives limit $y \in W^{1,\infty}(\omega)$ satisfying iso constraint and $D^2 y = H$ in $\omega \setminus C$
(ii) Extensions/restrictions $y|_{\omega_m^i \cap \omega^i}$ provide recovery sequence □

Avoid extension: Cut out diamonds along C_m so that $\widehat{\omega}_m^i \subset \omega^i$

- Babuška's paradox in variational formulation
- Impose simple support in corners only, avoid curved elements
- Paradox explains singularities in nonlinear bending problems

- Ongoing and future work:
  - Efficient numerics
  - Optimize crease lines
  - Transfer to applications

- More information

http://aam.uni-freiburg.de/bartels

Thank you !