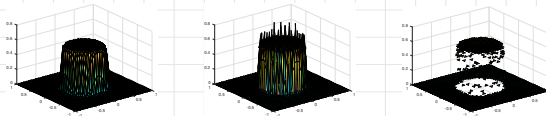


Error estimates for nonconforming and discontinuous discretizations of nonsmooth problems via convex duality



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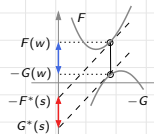
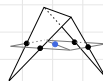
Joint seminar UMD College Park & Baltimore, George Mason U, U Delaware

Sayas Numerics Seminar, September 7, 2021

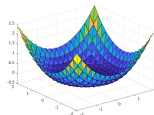
Joint work with Robert Tovey (INRIA), Zhangxian Wang, Friedrich Wassmer



- ① TV Model Problem
- ② Convexity and Nonstandard FE
- ③ Nonsmooth Examples
- ④ Generalization to dG
- ⑤ Mesh Grading and Adaptivity
- ⑥ Summary



$$(\Pi_h \mathcal{R} T_N^0)^\perp = \nabla_h(\ker \Pi_h|_{S_D^{1,\sigma}})$$



TV Model Problem

Example: Image processing via TV minimization [Rudin, Osher & Fatemi '92]

$$I(u) = |Du|(\Omega) + \frac{\alpha}{2} \|u - g\|^2$$

- ▶ nondifferentiable convex problem with discontinuous solutions
- ▶ formal Euler-Lagrange equations – $\operatorname{div} \frac{\nabla u}{|\nabla u|} + \alpha(u - g) = 0$

Experiment:

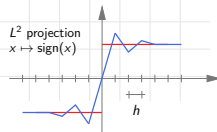
P1-FE
 $h \sim 2^{-5}$



Accuracy: Error estimates for L^2 error if $u \in L^\infty(\Omega) \cap BV(\Omega)$

- ▶ $\mathcal{O}(h^{1/4})$ for P1 FE [Wang & Lucier '11, B. '12, B., Nochetto & Salgado '14]
- ▶ $\mathcal{O}(h^{1/2})$ for anisotropic TV [B., Nochetto & Salgado '15]
- ▶ $\mathcal{O}(h^{1/2})$ for P0/CR FE [Chambolle & Pock '19+, B. '20+]

Note: Rate $\mathcal{O}(h^{1/2})$ optimal for generic BV functions



Variational problem: With convex functionals ϕ, ψ

$$I(u) = \int_{\Omega} \phi(\nabla u) \, dx + \int_{\Omega} \psi(x, u) \, dx$$

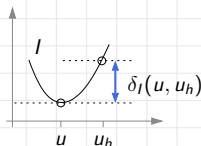
- formal Euler-Lagrange equations: $-\operatorname{div} \phi'(\nabla u) + \psi'(\cdot, u) = 0$

Coercivity: Optimality of u

$$\delta_I(u, u_h) \leq I(u_h) - I(u)$$

Error estimate: By minimality of u_h for every v_h

$$\delta_I(u, u_h) \leq I(v_h) - I(u)$$



Hence: Suitable interpolant $v_h = \mathcal{J}_h u$ leads to error estimate

- control of inconsistent $I_h \approx I$
- optimality and regularity
- use of duality $I(u) \geq D(z)$

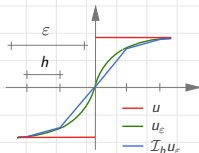
Model: ROF minimization in $BV(\Omega)$, discretization with $P1$ -FE

$$\frac{\alpha}{2} \|u - u_h\|^2 \leq I(v_h) - I(u)$$

Interpolant: With regularization u_ε set $v_h = \mathcal{I}_h u_\varepsilon$

$$\|u - \mathcal{I}_h u_\varepsilon\|_{L^1} \leq c(h^2 \varepsilon^{-1} + \varepsilon) |Du|(\Omega),$$

$$\|\nabla \mathcal{I}_h u_\varepsilon\|_{L^1} \leq (1 + ch\varepsilon^{-1} + c\varepsilon) |Du|(\Omega)$$



Continuity: With binomial formula and $\varepsilon = h^{1/2}$

$$\begin{aligned} \frac{\alpha}{2} \|u - u_h\|^2 &\leq |Dv_h|(\Omega) - |Du|(\Omega) + \frac{\alpha}{2} \int_{\Omega} (v_h - u)(v_h + u - 2g) \, dx \\ &\leq ch\varepsilon^{-1} + c(h^2 \varepsilon^{-1} + \varepsilon) = \mathcal{O}(h^{1/2}) \end{aligned}$$

Optimality via TVD: If $\tilde{\mathcal{I}}_h : W^{1,1}(\Omega) \rightarrow \mathcal{S}^1(\mathcal{T}_h)$ with $\|\nabla \tilde{\mathcal{I}}_h v\|_{L^1} \leq \|\nabla v\|_{L^1}$ then

$$\frac{\alpha}{2} \|u - u_h\|^2 = \mathcal{O}(h)$$

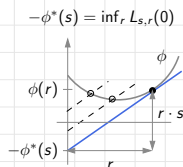
OK: Anisotropic ℓ^1 version of BV and structured grids [B., Nochetto & Salgado '15]

Convexity and Nonstandard FE

Conjugate: For convex $\phi : \mathbb{R}^\ell \rightarrow \mathbb{R}$ define *convex conjugate*

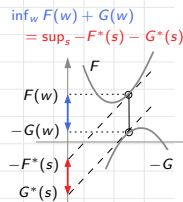
$$\phi^*(s) = \sup_{r \in \mathbb{R}^\ell} r \cdot s - \phi(r)$$

i.e., Young's ineq $r \cdot s \leq \phi(r) + \phi^*(s)$, equality for $s = \phi'(r)$



Duality: Use $\phi^{**} = \phi$, integrate by parts, exchange extrema

$$\begin{aligned} \inf_u I(u) &= \inf_u \int_{\Omega} \overbrace{\sup_z \nabla u \cdot z - \phi^*(z)}^{=\phi(\nabla u)} dx + \int_{\Omega} \psi(x, u) dx \\ &\geq \sup_z \inf_u - \int_{\Omega} \phi^*(z) dx - \int_{\Omega} u \operatorname{div} z - \psi(x, u) dx \\ &= \sup_z - \int_{\Omega} \phi^*(z) dx - \int_{\Omega} \psi^*(x, \operatorname{div} z) dx = \sup_z D(z) \end{aligned}$$



Strong duality: If saddle (u, z) exists then equality $I(u) = D(z)$ holds

- important implicit information in dual solution $z = \phi'(\nabla u)$

Motivation: Continuous $P1$ vector fields too stiff for divergence, e.g. in Stokes,

$$\operatorname{div} : \mathcal{S}^1(\mathcal{T}_h)^d \rightarrow \mathcal{L}^0(\mathcal{T}_h) \quad \text{NOT surjective}$$

CR-FEM: $\mathcal{S}^{1,cr}(\mathcal{T}_h)$ continuous (only) at side midpoints [Crouzeix–Raviart '73]



Quasi-interpolant: Averages over element sides

$$\mathcal{J}_h : H^1(\Omega) \rightarrow \mathcal{S}^{1,cr}(\mathcal{T}_h), \quad \mathcal{J}_h v(x_S) = |S|^{-1} \int_S v \, ds$$

Projection property: Elementwise gradient given by projection onto constants

$$\nabla_h \mathcal{J}_h v = \Pi_h \nabla v$$

Jensen's inequality: For convex function ϕ

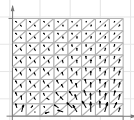
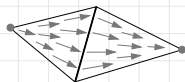
$$\int_{\Omega} \phi(\nabla_h \mathcal{J}_h v) \, dx \leq \int_{\Omega} \phi(\nabla v) \, dx$$

Motivation: Divergence operator on L^2 vector fields (instead H^1)

RT-FEM: $\mathcal{RT}^0(\mathcal{T}_h)$ affine vector fields with weak divergence [Raviart–Thomas '77]

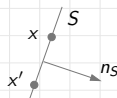
$$q_h|_T(x) = ax + b$$

$$a \in \mathbb{R}, b \in \mathbb{R}^d$$



Idea: Constant normal components on hyperplanes

$$q_h(x) \cdot n_S = c \quad \forall x \in S$$



Quasi-interpolant: Averages of normal components over sides

$$\mathcal{L}_h : W^{1,1}(\Omega; \mathbb{R}^d) \rightarrow \mathcal{RT}^0(\mathcal{T}_h), \quad \mathcal{L}_h q(x_S) \cdot n_S = |S|^{-1} \int_S q \cdot n_S \, ds$$

Projection property: Surjectivity of $\operatorname{div} : \mathcal{RT}^0(\mathcal{T}_h) \rightarrow \mathcal{L}^0(\mathcal{T}_h)$ via

$$\operatorname{div} \mathcal{L}_h q = \Pi_h \operatorname{div} q$$

Properties of CR and RT: On element sides $S \in \mathcal{S}_h$ with normal n_S

- ▶ jumps $\llbracket v_h \rrbracket$ of CR functions over sides have vanishing mean
- ▶ components $q_h \cdot n_S$ for RT fields constant and continuous

Integration by parts: For $v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ and $q_h \in \mathcal{RT}_N^0(\mathcal{T}_h)$

$$\begin{aligned} \int_{\Omega} v_h \operatorname{div} q_h \, dx &= - \sum_{T \in \mathcal{T}_h} \int_T \nabla v_h \cdot q_h \, dx + \sum_{S \in \mathcal{S}_h} \underbrace{\int_S \llbracket v_h \rrbracket q_h \cdot n_S \, ds}_{=0} \\ &= - \int_{\Omega} \nabla_h v_h \cdot q_h \, dx \end{aligned}$$

Projection onto constants: L^2 projection $\Pi_h : L^2(\Omega; \mathbb{R}^\ell) \rightarrow \mathcal{L}^0(\mathcal{T}_h)^\ell$

Duality connection: Within $\mathcal{L}^0(\mathcal{T}_h)^\ell$

$$\int_{\Omega} \Pi_h v_h \operatorname{div} q_h \, dx = - \int_{\Omega} \nabla_h v_h \cdot \Pi_h q_h \, dx$$

- ▶ projection onto constants important to apply Fenchel's inequality

Imitate arguments: For $u_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ and $z_h \in \mathcal{RT}_N^0(\mathcal{T}_h)$

$$\begin{aligned}
 I_h(u_h) &= \int_{\Omega} \phi(\nabla_h u_h) + \psi_h(\Pi_h u_h) \, dx \\
 &\geq \int_{\Omega} -\phi^*(\Pi_h z_h) + \Pi_h z_h \cdot \nabla_h u_h + \psi_h(\Pi_h u_h) \, dx \\
 &= \int_{\Omega} -\phi^*(\Pi_h z_h) - \operatorname{div} z_h \Pi_h u_h + \psi_h(\Pi_h u_h) \, dx \\
 &\geq \int_{\Omega} -\phi^*(\Pi_h z_h) - \psi_h^*(\operatorname{div} z_h) \, dx \\
 &= D_h(z_h)
 \end{aligned}$$

Weak duality: Discrete functionals I_h and D_h satisfy $I_h(u_h) \geq D_h(z_h)$

Strong duality: If $\phi, \psi_h \in C^1$ and u_h minimal for I_h then

$$z_h = D\phi(\nabla_h u_h) + D\psi_h(\cdot, \Pi_h z_h) d^{-1}(x - x_T) \implies I_h(u_h) = D_h(z_h)$$

Related: $z_h = D\phi(\nabla_h u_h) + d^{-1}(x - x_T) \Pi_h f$ [Marini '85, Carstensen & Liu '15]

Nonlinear Dirichlet problem: With $f_h = \Pi_h f$ consider

$$I_h(u_h) = \int_{\Omega} \phi(\nabla_h u_h) \, dx - \int_{\Omega} f_h \Pi_h u_h \, dx,$$

$$D_h(z_h) = - \int_{\Omega} \phi^*(\Pi_h z_h) \, dx - I_{\{-f_h\}}(\operatorname{div} z_h)$$

A priori analysis: Control discrete error $\delta_{I_h}(u_h, \mathcal{J}_h u)$ via

$$\begin{aligned} \delta_{I_h} &\leq I_h(\mathcal{J}_h u) - I_h(u_h) \leq I_h(\mathcal{J}_h u) - D_h(\mathcal{L}_h z) && \blacktriangleright \text{discrete coercivity \& duality} \\ &\leq \int_{\Omega} \phi(\nabla u) - \mathcal{L}_h z_h \cdot \nabla_h \mathcal{J}_h u + \phi^*(\Pi_h \mathcal{L}_h z) \, dx && \blacktriangleright \text{Jensen's inequality,} \\ & && \operatorname{div} \mathcal{L}_h z = -f_h \\ &\leq \int_{\Omega} -\phi^*(z) + (z - \Pi_h \mathcal{L}_h z) \cdot \nabla u + \phi^*(\Pi_h \mathcal{L}_h z) \, dx && \blacktriangleright \text{strong duality } I(u) = D(z), \\ & && \operatorname{div} z = -f, \nabla_h \mathcal{J}_h u = \Pi_h \nabla u \\ &\leq \int_{\Omega} (D\phi^*(z) - D\phi^*(\Pi_h \mathcal{L}_h z)) \cdot (z - \Pi_h \mathcal{L}_h z) \, dx && \blacktriangleright \text{convexity, } \nabla u = D\phi^*(z) \\ &= \mathcal{O}(h^2) && \blacktriangleright \text{interpolation estimates,} \\ & && \text{if, e.g., } \phi^* \text{ Lipschitz} \end{aligned}$$

Generalization: Convex variational problems on $W_D^{1,p}(\Omega)$

Theorem ([B. '20+]). If $z \in W_N^q(\operatorname{div}; \Omega) \cap W^{1,1}(\Omega; \mathbb{R}^d)$

$$\begin{aligned}\delta_{I_h}(u_h, \mathcal{J}_h u) &\leq \int_{\Omega} (D\phi^*(z) - D\phi^*(\Pi_h \mathcal{L}_h z)) \cdot (z - \Pi_h \mathcal{L}_h z) \, dx \\ &\quad + \int_{\Omega} (D\psi(u) - D\psi_h(\Pi_h \mathcal{J}_h u)) \cdot (u - \Pi_h \mathcal{J}_h u) \, dx \\ &\quad + \int_{\Omega} \psi_h(u) - \psi(u) \, dx + \int_{\Omega} \psi_h^*(\Pi_h \operatorname{div} z) - \psi^*(\operatorname{div} z) \, dx.\end{aligned}$$

Interpretation: CR discretization error controlled by RT interpolation error

- ▶ inconsistency $\nabla_h \approx \nabla$ and $I_h \approx I$ estimated via duality
- ▶ low order terms reduce to $(\gamma/2)\|u - \Pi_h \mathcal{J}_h u\|^2$ for quadratic ψ
- ▶ avoids use of Strang lemmas; guarantee feasibility of interpolants
- ▶ use of quadrature important and computationally efficient

Orthogonality relation: Within $(\mathcal{L}^0)^d \subset L^2(\Omega; \mathbb{R}^d)$ [B. & Wang '20+]

$$(\Pi_h \mathcal{R} T_N^0)^\perp = \nabla_h (\ker \Pi_h|_{\mathcal{S}_D^{1,cr}}) \iff A^\perp = B$$

Proof: (i) If $\Pi_h v_h = 0$ then for all $y_h \in \mathcal{R} T_N^0$

$$(\nabla_h v_h, \Pi_h y_h) = -(\Pi_h v_h, \operatorname{div} y_h) = 0 \implies A \subset B^\perp$$

(ii) Let $y_h \in B^\perp$. Choose $r_h \in Z_h = (\ker \Pi_h|_{\mathcal{S}_D^{1,cr}})^\perp \subset \mathcal{S}_D^{1,cr}$ with

$$(\Pi_h r_h, \Pi_h v_h) = (y_h, \nabla_h v_h) \quad \forall v_h \in Z_h.$$

True for all $v_h \in \mathcal{S}_D^{1,cr}$. Let $z_h \in \mathcal{R} T_N^0$ with $-\operatorname{div} z_h = \Pi_h r_h$

$$(y_h - z_h, \nabla_h v_h) = (\Pi_h r_h, \Pi_h v_h) + (\operatorname{div} z_h, v_h) = 0 \quad \forall v_h \in \mathcal{S}_D^{1,cr}$$

Define $\tilde{y}_h|_T = y_h|_T + \operatorname{div} z_h|_T (x - x_T)/d$ so that

$$(\tilde{y}_h - z_h, \nabla_h v_h) = (y_h - z_h, \nabla_h v_h) = 0 \quad \forall v_h \in \mathcal{S}_D^{1,cr}$$

Implies $\tilde{y}_h - z_h \in \mathcal{R} T_N^0$. Since $\Pi_h \tilde{y}_h = y_h$ deduce $y_h \in A$ and $B^\perp \subset A$. □

Remarks: Identity uses inf-sup condition and has consequences

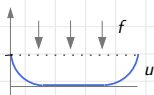
- ▶ strong discrete duality relation
- ▶ dual variant $\operatorname{div} \ker \Pi_h|_{\mathcal{R} T_N^0} = (\Pi_h \mathcal{S}_D^{1,cr})^\perp \implies \Pi_h \mathcal{S}_D^{1,cr} = \mathcal{L}^0$ if $\Gamma_D \neq \partial\Omega$
- ▶ alternative proof uses discrete Poincaré lemma [Chambolle & Pock '19+]

Nonsmooth Examples

Functionals: For $f \in L^2(\Omega)$ impose $u \geq 0$ via indicator functionals I_+

$$I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} f u dx + I_+(u),$$

$$D(z) = -\frac{1}{2} \int_{\Omega} |z|^2 dx - I_-(\operatorname{div} z + f)$$



Low order terms: Pointwise convex conjugation

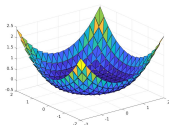
$$\psi(x, s) = -f(x)s + I_+(s) \quad \implies \quad \psi^*(x, t) = I_-(t + f(x))$$

Complementarity: $z = \nabla u$ and $\operatorname{div} z + f = 0$ if $u > 0$

Discretization: Use $f_h = \Pi_h f$ and impose constraint at midpoints of elements

$$I_h(u_h) = \frac{1}{2} \int_{\Omega} |\nabla_h u_h|^2 dx - \int_{\Omega} f_h u_h dx + I_+(\Pi_h u_h),$$

$$D_h(z_h) = -\frac{1}{2} \int_{\Omega} |\Pi_h z_h|^2 dx - I_-(\operatorname{div} z_h + f_h)$$



Proposition ([B. '20+]). If $z = \nabla u \in H^1(\Omega; \mathbb{R}^d)$ then

$$\|\nabla_h(u_h - u)\| \leq ch(\|D^2 u\| + \|f + \operatorname{div} z\|).$$

Proof: Since $\mathcal{J}_h u$ and $\mathcal{L}_h z$ admissible with $\mu = f + \operatorname{div} z$

$$\begin{aligned} \delta_h^2 &\leq \int_{\Omega} (f + \operatorname{div} z)(u - \Pi_h \mathcal{J}_h u) \, dx + \frac{1}{2} \int_{\Omega} |z - \Pi_h \mathcal{L}_h z|^2 \, dx \\ &= \int_{\Omega} \mu(\mathcal{J}_h u - \Pi_h \mathcal{J}_h u) \, dx + \int_{\Omega} \mu(u - \mathcal{J}_h u) \, dx + \frac{1}{2} \int_{\Omega} |z - \Pi_h \mathcal{L}_h z|^2 \, dx. \end{aligned}$$

For element contact set $\mathcal{C}_T = \{x \in T : u(x) = 0\}$

$$\mu|_{T \setminus \mathcal{C}_T} = 0, \quad \nabla u|_{\mathcal{C}_T} = 0.$$

With $\mathcal{J}_h u(x) = \mathcal{J}_h u(x_T) + \nabla_h \mathcal{J}_h u|_T \cdot (x - x_T)$ for critical term **A**

$$\mathbf{A}|_T = \int_{\mathcal{C}_T} \mu \nabla_h(\mathcal{J}_h u - u) \cdot (x - x_T) \, dx \leq h_T \|\mu\|_{L^2(T)} \|\nabla_h(\mathcal{J}_h u - u)\|_{L^2(T)}. \quad \square$$

► note constant-free intermediate estimate

Primal problem: For noisy image $g \in L^\infty(\Omega)$ find u via minimizing

$$I(u) = |Du|(\Omega) + \frac{\alpha}{2} \|u - g\|^2$$

Total variation: Extends $\|\nabla u\|_{L^1(\Omega)}$ via operator norm of Du on $C(\bar{\Omega})$

$$|Du|(\Omega) = \sup \left\{ - \int_{\Omega} u \operatorname{div} z \, dx : z \in C_c^\infty(\Omega; \mathbb{R}^d), |z|_{\ell^2} \leq 1 \text{ in } \Omega \right\}$$

Dual problem: By exchanging extrema $u = g + \alpha^{-1} \operatorname{div} z$

$$D(z) = -\frac{1}{2\alpha} \|\operatorname{div} z + \alpha g\|^2 + \frac{\alpha}{2} \|g\|^2 - I_{K_1(0)}(z)$$

Optimality: $z \in \partial|Du|$ and strong duality $I(u) = D(z)$ [Hintermüller & Kunisch '04]

Discretization: With $g_h = \Pi_h g$ consider inconsistent I_h and D_h

$$I_h(u_h) = \int_{\Omega} |\nabla_h u_h| \, dx + \frac{\alpha}{2} \|\Pi_h u_h - g_h\|^2,$$

$$D_h(z_h) = -\frac{1}{2\alpha} \|\operatorname{div} z_h + \alpha g_h\|^2 + \frac{\alpha}{2} \|g_h\|^2 - I_{K_1(0)}(\Pi_h z_h)$$

Proposition ([Chambolle & Pock '19+, B. '20+]). If $z \in W^{1,\infty}(\Omega; \mathbb{R}^d)$ then

$$\|u - \Pi_h u_h\| \leq ch^{1/2} (\|u\|_{L^\infty} |Du|(\Omega) + \|g\| \|\nabla z\|_{L^\infty} \|\operatorname{div} z\|)^{1/2}.$$

Proof: For admissible $\tilde{u}_h$ and $\tilde{z}_h$

$$\frac{1}{2} \|\Pi_h(u_h - \tilde{u}_h)\|^2 \leq I_h(\tilde{u}_h) - I_h(u_h) \leq I_h(\tilde{u}_h) - D_h(\tilde{z}_h)$$

For $\tilde{u}_h = \lim_{\varepsilon \rightarrow 0} \mathcal{J}_h u_\varepsilon$ have TVD and approximation

$$\|\nabla_h \tilde{u}_h\|_{L^1} \leq |Du|(\Omega), \quad \|\tilde{u}_h - u\|_{L^1} \leq ch |Du|(\Omega),$$

and hence

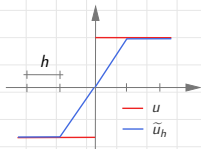
$$I_h(\tilde{u}_h) \leq I(u) + \frac{\alpha}{2} \|\Pi_h \tilde{u}_h - u\|_{L^1} 4 \|g\|_{L^\infty} - \frac{\alpha}{2} \|g - g_h\|^2.$$

For $\tilde{z}_h = \gamma_h^{-1} \mathcal{L}_h z$ with $\gamma_h = \|\mathcal{L}_h z\|_{L^\infty}$ have

$$D_h(\tilde{z}_h) \geq D(z) - (1 - \gamma_h^{-1}) \|g\| \|\operatorname{div} z\| - \frac{\alpha}{2} \|g - g_h\|^2.$$

Strong duality $I(u) = D(z)$ gives estimate. □

► note canceling low order contributions; no regularity conditions on g

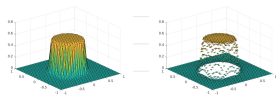


Setup: For Dirichlet BC, $g = \chi_{B_r(0)}$ have

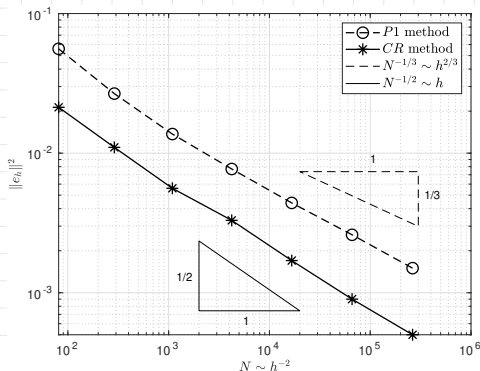
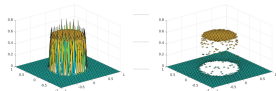
$$u = \max\{0, 1 - d/(\alpha r)\} \chi_{B_r(0)}, \quad z \in W^{1,\infty}(\Omega; \mathbb{R}^d)$$

- ▶ choose $r = 1/2$, $d = 2$, $\Omega = (-1, 1)^2$, and $\alpha = 10$
- ▶ iterative solution via semi-implicit gradient descent [B., Diening & Nochetto '18]

P1 solution u_h^{p1} and $\Pi_h u_h^{p1}$



CR solution u_h^{cr} and $\Pi_h u_h^{cr}$



Single disc phantom [Chambolle et al. '10]: For $B = B_1(0) \subset \subset \Omega$ and $g = \chi_B$, Dirichlet BC, ROF minimizer $u = c_\alpha g$ with Lipschitz continuous dual

$$z(x) = -c'_\alpha \begin{cases} x, & |x| \leq 1 \\ x/|x|^2, & |x| \geq 1 \end{cases}$$

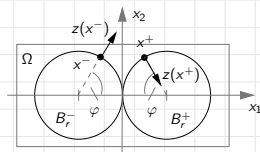
Optimality: Dual solutions satisfy $z \in \partial|Du|$ and $\operatorname{div} z = -\alpha(u - g)$

Touching discs [B., Tovey & Wassmer '21+]:

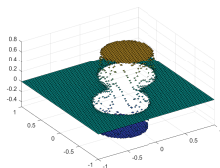
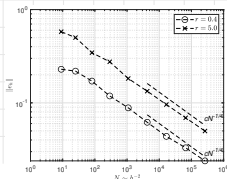
For symmetric Ω , $g = \chi_{B^+} - \chi_{B^-}$, Dirichlet BC, have $u = c'_{1,\alpha} g$ and $z = \mp \nu$ on $\partial B^\pm$ and

$$|x^+ - x^-| \approx \varphi^2, \quad |z(x^+) - z(x^-)| \approx 2\varphi$$

$\Rightarrow z$ not Lipschitz continuous



**No reduction of
CR convergence
rate:**

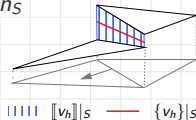


Generalization to dG

Jumps and averages: On inner sides $S \in \mathcal{S}_h$ with normal n_S

$$[[v_h]](x) = \lim_{\varepsilon \rightarrow 0} (v_h(x - \varepsilon n_S) - v_h(x + \varepsilon n_S)),$$

$$\{v_h\}(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2} (v_h(x - \varepsilon n_S) + v_h(x + \varepsilon n_S))$$



► midpoint evaluation $[[v_h]]_h|_S = [[v_h]](x_S)$ and $\{v_h\}_h|_S = \{v_h\}(x_S)$

Broken CR and RT spaces: Omit all continuity requirements

$$\mathcal{S}^{1,dg}(\mathcal{T}_h) = \mathcal{L}^1(\mathcal{T}_h), \quad \mathcal{RT}^{0,dg}(\mathcal{T}_h) = \mathcal{L}^0(\mathcal{T}_h)^d + (\text{id} - x_T)\mathcal{L}^0(\mathcal{T}_h)$$

Integration by parts: With $[[v_h y_h]] = [[v_h]]\{y_h\} + \{v_h\}[[y_h]]$

$$\begin{aligned} \int_{\Omega} v_h \operatorname{div}_h y_h \, dx + \int_{\Omega} \nabla_h v_h \cdot y_h \, dx \\ = \int_{\mathcal{S}_h \setminus \Gamma_N} [[v_h]]_h \{y_h \cdot n_S\} \, ds + \int_{\mathcal{S}_h \setminus \Gamma_D} \{v_h\}_h [[y_h \cdot n_S]] \, ds \end{aligned}$$

► reveals duality of jumps and averages

Stabilizing and regularizing terms: For $u_h \in \mathcal{S}^{1,dg}(\mathcal{T}_h)$ and $z_h \in \mathcal{RT}^{0,dg}(\mathcal{T}_h)$

$$J_h(u_h) = \frac{1}{r} \|\alpha_S^{-1} \llbracket u_h \rrbracket_h\|_{L^r(S_h \setminus \Gamma_N)}^r + \frac{1}{s} \|\beta_S \{u_h\}_h\|_{L^s(S_h \setminus \Gamma_D)}^s,$$

$$K_h(z_h) = \frac{1}{r'} \|\alpha_S \{z_h \cdot n_S\}\|_{L^{r'}(S_h \setminus \Gamma_N)}^{r'} + \frac{1}{s'} \|\beta_S^{-1} \llbracket z_h \cdot n_S \rrbracket\|_{L^{s'}(S_h \setminus \Gamma_D)}^{s'}$$

Proposition ([B. '20+]). $I_h(u_h) \geq D_h(z_h)$ for primal and dual dG functionals

$$I_h(u_h) = \int_{\Omega} \phi(\nabla_h u_h) + \psi_h(x, \Pi_h u_h) \, dx + J_h(u_h),$$

$$D_h(z_h) = - \int_{\Omega} \phi^*(\Pi_h z_h) + \psi_h^*(x, \operatorname{div}_h z_h) \, dx - K_h(z_h).$$

If $r = s = 2$ and $\beta_S = 0$ and $\phi, \psi \in C^1$ then strong duality applies.

- use of quadrature in jumps and averages essential

Nonlinear Dirichlet problems: Consider low order term $\psi(x, s) = -f(x)s$

Proposition ([B. '20+]). If $z \in W^{1,1}(\Omega; \mathbb{R}^d)$

$$\begin{aligned} \delta_{I_h}(u_h, \mathcal{J}_h u) &\leq \int_{\Omega} (D\phi^*(z) - D\phi^*(\Pi_h \mathcal{L}_h z)) \cdot (z - \Pi_h \mathcal{L}_h z) \, dx \\ &\quad + J_h(\mathcal{J}_h u) + K_h(\mathcal{L}_h z), \end{aligned}$$

where

$$\begin{aligned} J_h(\mathcal{J}_h u) &\leq c_{\mathcal{T}} \|h_S^{-1} \beta_S^s\|_{L^\infty(S)} \|\mathcal{J}_h u\|_{L^s(\Omega)}^s, \\ K_h(\mathcal{L}_h z) &\leq c_{\mathcal{T}} \|h_S^{-1} \alpha_S^{r'}\|_{L^\infty(S)} \|\mathcal{L}_h z\|_{L^{r'}(\Omega)}^{r'}. \end{aligned}$$

Examples: Strength of jump terms

- ▶ Poisson: $r = 2$ and $\alpha_S = h^{3/2}$ (sharp)
- ▶ TV: $r = 1$ and $\alpha_S \leq 1$ or $r = 2$ and $\alpha_S = h^p$, $p \geq 2$

Consistency: Discretize then optimize

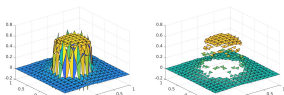
- ▶ note disappearing jump terms for CR and RT comparison functions
- ▶ lack of consistency terms for elliptic problems requires overpenalization

Regularized functional: With $|a|_\varepsilon = (|a|^2 + \varepsilon^2)^{1/2}$

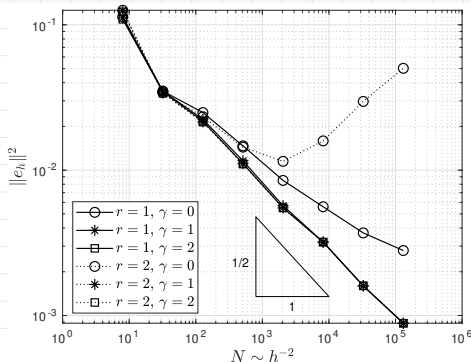
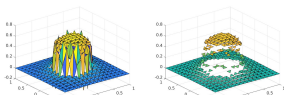
$$I_{h,\varepsilon}(u_h) = \int_{\Omega} |\nabla_h u_h|_\varepsilon \, dx + \frac{\alpha}{2} \|\Pi_h u_h - g_h\|^2 + \frac{c_\alpha^{-r}}{r} \int_S h_S^{-\gamma r} \llbracket u_h \rrbracket_\varepsilon^r \, ds$$

► uniform estimate $0 \leq |a|_\varepsilon - |a| \leq \varepsilon$ justifies $\varepsilon = h$

Linear: $r = 1, \gamma = 1, c_\alpha = 1$



Quadratic: $r = 2, \gamma = 1, c_\alpha = 1$



Mesh Grading and Adaptivity

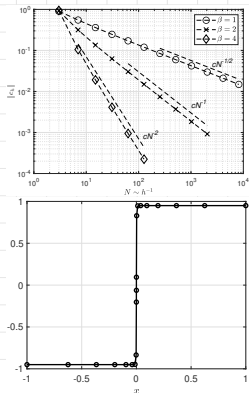
L^2 Best approximation: For $u = \text{sign} + v$, $v \in H^1(a, b)$

$$c_1 h_0^{1/2} \leq \inf_{u_h \in \mathcal{S}^1(\mathcal{T}_h)} \|u - u_h\| \leq c_2 (h_0^{1/2} + h_{\max} \|v'\|)$$

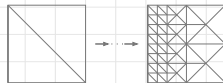
Idea: Use β -graded mesh with $h_{\min} \sim h_{\max}^\beta$

1D ROF model: If $u \in W^{2,1}((a, b) \setminus J_u)$

$$\begin{aligned} \frac{\alpha}{2} \|u - u_h\|^2 &\leq I(\mathcal{I}_h u) - I(u) \\ &= \underbrace{\|(\mathcal{I}_h u)'\|_{L^1(a,b)} - |Du|(a,b)}_{\leq 0 \text{ (TVD property of } \mathcal{I}_h)} \\ &\quad + \frac{\alpha}{2} \underbrace{\int_a^b (\mathcal{I}_h u - g)^2 - (u - g)^2 dx}_{\leq 4 \|u - \mathcal{I}_h u\|_{L^1} \|g\|_{L^\infty}} \\ &\leq c\alpha (h_{\min} |Du|(a, b) \\ &\quad + h_{\max}^2 \|u''\|_{L^1((a,b) \setminus J_u)} \|g\|_{L^\infty}) \end{aligned}$$



Graded triangulations: Shape regularity and conformity limit grading strength in $\mathbb{R}^d$, $d > 1$



CR discretization: Use discrete TVD property and discrete duality as above

$$\frac{\alpha}{2} \|\Pi_h(\tilde{u}_h - u_h)\|^2 \leq 2\alpha \|\Pi_h \tilde{u}_h - u\|_{L^1(\Omega)} \|g\|_{L^\infty(\Omega)} + (1 - \gamma_h^{-1}) \|g\| \|\operatorname{div} z\|$$

Difficulty: Local error $\mathcal{O}(h_T)$ even in regular regions, $\gamma_h \geq \max_{T \in \mathcal{T}_h} |J_{RT} z(x_T)|$

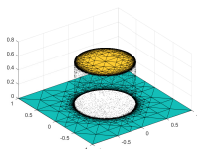
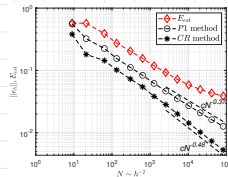
Proposition ([B., Tovey & Wassmer '21+]). If ROF sol. $u \in BV(\Omega)$ p/w constant, dual $z \in W^{1,\infty}(\Omega; \mathbb{R}^d)$ s.t.

$$|z(x)| \leq 1 - \ell d_{J_u}(x)$$

and triangulations quadratically graded

$$h_T \leq c \begin{cases} h d_{J_u}(T)^{1/2} & \text{if } d_{J_u}(T) \geq h^2, \\ h^2 & \text{otherwise} \end{cases}$$

then $\|\Pi_h(u - \tilde{u}_h)\| \leq h |\log h| M(\alpha, u, z, g)$.



Continuous coercivity: For conforming approximation $u_h^c \in W^{1,p}(\Omega)$

$$\begin{aligned}\delta_I(u, u_h^c) &\leq I(u_h^c) - I(u) \leq I(u_h^c) - D(z_h) \\ &= \int_{\Omega} \phi(\nabla u_h^c) + \psi(u_h^c) + \phi^*(z_h) + \psi^*(\operatorname{div} z_h) \, dx\end{aligned}$$

Poisson problem: Assume $f_h = f$ and use $\operatorname{div} z_h = -f$

$$\frac{1}{2} \|\nabla(u - u_h^c)\|^2 \leq \frac{1}{2} \int_{\Omega} |\nabla u_h^c|^2 + |z_h|^2 + \operatorname{div} z_h u_h^c \, dx = \frac{1}{2} \|\nabla u_h^c - z_h\|^2$$

TV minimization: Direct calculations and integration by parts for $|z_h| \leq 1$

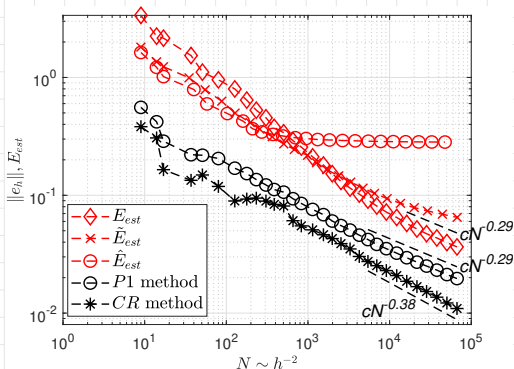
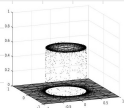
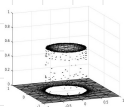
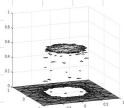
$$\begin{aligned}\frac{\alpha}{2} \|u - u_h^c\|^2 &\leq \int_{\Omega} |\nabla u_h^c| + \frac{\alpha}{2} (u_h^c - g)^2 + \frac{1}{2\alpha} (\operatorname{div} z_h + \alpha g)^2 - \frac{\alpha}{2} g^2 \, dx \\ &= \int_{\Omega} (|\nabla u_h^c| - \nabla u_h^c \cdot z_h) + \frac{1}{2\alpha} (\operatorname{div} z_h + \alpha(u_h^c - g))^2 \, dx\end{aligned}$$

Interpretation: Residual type estimators, u_h^c and z_h only admissible

- ▶ flux equilibration, hypercircle method: [Repin '00, Braess '07, B. '15]
- ▶ nearly optimal z_h via postprocessing primal CR approximation u_h

ROF estimator: Use localized error bound to refine mesh [B. & Milicevic '19]

$$e_h^2 \leq \eta_h^2 = \sum_{T \in \mathcal{T}_h} \int_T (|\nabla u_h^c| - \nabla u_h^c \cdot z_h) + \frac{1}{2\alpha} (\operatorname{div} z_h + \alpha(u_h^c - g))^2 dx$$



- ▶ use of $H(\operatorname{div}; \Omega)$ -conforming dual variable avoids oscillations
- ▶ open: construction of admissible, nearly optimal $z^{(h)}$

Summary

- ▶ Optimal error estimates via convex duality
- ▶ Nonsmooth problems, only first order systems
- ▶ Dual problem provides right regularity condition
- ▶ A posteriori error estimates via primal-dual gap
- ▶ Solve dual problem via generalized Marini formula

- ▶ Future aspects/open problems:
 - ▷ general regularity results, e.g., dual form TV
 - ▷ higher order nonstandard finite elements
 - ▷ optimal iterative solution

- ▶ More information

<http://aam.uni-freiburg.de/bartels>



Thank you !