Numerical Solution of Nonsmooth Problems and Application to Damage Evolution and Optimal Insulation

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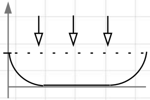
Nonsmooth Models

- Obstacle problem

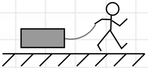
  Minimize $\displaystyle \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega f u \,\mathrm{d}x$

  subject to $u \geq \chi$



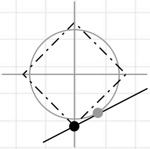
- Placticity (friction) models

  $$\partial_t p \in \partial I_K(\sigma),$$
  $$-\operatorname{div} \mathbb{C}\big(\varepsilon(u) - p\big) = f$$



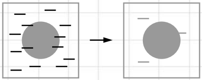
- Sparse reconstruction

  Minimize $|x|_{\ell^1}$ subject to $Ax = b$



- Image denoising

  Minimize $\displaystyle \int_\Omega |Du| + \frac{\alpha}{2}\|u-g\|^2$



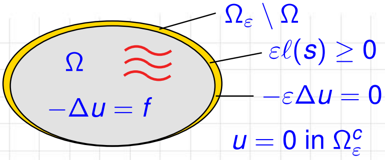
Outline

Thin Insulating Films

Thin Insulation

Model of insulation:

- $\Omega \subset \mathbb{R}^d$ conducting body
- insulating layer of width $\varepsilon\ell(s)$
- conductivity of layer is ε



PDE system: $u \in C(\overline{\Omega}_\varepsilon)$ with

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ -\varepsilon\Delta u &= 0 \quad \text{in } \Omega_\varepsilon \setminus \overline{\Omega} \\ u &= 0 \quad \text{on } \partial\Omega_\varepsilon \end{aligned}$$

Limit $\varepsilon \to 0$ leads to Robin BC:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u + \ell\,\partial_n u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

See: Brézis, Caffarelli & Friedman 80, Acerbi & Buttazzo '86

Operator form of limit model

$$\mathcal{A}_\ell u = f$$

with $\mathcal{A}_\ell : H^1(\Omega) \to H^1(\Omega)^*$

$$\langle \mathcal{A}_\ell u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\partial\Omega} \ell^{-1} uv \,\mathrm{d}s$$

- $\mathcal{A}_\ell$ coercive & symmetric

Loss of Heat

Associated heat equation

$$\partial_t u + \mathcal{A}_\ell u = 0, \quad u(0) = u_0$$

Prinicipal eigenvalue determines long-time behaviour

$$\lambda_{m,\ell} = \inf_{\|u\|=1} \langle \mathcal{A}_\ell u, u\rangle \quad \Longleftrightarrow \quad \lambda_{m,\ell}\|u\|^2 \le \langle \mathcal{A}_\ell u, u\rangle$$

Decay of variance of enthalpy

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = -\langle \mathcal{A}_\ell u, u\rangle \le -\lambda_{m,\ell}\|u\|^2 \quad \Longrightarrow \quad \|u(t)\| \le \|u_0\|\, e^{-\lambda_{m,\ell} t}$$

- $\lambda_{m,\ell}$ smallest eigenfrequency
- Dependence on available mass $m = \int_{\partial\Omega} \ell \, \mathrm{d}s$?
- Optimal distribution ℓ ?

Class of admissible mass distributions

$$\mathcal{I}_m = \left\{ \ell \in L^1(\partial\Omega) : \ell \ge 0, \int_{\partial\Omega} \ell \, \mathrm{d}s = m \right\}$$

Optimal Distribution

For fixed m optimize $\lambda_{m,\ell}$ over $\ell \in \mathcal{I}_m$ and note interchange

$$\lambda_m = \inf_{\ell \in \mathcal{I}_m} \lambda_{m,\ell} = \inf_{\|u\|=1} \inf_{\ell \in \mathcal{I}_m} \langle \mathcal{A}_\ell u, u \rangle$$

Given $u \in H^1(\Omega)$ optimal thickness $\ell \in \mathcal{I}_m$ is

$$\ell(s) = m \frac{|u(s)|}{\|u\|_{L^1(\partial\Omega)}}$$

Problem reduces to

$$\lambda_m = \inf_{\|u\|=1} J_m(u), \qquad J_m(u) = \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{1}{m} \left(\int_{\partial\Omega} |u| \,\mathrm{d}s \right)^2$$

- Existence of eigenfunctions u_m
- Nonlocal, nondifferentiable, constrained problem
- Thickness ℓ proportional to trace of $|u|$

Symmetry Break

Recall

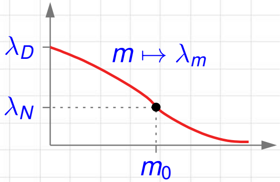
$$\lambda_m = \min_{\|u\|=1} J_m(u), \quad J_m(u) = \|\nabla u\|^2 + \frac{1}{m}\|u\|^2_{L^1(\partial\Omega)}$$

- $m \mapsto \lambda_m$ continuous, strictly decreasing, $\lambda_m \to 0$ as $m \to \infty$

Dirichlet and Neumann eigenvalues for $-\Delta$

$$\lambda_D = \min_{\|u\|=1,\, u|_{\partial\Omega}=0} \|\nabla u\|^2$$

$$> \lambda_N = \min_{\|u\|=1,\, \int_\Omega u\,\mathrm{d}x=0} \|\nabla u\|^2$$



- $\lambda_m \to \lambda_D$ for $m \to 0$ and $\lambda_{m_0} = \lambda_N$ for some $m_0 > 0$

Theorem (Bucur, Buttazzo & Nitsch '16)**.**
Nonradial eigenfunctions u_m for $\Omega = B_R(0)$ iff $m < m_0$, particularly, optimal mass distributions ℓ leave gaps.

- Gap symmetric and/or connected?

Proof (sketched)

- radial u_m solves 1D Sturm–Liouville problem and has no roots
- u_m satisfies PDE

$$(\nabla u_m, \nabla v) + \frac{1}{m}\|u_m\|_{L^1(\partial\Omega)}\big(\sigma(u_m), v\big)_{\partial\Omega} = \lambda_m(u_m, v)$$

  with $\sigma(u_m) \in \partial|u_m|$, i.e., $\sigma(u_m) = 1$ since $u_m > 0$

- testing u_N with $v(x) = c_1|x|^2 - c_2$ gives

$$(u_N, 1)_{\partial\Omega} = (\nabla u_N, \nabla v) = \lambda_N(u_N, v) = 0$$

- $u_m \perp_{L^2, H^1} u_N$ since $\lambda_m(u_m, u_N) = (\nabla u_m, \nabla u_N) = \lambda_N(u_N, u_m)$
- for ε small $\|u_m + \varepsilon u_N\|_{L^1(\partial\Omega)} = \|u_m\|_{L^1(\partial\Omega)}$
- implies

$$\lambda_m \leq J_m\left(\frac{u_m + \varepsilon u_N}{\|u_m + \varepsilon u_N\|}\right) = \frac{J_m(u_m) + \varepsilon^2\|\nabla u_N\|^2}{1 + \varepsilon^2} = \frac{\lambda_m + \varepsilon^2\lambda_N}{1 + \varepsilon^2}$$

- contradicts $\lambda_m > \lambda_N$ hence u_m not radial

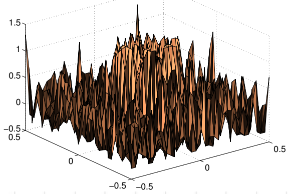
Total Variation Minimization

BV Model Problem

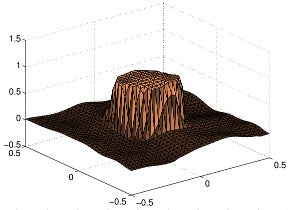
For noisy image $g : \Omega \to \mathbb{R}$ find $u \in BV(\Omega)$ minimal for

$$I(u) = \int_\Omega |Du| + \frac{\alpha}{2}\|u - g\|^2$$

- Proposed by Rudin, Osher & Fatemi '92
- Convex, nondifferentiable minimization problem
- Captures discontiniuties, staircasing effect



$\Longrightarrow$



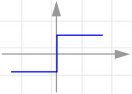
Contributions: Elliott & Mikelić '91, Chambolle & P.-L. Lions '97, Feng & Prohl '03, Hintermüller & Kunisch '04, Tsai & Osher '05, Osher, Burger, Goldfarb, Xu & Yin '05, Wu & Tai '09, Chambolle & Pock '11, Wang & Lucier '11, Stamm & Wihler '15, ...

Properties of BV

Function $v \in BV(\Omega)$ if $v \in L^1(\Omega)$ and

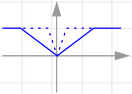
$$\int_\Omega |Dv| = |Dv|(\Omega) = \sup_{\xi \in C_0^\infty(\Omega;\mathbb{R}^d),\, |\xi|\le 1} -\int_\Omega v \operatorname{div} \xi \,\mathrm{d}x < \infty$$

Examples: (i) $v(x) = \operatorname{sign}(x)$, $Dv = 2\delta_0$, $|Dv|(\Omega) = 2$



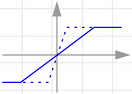
(ii) $v(x) = \chi_A(x)$, $|Dv|(\Omega) = \mathrm{Per}_\Omega(A)$

- Existence of solutions by convexity and compactness
- $W^{1,1}(\Omega) \subset BV(\Omega)$ strictly and $\|\nabla v\|_{L^1(\Omega)} = |Dv|(\Omega)$
- $C^\infty(\overline{\Omega}) \subset BV(\Omega)$ weakly but not strongly dense



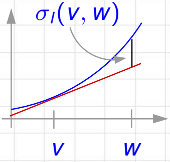
Intermediate convergence in $BV(\Omega)$:

$$v_j \to v \text{ in } L^1(\Omega) \quad \& \quad |Dv_j|(\Omega) \to |Dv|(\Omega)$$



Aspects and Tools

- Choice of finite element space $V_h \subset BV(\Omega)$
- Convergence rates unter appropriate regularity conditions
- A posteriori error estimates and local mesh refinement
- Effective iterative numerical solution



Strong convexity: For arbitrary $v \in BV(\Omega) \cap L^2(\Omega)$

$$\frac{\alpha}{2}\|u - v\|^2 \le I(v) - I(u)$$

Strong duality: Sharp relation $I(v) \ge D(q)$ with $q \in H_N(\mathrm{div};\Omega)$ and

$$D(q) = -\frac{1}{2\alpha}\|\operatorname{div} q + \alpha g\|^2 + \frac{\alpha}{2}\|g\|^2 - I_{K_1(0)}(q)$$

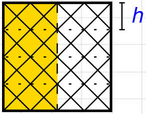
Combination: For minimizer u and arbitrary v and q

$$\frac{\alpha}{2}\|u - v\|^2 \le I(v) - I(u) \le I(v) - D(q)$$

Convergence Rates

- $P0$ FE fails as perimeter approximated incorrectly:

$$u_h \to_{L^1} \chi_{\{x<0\}} \implies |Du_h|(\Omega) \to \sqrt{2}$$



- $P1$ FE quasi-interpolant via regularization and interpolation

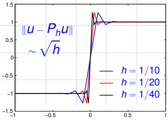
Lemma (B., Nochetto & Salgado '12). There exists $\widetilde{u}_{h,\varepsilon} \in \mathcal{S}^1(\mathcal{T}_h)$ with

$$\|\nabla \widetilde{u}_{h,\varepsilon}\|_{L^1} \leq (1 + c\varepsilon + ch\varepsilon^{-1})|Du|(\Omega),$$
$$\|u - \widetilde{u}_{h,\varepsilon}\|_{L^1} \leq c(h^2\varepsilon^{-1} + \varepsilon)|Du|(\Omega).$$

Implies rate $\mathcal{O}(h^{1/4})$ with $\varepsilon = h^{1/2}$:

$$\frac{\alpha}{2}\|u - u_h\|^2 \leq I(u_h) - I(u) \leq \inf_{v_h \in \mathcal{S}^1(\mathcal{T}_h)} I(v_h) - I(u) \leq ch^{1/2}$$

- Optimal rate is $\mathcal{O}(h^{1/2})$ for generic $u \in BV(\Omega)$
- TVD property $\|\nabla \widetilde{u}_{h,\varepsilon}\| \leq |Du|(\Omega)$ yields optimality



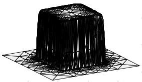
Adaptivity

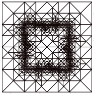
Computable localized error control via approximation of dual (B. '15):

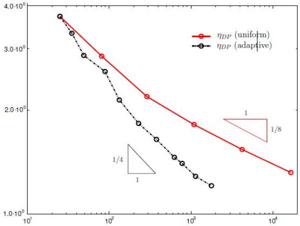
$$
\begin{aligned}
\frac{\alpha}{2}\|u-u_h\|^2 &\le I(u_h) - D(p_h) \\
&= \|\nabla u_h\|_{L^1} - \int_\Omega p_h \cdot \nabla u_h \,\mathrm{d}x + \frac{1}{2\alpha}\|\operatorname{div} p_h - \alpha(u_h - g)\|^2 \\
&= \sum_{T\in\mathcal{T}_h} \eta_T(u_h, p_h),
\end{aligned}
$$

provided $p_h \in H_N(\mathrm{div};\Omega)$ with $|p_h| \le 1$; note $\eta_T(u_h,p_h) \ge 0$

- Nearly optimal rate $h^{1/2} \sim N^{-1/4}$
- Dual solution in $P1$ vector fields
- Better: $H(\mathrm{div};\Omega)$ FE spaces







Solution via ADMM

Decouple nondifferentiability from gradient via $p = \nabla u$

$$L_\tau(u, p; \lambda) = \int_\Omega |p| \, \mathrm{d}x + \frac{\alpha}{2}\|u - g\|^2 + (\lambda; \nabla u - p)_H + \frac{\tau}{2}\|\nabla u - p\|_H^2$$

- Formally, e.g., on discrete level, for arbitrary $\tau \geq 0$

$$\inf_{u,p} \sup_{\lambda} L_\tau(u, p; \lambda) = \inf_u I(u)$$

- Choice of Hilbert space H: weighted L^2 norm on FE spaces

Iterative solution via decoupled minimization:

(Glowinski & Morocco '76, Gabay & Mercier '76, He & Yuan '12, Deng & Yin '16)

Algorithm. Choose (u^0, λ^0) and $\tau > 0$; set $j = 1$.

(1) Compute minimizer p^j for $p \mapsto L_\tau(u^{j-1}, p; \lambda^{j-1})$.

(2) Compute minimizer u^j for $u \mapsto L_\tau(u, p^j; \lambda^{j-1})$.

(3) Update λ^{j-1} via $\lambda^j = \lambda^{j-1} + \tau(\nabla u^j - p^j)$.

Properties

- Unconditional convergence if $\tau > 0$
- Proof holds for sequence of decreasing step sizes $\tau_{j+1} \leq \tau_j$
- Monotonicity of residuals

$$R_j = \|\lambda^j - \lambda^{j-1}\|_H^2 + \tau_j^2 \|\nabla(u^j - u^{j-1})\|_H^2$$

- Error control via residual

$$\|u - u^j\|^2 + \|u^j - u^{j-1}\|^2 \leq C_0 \tau_j^{-1} R_j^{1/2}$$

- Linear convergence of residuals

$$R_{j+1} \leq \gamma R_j, \quad 0 < \gamma < 1$$

Motivates adaptive step size adjustment (B. & Milicevic ’17):

- Start with $\tau \gg 1$ and $\gamma \in (0, 1)$
- Accept τ if γ-reduction satisfied, decrease τ otherwise
- If $\tau = 1$ increase γ and re-start

Experiment

- Comparison to ADMM and accelerated version (Goldfarb et al. '13)
- Hyphens indicate more than 10^4 steps
- Variable step size variant best for large initial step sizes
- Restricted to convex problems

	ADMM (ROF; $\varepsilon_{stop}^{(2)} = h$, $\tau_j \equiv \overline{\tau}$)							
	$\overline{\tau} = 1$		$\overline{\tau} = h^{-1}$		$\overline{\tau} = h^{-2}$		$\overline{\tau} = h^{-3}$	
ℓ	N	$E_h/\sqrt{h}$	N	$E_h/\sqrt{h}$	N	$E_h/\sqrt{h}$	N	$E_h/\sqrt{h}$
3	27	0.2242	7	0.1531	8	0.1479	49	0.1836
4	137	0.1690	14	0.1455	47	0.0558	521	0.0569
5	691	0.1657	32	0.1614	217	0.0248	4878	0.0251
6	3882	0.1739	87	0.1729	740	0.0529	–	–
7	–	–	286	0.1569	2037	0.0169	–	–
8	–	–	920	0.1169	7598	0.0131	–	–
9	–	–	2585	0.0834	–	–	–	–
ℓ	Fast-ADMM ($\gamma = 0.999$)							
3	15	0.1188	6	0.1013	7	0.0757	26	0.1625
4	63	0.1644	13	0.1080	26	0.0342	326	0.0554
5	246	0.1662	28	0.0975	107	0.0217	8178	0.0251
6	2370	0.1746	49	0.1693	384	0.0526	–	–
7	–	–	153	0.1561	1145	0.0168	–	–
8	–	–	537	0.1154	–	–	–	–
9	–	–	1669	0.0829	–	–	–	–
ℓ	Variable-ADMM ($\underline{\tau} = 1$, $\gamma = 0.5$, $\overline{\gamma} = 0.999$, $\delta = 0.5$)							
3	27	0.2242	14	0.1531	6	0.1244	20	0.1310
4	137	0.1690	34	0.1455	26	0.0957	33	0.1006
5	691	0.1657	78	0.1614	60	0.0568	81	0.0529
6	3882	0.1739	331	0.1729	137	0.0426	181	0.0178
7	–	–	995	0.1569	319	0.0156	375	0.0162
8	–	–	2272	0.1169	365	0.0161	430	0.0202
9	–	–	–	–	834	0.0140	1763	0.0080

Subdifferential Flow

- Gradient flows most general and robust tool to minimize
- Formal PDE for TV flow and backward Euler method

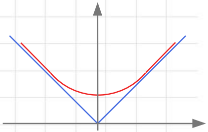
$$\partial_t u = \operatorname{div} \frac{\nabla u}{|\nabla u|}, \quad d_t u^k = \operatorname{div} \frac{\nabla u^k}{|\nabla u^k|}$$

- Unconditional energy stability by convexity

$$\|d_t u^k\|^2 + d_t \int_\Omega |\nabla u^k| \,\mathrm{d}x \le \|d_t u^k\|^2 + \int_\Omega \frac{\nabla u^k}{|\nabla u^k|} \cdot \nabla d_t u^k \,\mathrm{d}x = 0$$

- Practical variant semiimplicit and regularized

$$d_t u^k = \operatorname{div} \frac{\nabla u^k}{|\nabla u^{k-1}|_\varepsilon}$$



- Violates monotonicity property and might depend critically on ε

Unconditional Stability

- Continuous differentiation formula

$$\frac{d}{dt}|a(t)| = \frac{a(t)a'(t)}{|a(t)|} = \frac{1}{2}\frac{\frac{d}{dt}|a(t)|^2}{|a(t)|}$$

- Discrete product and quotient rule

$$d_t a^k b^k = (d_t a^k) b^k + a^{k-1}(d_t b^k), \qquad d_t \frac{1}{c^k} = \frac{-d_t c^k}{c^{k-1}c^k}$$

- Binomial formula

$$2a^k \cdot d_t a^k = d_t |a^k|^2 + \tau |d_t a^k|^2$$

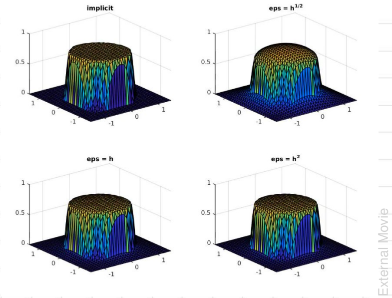
- Crucial formula

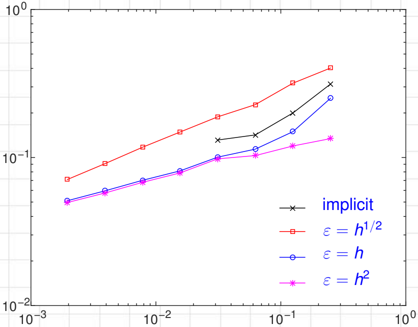
$$d_t|a^k| = d_t \frac{|a^k|^2}{|a^k|} = \frac{d_t|a^k|^2}{|a^{k-1}|} + |a^k|^2 d_t \frac{1}{|a^k|}$$
$$= \frac{d_t|a^k|^2}{|a^{k-1}|} - \frac{|a^k| d_t |a^k|}{|a^{k-1}|} = \frac{1}{2}\frac{d_t|a^k|^2}{|a^{k-1}|} - \frac{\tau|d_t a^k|^2}{2|a^{k-1}|} \leq \frac{1}{2}\frac{d_t|a^k|^2}{|a^{k-1}|}$$

- Implies unconditional convergence of semi-implicit scheme

Experiments

- Change of height proportional to mean curvature of level set
- Evolution of characteristic function of disk $u_0 = \chi_{B_1}$
- Exact solution given by $u(t,x) = (1-2t)_+ \chi_{B_1}(x)$
- Error $\|u - u_h\|_{L^\infty(L^2)}$ depends on ε^{-1}





BV Regularized Damage Evolution

BV Regularized Damage Model

Energy and dissipation for displacement φ and damage variable z:

$$\mathcal{E}(\varphi, z) = \int_\Omega f(z)\varepsilon(\varphi) : \mathbb{C}\varepsilon(\phi)\,\mathrm{d}x + |Dz|(\Omega) + \int_\Omega I_{[0,1]}(z)\,\mathrm{d}x$$

$$\mathcal{R}(v) = \int_\Omega R(v)\,\mathrm{d}x, \qquad R(v) = \begin{cases} \varrho|v| & \text{if } v \in (-\infty, 0], \\ +\infty & \text{if } v > 0. \end{cases}$$

Evolution of $q = (\varphi, z)$ via Biot's equation $0 \in \partial_q \mathcal{E}(q) + \partial_{\dot{q}} \mathcal{R}(\dot{q})$

Energetic solutions are globally stable and fulfill energy balance

$$\mathcal{E}\big(q(t)\big) \le \mathcal{E}\big(\tilde{q}\big) + \mathcal{R}\big(\tilde{z} - z(t)\big)$$

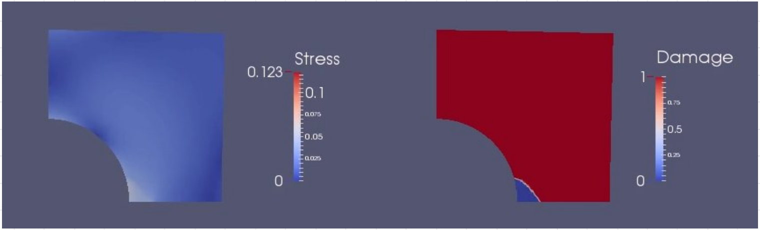
$$\mathcal{E}\big(q(t)\big) + \mathrm{Diss}_{\mathcal{R}}\big(z, [s,t]\big) = \mathcal{E}\big(q(s)\big) + \int_s^t \partial_r \mathcal{E}\big(q(r)\big)\,\mathrm{d}r$$

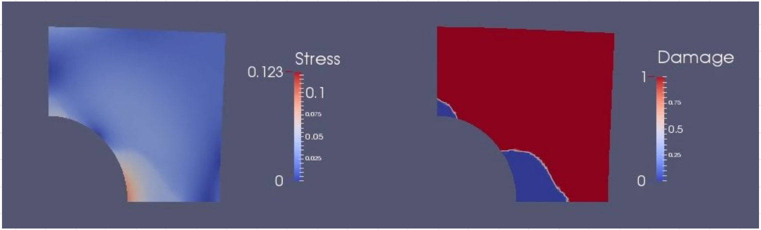
with $\mathrm{Diss}_{\mathcal{R}}(z, [s,t]) = \sup\big\{\sum_{j=1}^N \mathcal{R}\big(z(t_j) - z(t_{j-1})\big),\ t_0 < \ldots < t_N\big\}$.

- derivative-free formulation
- existence theories: Mielke '05; partial damage: Thomas '13

Experiment

- Time-stepping via decoupled minimization





External Link

Computing
Insulating Films

Gradient Flow

Minimization problem

$$J_m(u) = \|\nabla u\|^2 + \frac{1}{m}\|u\|^2_{L^1(\partial\Omega)} \quad \text{subject to} \quad \|u\| = 1$$

Regularized gradient flow with $|u|_\varepsilon = (u^2 + \varepsilon^2)^{1/2}$

$$(\partial_t u, v)_* + (\nabla u, \nabla v) + \frac{1}{m}\|u\|_{L^1_\varepsilon}\Big(\frac{u}{|u|_\varepsilon}, v\Big)_{\partial\Omega} = \mu(u, v)$$

- $\|u(t)\| = 1$ implies $\partial_t u \perp_{L^2} u$; RHS disappears if $v \perp_{L^2} u$

Algorithm (B. '17). Given u^{k-1} compute u^k with $d_t u^k \perp_{L^2} u^{k-1}$ and

$$(d_t u^k, v)_* + (\nabla u^k, \nabla v) + \frac{1}{m}\|u^{k-1}\|_{L^1_\varepsilon}\Big(\frac{u^k}{|u^{k-1}|_\varepsilon}, v\Big)_{\partial\Omega} = 0$$

for all v with $v \perp_{L^2} u^{k-1}$ with quotient $d_t u^k = (u^k - u^{k-1})/\tau$.

- Iteration (essentially) unconditionally stable

Spatial Discretization

For triangulation $\mathcal{T}_h$ of Ω use $P1$ FE-space

$$\mathcal{S}^1(\mathcal{T}_h) = \{v_h \in C(\overline{\Omega}) : v_h|_T \in P_1(T) \text{ for all } T \in \mathcal{T}_h\}.$$

and discretized functional

$$J_{m,\varepsilon,h}(u_h) = \|\nabla u_h\|^2 + \frac{1}{m}\Big(\sum_{z \in \mathcal{N}_h \cap \partial\Omega} \beta_z |u_h(z)|_\varepsilon\Big)^2$$

- Stability estimate carries over to $J_{m,\varepsilon,h}$

Lemma (B. '17). If eigenfunction $u = u_m$ satisfies $u \in H^2$ then

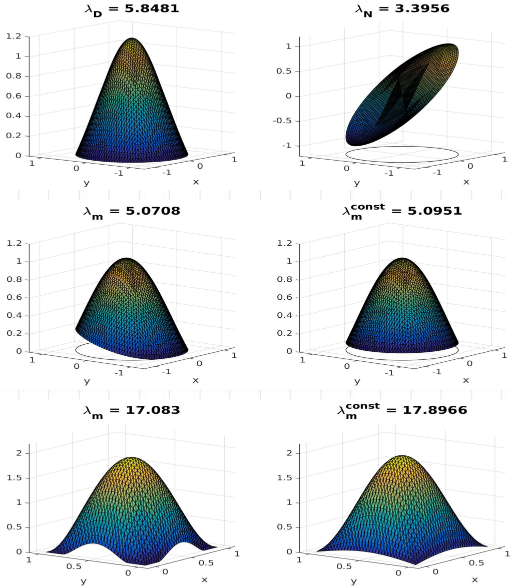
$$0 \le \min_{u_h \in \mathcal{S}^1(\mathcal{T}_h)} J_m(u_h) - J_m(u) \le ch\|u\|^2_{H^2(\Omega)},$$

$$\big|J_{m,\varepsilon,h}(u_h) - J_m(u_h)\big| \le c(\|u_h\|_{H^1(\Omega)} + 1)^2(\varepsilon + h^{1/2}).$$

- Consistency estimates imply Γ convergence via density

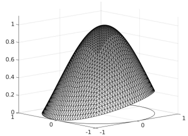
Experiments

- Dirichlet and Neumann eigenfunctions on unit disk
- Optimal and constant ℓ with $m = 0.4$ on unit disk
- Optimal and constant ℓ with $m = 0.1$ on unit square ($\lambda_D = 2\pi^2$, $\lambda_N = \pi^2$)



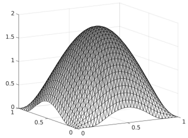
Iteration Numbers

- $\Omega = B_1(0) \subset \mathbb{R}^2$, $m = 1/2$, $h = 2^{-\ell}$, $\varepsilon = h/10$, $\tau = 1$



ℓ	$\#\mathcal{N}_h$	$\#\mathcal{T}_h$	K_{stop}	$\lambda_{m,h}$
3	145	256	40	4.9070
4	545	1024	84	4.9115
5	2113	4096	99	4.9115
6	8321	16384	92	4.9110
7	33025	65536	102	4.9106

- $\Omega = (0,1)^2 \subset \mathbb{R}^2$, $m = 1/8$, $h = 2^{-\ell}$, $\varepsilon = h/10$, $\tau = 1$

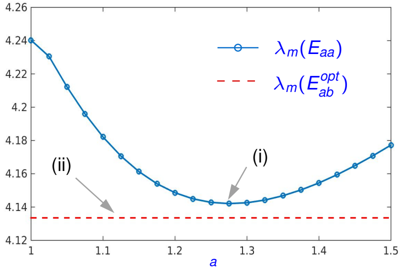


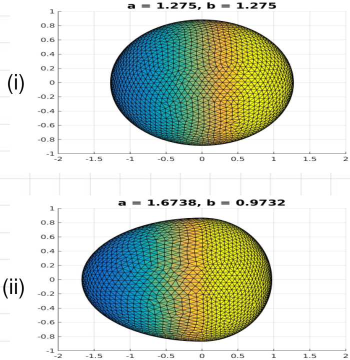
ℓ	$\#\mathcal{N}_h$	$\#\mathcal{T}_h$	K_{stop}	$\lambda_{m,h}$
3	81	128	23	16.5093
4	289	512	23	16.5828
5	1089	2048	18	16.5938
6	4225	8192	18	16.5942
7	16641	32768	14	16.5950

- Fixed stopping criterion: $\|d_t u_h^k\| \leq \varepsilon_{\text{stop}} = 10^{-3}$

Rotational Shapes

- Symmetrical ellipsoids E_{aa} with radii (a, a, r_a), $a \in [1, 1.5]$
- Optimal assembled ellipsoid E_{ab} with radii (a, b, r_{ab})
- Fixed volume $c_0 = |B_1|$ and mass $m = 6.0$





- High resolution via two-dimensional reduction
- Egg shape optimal among convex shapes ?

Summary

Summary

- Reduced convergence rates for nondifferentiable problems
- Iterative solution via regularized gradient flows
- Semi-implicit discretization unconditionally stable

- Future topics
  - Convergence of adaptive schemes
  - Optimal isolating bodies
  - Error estimates for damage evolution

- Further information

  `http://aam.uni-freiburg.de/bartels`