# Finite Element Approximation of Harmonic Maps between Surfaces

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# Introduction

*"Harmony and Harmonicity — If you fall in love with harmonic functions your mathematician's soul will never come to rest unless you comprehend the origin of their irresistible appeal and beauty. And if you are bent on spaces, manifolds and maps you start researching for the geometric habitat of harmonicity."*
*– Misha Gromov, May 2000, Preface of [EF01]*

Geometric partial differential equations and their analysis as well as numerical simulation have recently attracted considerable attention among pure and applied mathematicians. Motivated by interesting applications such as general relativity, micromagnetics, liquid crystal theory, biophysics, and medical image processing, significant progress has been made in the mathematical understanding of evolutionary and stationary partial differential equations from, into, and between surfaces within the last two decades. While the properties of solutions of partial differential equations with values into surfaces with symmetries such as the unit sphere are now relatively well understood, only few results are available in the general case. This thesis aims at contributing to the development and analysis of approximation schemes for such problems. In the remaining part of this introduction, we present geometric partial differential equations as mathematical models of certain physical processes, discuss analytical properties of solutions, indicate difficulties in the numerical approximation, and summarize the main contributions of this work.

### *Mathematical models leading to geometric partial differential equations*

*Micromagnetics.* A good understanding of magnetic material behavior at small scales is important for the development of new storage media. The magnetization field of a ferromagnetic body occupying the domain $\Omega \subset \mathbb{R}^3$ describes the orientation of the elementary magnets in an averaged or statistical sense and is in a constant temperature scenario modeled as a unit length vector field $m\colon \Omega \to S^2$. Following the argumentation of Landau and Lifshitz [LL35], the actual magnetization field minimizes the energy functional

$$E^{LL}(m) = A \int_\Omega |\nabla m|^2 \,\mathrm{d}x + K_a \int_\Omega \varphi(m) \,\mathrm{d}x + \frac{\mu_0}{2} \int_{\mathbb{R}^3} |H_{ind}|^2 \,\mathrm{d}x - \int_\Omega H_{ext} \cdot m \,\mathrm{d}x$$

among all possible magnetizations. The factors $A, K_a, \mu_0$ are given constants, the function $\varphi\colon S^2 \to \mathbb{R}$ is an anisotropy energy density that models preferred directions of the magnetization, and $H_{ext}$

represents an applied magnetic field. The induced magnetic field $H_{ind}$ generated by the magnetized body $\Omega$ is given by $H_{ind} = -\nabla U$ where the scalar function $U$ solves

$$-\Delta U = \mu_0^{-1} \operatorname{div} m$$

in the distributional sense in $\mathbb{R}^3$. The micromagnetic energy $E^{LL}$ is capable of describing various fascinating phenomena such as thin film magnetization patterns observed in practice; the article [DKMO05] surveys recent analytical developments in this direction. The computation of stationary points of $E^{LL}$ is a difficult task owing to the non-convex side-constraint $m(x) \in S^2$ for almost every $x \in \Omega$. Popular numerical strategies penalize the constraint or employ projection methods to overcome this problem; we refer the reader to [Pro01, KP06] for an overview of numerical methods in micromagnetics. Stationary points of $E^{LL}$ can also be detected through the Landau-Lifshitz-Gilbert dynamics defined by the time-dependent, non-linear partial differential equation

$$\partial_t m = -m \times \nabla E^{LL}(m) - \gamma\, m \times \big(m \times \nabla E^{LL}(m)\big),$$

with the Gâteaux differential $\nabla E^{LL}$ of $E^{LL}$, see [Vis85]. The precise mathematical properties of solutions of this evolutionary geometric partial differential equation are not entirely understood, see [Mel05, Ko05] for partial regularity results, and numerical simulations can provide valuable insight [BKP07]. While the Landau-Lifshitz energy $E^{LL}$ is widely accepted as an appropriate mathematical model of certain micromagnetic effects, more realistic models have to take temperature variations into account and then the unit sphere may not be the right target manifold.

*Liquid crystals.* Another interesting energy functional that acts on vector fields with values in a surface arises in liquid crystal theory. In a mathematical modeling due to Oseen [Ose33] and Frank [Fra58], see also [Vir94, dGP93], the vectorial quantity $u\colon \Omega \to S^2$ models the orientation of the rod-like molecules that constitute the liquid crystal which occupies the domain $\Omega \subset \mathbb{R}^3$. A penalization of high energy states such as bend, splay, and twist configurations and consideration of certain symmetries lead to the functional

$$E^{OF}(u) = \int_\Omega k_1 |\operatorname{div} u|^2 + k_2 |u \cdot \operatorname{Curl} u|^2 + k_3 |u \times \operatorname{Curl} u|^2 + (k_2 + k_4)\big(|\nabla u|^2 - (\operatorname{div} u)^2\big)\, dx.$$

Low energy configurations are preferred and this naturally leads to searching for stationary points of the energy functional $E^{OF}$. The mathematical model captures important effects of liquid crystals such as point singularities, but it cannot predict other important phenomena such as line singularities. This latter deficiency can be overcome by replacing the target manifold $S^2$ by the real projective plane $\mathbb{R}P^2$ which identifies opposite points on $S^2$ and thereby respects the head to tail symmetry of certain liquid crystals, see [Jos86, EL80, Bal07] for related analytical aspects.

*Biomembranes.* A much less understood energy functional associated to a geometric partial differential equation, is the Willmore energy [Wil93] of surfaces, defined for a sufficiently smooth submanifold $S$ by

$$E^W(S) = \frac{1}{2} \int_S H^2 \, ds$$

for the mean curvature $H$ of $S$. We refer the reader to [KS02a, KS02b] and [DD06, Rus05, CDD$^+$04] for related analytical and numerical aspects. The Willmore functional gives the elastic bending energy of the surface $S$ which is the unknown in the model. Slightly more general models in biophysics incorporate “spontaneous curvature” defined by a constant $H_0$ in the model and then an optimal $S$ models the shape of a membrane. A simple mathematical description of the shape of a membrane and its interaction with the surfactants in the two lipid bilayers is due to [FG97] and seeks stationary points of the functional

$$E^{FG}(S, Q^+, Q^-) = \frac{1}{2}\int_S \kappa_M (H - H_0)^2 + 2\kappa_G K + \alpha \mathbf{K} : (Q^+ - Q^-) + \beta\left|\nabla Q^+\right|^2 + \beta\left|\nabla Q^-\right|^2 \,\mathrm{d}s.$$

In this model, $Q^\pm = n^\pm \otimes n^\pm - \mathbf{I}_{3\times 3}$ for $n^\pm \in S^2$ are the order parameters corresponding to the director fields $n^\pm$ of the upper and lower sheet of the bilayer, $K$ is the Gaussian curvature of $S$, $\mathbf{K}$ a curvature tensor, and $\alpha, \beta$ are given constants. This energy generalizes mathematical models from [Can70, Hel73, Sei97] by combining the Willmore functional with a simplified liquid crystal energy and coupling the unknown quantities through the third term in the functional.

The first two described mathematical models defined through the functionals $E^{LL}$ and $E^{OF}$ reveal that partial differential equations that define functions with values in a surface have important applications and that the unit sphere as a target manifold may be too restrictive in some situations. The Willmore functional is beyond the scope of this work but the coupled energy $E^{FG}$ motivates to study energy functionals that are defined on surfaces.

***Harmonic maps between surfaces and their properties***

In the so-called equal-constant setting $k_1 = k_2 = k_3 = k_4$ in liquid crystal theory or in models of ferromagnetic bodies of small diameter, the aforementioned energy functionals $E^{OF}$ and $E^{LL}$ reduce to the Dirichlet energy

$$E(v) = \frac{1}{2}\int_M \left|\nabla_M v\right|^2 \mathrm{d}s. \tag{1}$$

Here we replace the physical domain $\Omega$ by a submanifold $M \subset \mathbb{R}^m$ and introduce the tangential gradient $\nabla_M$ on $M$. Given another compact submanifold $N \subset \mathbb{R}^n$ without boundary which serves as the target manifold, a weakly differentiable vector field $u\colon M \to N$ is called a *harmonic map into* $N$ if it is stationary for $E$ with respect to compactly supported, tangential perturbations. This is true if $u$ is a weak solution of the non-linear partial differential equation

$$-\Delta_M u = A_N(u)\left[\nabla_M u; \nabla_M u\right] \tag{2}$$

with the second fundamental form $A_N$ on $N$; see, e.g., [Str00, FMS98] for a derivation of (2).

The existence of global minimizers for (1) and hence of harmonic maps follows with the direct method in the calculus of variations for non-empty sets of admissible vector fields. The related problem for an open domain $M \subset \mathbb{R}^m$ and $N = \mathbb{R}$ leads to harmonic functions with all their well-understood properties. When $N$ has a boundary and its interior is non-empty then (1) can be understood as a variational inequality for which existence, uniqueness, and regularity as well as numerical approximation have been investigated intensively in the literature. We will focus on the case that $N$ is a surface, i.e., restrict to sufficiently smooth, compact submanifolds $N$ without boundary which excludes variational inequalities and harmonic functions. The simplest choice of such a surface, namely $N = S^{n-1} \subset \mathbb{R}^n$, the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$, already reveals a variety of intriguing phenomena captured by the mathematical model (1): If $M = B_1(0) \subset \mathbb{R}^m$ is the open unit ball in $\mathbb{R}^m$ and $N = S^{n-1}$ then harmonic maps are known to be smooth if $m = 2$, see [Hél91, Mor66]. In higher dimensions the picture changes drastically. For $m \geq 3$ harmonic maps into the sphere are partially regular if they are energy minimizing or, more generally, stationary with respect to spatial variations [Eva91, Bet93a, Har97]. Those results are sharp in the sense that (a) the function $x \mapsto x/|x|$ is an energy minimizing harmonic map into the sphere if $m \geq 3$ [Lin87, BCL86, JK83, SU82] and (b) there exist harmonic maps into $S^{n-1}$ which are everywhere discontinuous if $m, n \geq 3$ [Riv95]. For general, compact $C^2$ target manifolds $N$ without boundary it is still known that harmonic maps into $N$ are smooth if $M$ is two-dimensional, see [Hél91, Hél02, Riv07].

Of particular importance for this work are weak compactness results for harmonic maps. Given a bounded sequence of harmonic maps into $S^{n-1}$, it is known and straightforward to show that every accumulation point of the sequence is again a harmonic map into $S^{n-1}$, see [Sha88] for relevant identities. For submanifolds $N$ without symmetry this is only known if $M$ is two-dimensional and $N$ is compact, $C^2$ regular, and without boundary [Bet93b, FMS98, Riv07].

Most of the approximation schemes devised below are motivated and based on gradient flows of harmonic maps. The $L^2$ gradient flow of a harmonic map and its suitability of defining a topological homotopy to smoothly deform a given surface into another one, have been studied intensively within the last two decades. In general, topological changes have to be expected and finite-time blow-up of weak solutions is known to occur even for two-dimensional domains $M$. The survey article [Str96] provides an overview of related results.

We remark that harmonic maps between surfaces are also used to compute closed geodesics, that they arise in Teichmüller theory and rigidity assertions for Kähler manifolds, and that they have applications when looking for conformal or harmonic parameterizations of surfaces. For more details and examples of other applications we refer the reader to [EL78, EL95, Jos84, Jos85, Hil85] and references therein.

### *Difficulties in the approximation of harmonic maps*

The properties of harmonic maps outlined above indicate that approximation schemes have to be developed carefully in order to deal with the limited regularity. The three major difficulties in the numerical approximation are that numerical schemes have to (i) cope with the critical nonlinearity in the right-hand side of (2), (ii) satisfy the constraint $u(x) \in N$ appropriately, and (iii) lead to approximations of low energy.

The first issue (i) can be effectively solved by noting that $A_N$ assumes values in the normal bundle of $N$ and restricting to test functions that are tangential along the unknown $u$. This results in the equivalent weak formulation of finding a weakly differentiable $u\colon M \to N$ such that

$$\big(\nabla_M u; \nabla_M v\big) = 0$$

for all smooth vector fields $v\colon M \to \mathbb{R}^n$ satisfying $v(x) \in T_{u(x)}N$ for almost every $x \in M$ and where $(\cdot;\cdot)$ denotes the inner product in $L^2(M)$. The practical realization (ii) of the constraint $u(x) \in N$ is by no means a straightforward task. Even for the simplest case $N = S^{n-1}$ one easily verifies that a continuous, piecewise polynomial function $w_h$ satisfies $w_h(x) \in S^{n-1}$ for almost every $x \in M$, i.e., $|w_h| = 1$ almost everywhere in $M$, if and only if $w_h$ is constant. Therefore, approximation schemes relax the constraint and finite element approximations are only required to assume their nodal values in $N$. In this way, the constraint may be satisfied almost nowhere but for a bounded sequence of such finite element functions every accumulation point satisfies the constraint almost everywhere. Ginzburg-Landau approximations provide another way of imposing the constraint in a relaxed, practical way. This, however, requires the introduction of a small penalization parameter and the resulting regularized problem does usually not show the desired sharp topological effects. Moreover, error estimates for the approximation of Ginzburg-Landau equations may depend exponentially on the inverse of the penalization parameter. Finally, the problem (iii) of computing harmonic maps of low energy can be solved by discretizing gradient flows of harmonic maps and choosing discretization parameters such that discrete energy laws are satisfied.

So far we have not addressed the discretization of the possibly non-flat domain $M$. Following the work of [Dzi88] this can be effectively done and analyzed by approximating $M$ with a polyhedral surface. The tangential gradient $\nabla_{M_h}$ along the approximate, Lipschitz-continuous surface $M_h$ is then defined only piecewise but can be shown to provide a good approximation of the continuous tangential gradient $\nabla_M$ by making use of lifting operators which associate to a given function $v\colon M_h \to \mathbb{R}$ a function $\widetilde{v}\colon M \to \mathbb{R}$.

Combining the ideas of employing test functions that assume their nodal values in the tangent space of $N$ and imposing the constraint $u(x) \in N$ only at the nodes of a given triangulation $\mathcal{T}_h$ of $M_h$ with vertices $\mathcal{N}_h$ and a subordinated lowest order finite element space $\mathcal{S}^1(\mathcal{T}_h)^n$, we are led to the following definition: The vector field $u_h \in \mathcal{S}^1(\mathcal{T}_h)^n$ is called a *discrete harmonic map into $N$* subject to the boundary data $u_{\mathrm{D}}$ if $u_h|_{\Gamma_{\mathrm{D}}} = u_{\mathrm{D},h}$, $u_h(z) \in N$ for all $z \in \mathcal{N}_h$, and

$$\big(\nabla_{M_h} u_h; \nabla_{M_h} v_h\big) = 0$$

is satisfied for all vector fields $v_h \in \mathcal{S}^1(\mathcal{T}_h)^n$ such that $v_h|_{\Gamma_{\mathrm{D}}} = 0$ and $v_h(z) \in T_{u_h(z)}N$ for all $z \in \mathcal{N}_h$. Here, $(\cdot;\cdot)$ denotes the $L^2$ inner product on $M_h$ and we included Dirichlet conditions on the possibly empty subset $\Gamma_{\mathrm{D}} \subseteq \partial M$; we always assume that $M$ has either a polyhedral boundary which is matched exactly by $\partial M_h$ or $M$ is a closed surface without boundary.

This formulation of the problem is still difficult to solve directly owing to the constraint on $u_h$ and the fact that the nodewise restriction on the test functions is defined through the unknown $u_h$. The major goals of this work are to discuss how to reliably and efficiently compute discrete harmonic maps into $N$ and to investigate whether they accumulate at weak solutions of (2) as the maximal mesh-size $h$ of a sequence of regular triangulations $\big(\mathcal{T}_h\big)_{h>0}$ tends to zero.

The approximation of harmonic maps into spheres started with the work [LL89] that studied point relaxation methods. An energy decreasing iterative algorithm that linearizes the constraint in each step has been introduced and shown to converge in a continuous setting in [Alo94, Alo97, AG97]. Convergence of a finite element discretization of that algorithm on weakly acute triangulations has been established in [Bar05a]. The authors of [VO02] discuss parametric approaches for the approximation of $p$-harmonic maps into spheres that lead to unconstrained discrete problems and successfully employ them to denoise color images. Convergence of projection and penalization approaches to local strong solutions is proved in [Pro01] and a careful analysis of the dependence of error estimates on penalization parameters in the approximation of Ginzburg-Landau type equations can be found in [FP03, Bar05a]. An interesting saddle-point formulation for the computation of discrete harmonic maps into spheres that leads to a separately convex optimization problem has been proposed in [CD03]. Various methods for the discretization of the harmonic map heat flow into spheres have recently been developed and analyzed in [AJ06, BBFP07, BP07, Alo07]. For numerical algorithms for the related problems of approximating minimal surfaces and conformal structures of surfaces we refer the reader to [Dzi91, PP93, GY02, DH99, DH06] and references therein.

Apart from the convergence result in [MSŠ97] of discrete harmonic maps on planar, regular lattices to harmonic maps into compact $C^4$ submanifolds $N \subset \mathbb{R}^n$ without boundary, the author is unaware of algorithms or approximation results for discrete harmonic maps into general target manifolds.

***Contributions of this work***

Motivated by the definition of a discrete harmonic map into a given surface $N$ and generalizing work of [Alo97, Bar05b, BBFP07, BP07] for $N = S^{n-1}$, we employ the following iteration to compute discrete harmonic maps of low energy. We denote by $\overset{\circ}{\mathcal{S}}{}^1(\mathcal{T}_h)$ the subspace of $\mathcal{S}^1(\mathcal{T}_h)$ consisting of functions that vanish on $\Gamma_\mathrm{D}$ if this set is non-empty or have zero integral mean otherwise; $\pi_N$ denotes the orthogonal or nearest-neighbor projection onto $N$ which is well-defined in a small, tubular neighborhood of $N$ provided that $N$ is $C^2$.

**Algorithm A.** *Input:* triangulation $\mathcal{T}_h$, damping parameter $\kappa > 0$, stopping criterion $\varepsilon > 0$.

1. Choose $u_h^0 \in \mathcal{S}^1(\mathcal{T}_h)^n$ such that $u_h^0|_{\Gamma_\mathrm{D}} = u_{\mathrm{D},h}$ and $u_h^0(z) \in N$ for all $z \in \mathcal{N}_h \setminus \Gamma_\mathrm{D}$. Set $i := 0$.
2. Compute $w_h^i \in \overset{\circ}{\mathcal{S}}{}^1(\mathcal{T}_h)^n$ such that $w_h^i(z) \in T_{u_h^i(z)}N$ for all $z \in \mathcal{N}_h$ and
$$\big(\nabla_{M_h} w_h^i; \nabla_{M_h} v_h\big) = -\big(\nabla_{M_h} u_h^i; \nabla_{M_h} v_h\big)$$
for all $v_h \in \overset{\circ}{\mathcal{S}}{}^1(\mathcal{T}_h)^n$ such that $v_h(z) \in T_{u_h^i(z)}N$ for all $z \in \mathcal{N}_h$.
3. Stop if $\big\|\nabla_{M_h} w_h^i\big\| \le \varepsilon$.
4. Define $u_h^{i+1} \in \mathcal{S}^1(\mathcal{T}_h)^n$ by setting
$$u_h^{i+1}(z) := \pi_N\big(u_h^i(z) + \kappa w_h^i(z)\big)$$
for all $z \in \mathcal{N}_h$.
5. Set $i := i+1$ and go to (2).

*Output:* $u_h^* := u_h^i$.

Algorithm A can be derived by discretizing the $H^1$ gradient flow of harmonic maps into $N$ and then $\kappa$ is a time-step size while $w_h^i$ serves as an approximation of the time-derivative. Another motivation of the iteration results from regarding $w_h^i$ as a correction of the given approximation $u_h^i$ and a linearization of the condition $u_h^i(z) + \kappa w_h^i(z) \in N$. In the latter derivation, $\kappa$ is a damping parameter rather than a time-step size. In both cases the steps of the algorithm can be summarized by the loop:

$$\texttt{linearize} \quad \longrightarrow \quad \texttt{update} \quad \longrightarrow \quad \texttt{project}$$

The damping parameter $\kappa$ is needed to guarantee that the nodal values of the update $u_h^i + \kappa w_h^i$ belong to $U_{\delta_N}(N)$ so that the final step in the loop is well defined. The following theorem states that the iteration converges if $\kappa = \mathcal{O}(h)$. We always suppose that $M$ is a smooth, compact, orientable hypersurface in $\mathbb{R}^m$ which is either without boundary or a polyhedral subset of $\mathbb{R}^{m-1} \times \{0\}$. For brevity, we let $d = m - 1$ denote the dimension of $M$. The $k$-dimensional target manifold $N \subset \mathbb{R}^n$ is assumed to be compact and without boundary but not necessarily orientable.

**Theorem I.** *Suppose that $d \le 4$ and $N$ is $C^3$. There exist $h$-independent constants $C', C'' > 0$ such that if $\kappa \le C' h_{min}$ and $\varepsilon > 0$ then Algorithm A is feasible and terminates within a finite number of iterations. The output $u_h^*$ satisfies $u_h^*(z) \in N$ for all $z \in \mathcal{N}_h$, $\|\nabla_{M_h} u_h^*\| \le C'' \|\nabla_{M_h} u_h^0\|$, and*

$$\big(\nabla_{M_h} u_h; \nabla_{M_h} v_h\big) = \mathcal{R}es_h(v_h)$$

*for all $v_h \in \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^n$ such that $v_h(z) \in T_{u_h^*(z)}N$ for all $z \in \mathcal{N}_h$ and a bounded linear functional $\mathcal{R}es_h \colon \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^n \to \mathbb{R}$ which satisfies $|\mathcal{R}es_h(w_h)| \le \varepsilon \|\nabla_{M_h} w_h\|$ for all $w_h \in \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^n$.*

The assumptions and assertion of Theorem A can be significantly improved if $N = \partial\mathcal{C}$ for a bounded, open, convex set $\mathcal{C} \subset \mathbb{R}^n$ and if $\mathcal{T}_h$ is *weakly acute*, e.g., if $d = 2$, $\mathcal{T}_h$ consists of triangles, and sums of angles opposite to inner edges in $\mathcal{T}_h$ are bounded by $\pi$ whereas angles of triangles opposite to boundary edges do not exceed $\pi/2$. In this case, using that $\pi_N \colon \mathbb{R}^n \setminus \mathcal{C} \to N$ is non-expanding, $\kappa$ can be chosen of order one and we have $C'' = 1$. It is remarkable that Algorithm A is globally convergent. While this guarantees that any choice of $u_h^0$ will lead to a discrete harmonic map into $N$, it also explains that the iteration can be very slowly convergent. Nevertheless, once a good approximation of a discrete harmonic map is available then local schemes such as Newton iterations can be employed and we devise a scheme based on a combination of global and local iterations which performs very efficiently in practice.

The proof of Theorem I exploits the fact that Algorithm A can be understood as a discretization of the $H^1$ gradient flow of harmonic maps and that $\pi_N$ is $C^2$ with $D\pi_N(p)|_{T_pN} = \mathrm{id}|_{T_pN}$ for all $p \in N$. Proving convergence of a sequence of outputs of Algorithm A to a harmonic map into $N$ as $h, \varepsilon \to 0$ is more involved and we provide a positive answer if $M$ is two-dimensional and the sequence of triangulations satisfies a restrictive angle condition: $\big(\mathcal{T}_h\big)_{h>0}$ is said to be *logarithmically right-angled* if for every $\varepsilon > 0$ there exists $h_0 > 0$ such that for all $0 < h \le h_0$ and every triangle $K \in \mathcal{T}_h$ with inner angles $\alpha_{K,j} \in [0, \pi]$, $j = 1, 2, 3$, we have $\min_{j=1,2,3} \log h_{min}^{-1} |\cos \alpha_{K,j}| \le \varepsilon$. Sufficient for this is that for all $h > 0$ each $K \in \mathcal{T}_h$ has a right angle. This notion of structured triangulations permits us to prove the following result which guarantees that a bounded sequence of outputs $\big(u_h^*\big)_{h>0}$ of Algorithm A accumulates at harmonic maps as $h \to 0$.

**Theorem II.** *Suppose that $d = 2$ and $N$ is $C^4$. Let $\big(\mathcal{T}_h\big)_{h>0}$ be a sequence of logarithmically right-angled triangulations and for each $h > 0$ let $u_h \in \mathcal{S}^1(\mathcal{T}_h)^n$ satisfy $u_h|_{\Gamma_{\mathrm{D}}} = u_{\mathrm{D},h}$ and $u_h(z) \in N$ for all $z \in \mathcal{N}_h$. Assume that for each $h > 0$ there exists a linear functional $\mathcal{R}es_h \colon \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^n \to \mathbb{R}$ which satisfies $|\mathcal{R}es_h(w_h)| \le \varepsilon(h) \|\nabla_{M_h} w_h\|$ for all $w_h \in \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^n$ and such that*

$$\big(\nabla_{M_h} u_h; \nabla_{M_h} v_h\big) = \mathcal{R}es_h(v_h)$$

*for all $v_h \in \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^n$ satisfying $v_h(z) \in T_{u_h^*(z)}N$ for all $z \in \mathcal{N}_h$. If $\|\nabla_{M_h} u_h\| \le C$, $u_{\mathrm{D},h} \to u_{\mathrm{D}}$ in $L^2(\Gamma_{\mathrm{D}}; \mathbb{R}^n)$, and $\varepsilon(h) \to 0$ as $h \to 0$ then every weak accumulation point $u \in W^{1,2}(M; \mathbb{R}^n)$ of the lifted sequence $\big(\widetilde{u}_h\big)_{h>0} \subset W^{1,2}(M; \mathbb{R}^n)$ is a harmonic map into $N$ satisfying $u|_{\Gamma_{\mathrm{D}}} = u_{\mathrm{D}}$.*

The asymptotic right-angled condition allows the usage of highly graded, locally refined triangulations but is restrictive in case of non-flat surfaces $M$. We show however that the condition is not necessary if the lifted sequence $(\widetilde{u}_h)_{h>0}$ is uniformly bounded in $W^{1,2+\sigma}(M;\mathbb{R}^n)$ for some positive $\sigma$. In view of this result, the angle condition appears to be a technical deficiency of the method of proof rather than a necessary condition. If $N = S^{n-1}$ is the $(n-1)$-dimensional unit sphere then $M$ does not have to be two-dimensional and any sequence of regular triangulations leads to the assertion of Theorem II. We note that the existence of a harmonic map into $N$ is implicitly assumed by requiring that a sequence of lifted initial discrete vector fields $(\widetilde{u}_h^0)_{h>0}$ is bounded in $W^{1,2}(M;\mathbb{R}^n)$.

The proof of Theorem II follows ideas from [FMS98, MSŠ97] and employs a discrete moving frame to rewrite the discrete Euler-Lagrange equations as an equivalent Hodge system. A discrete Coulomb gauge of the orthonormal frame leads to connection forms that are discrete divergence-free if the underlying triangulation is right-angled. Therefore, on general triangulations a discrete Hodge (or Helmholtz) decomposition of the connection forms that makes use of non-conforming finite element spaces leads to non-vanishing gradient contributions. To show that corresponding terms in the Hodge system vanish as $h \to 0$ we require the sequence of triangulations to be logarithmically right-angled. The limit of the remaining part can be, owing to a Jacobian structure, identified with weak concentration and compensation compactness principles based on results in [Lio85, Mül90, CLMS93] together with the fact that the set of (discrete) harmonic fields on $M$ is finite dimensional. This non-trivial and critical limit passage implies the theorem.

We remark that a more direct weak compactness result recently given in [Riv07] may lead to a sharper result than Theorem II since it avoids the use of a moving frame and only requires $N$ to be $C^2$ regular. The repeated use of the product rule, however, makes it difficult to adapt the proof to the finite element setting considered here.

### ***Overview of the thesis***

The outline of this work is as follows. In Chapter 1 we provide a variety of tools for the development and analysis of numerical algorithms. Weak compactness results are the key towards proving convergence of discrete harmonic maps and are discussed and appropriately adapted to finite element settings in Chapter 2. Various local and global, fully practical iterative schemes that compute discrete harmonic maps are devised and analysed in Chapter 3. Implementation issues together with numerical experiments that study the efficiency as well as the reliability of the proposed algorithms and investigate important effects such as finite-time blow-up and (discrete) geometric changes are reported in Chapter 4. Short MATLAB realizations of some routines are displayed in Appendix A; a summary of frequently used notation is provided in Appendix B.

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# Contents

# Chapter 1

# Analytical and numerical tools

Finite element methods for the approximation of partial differential equations on surfaces have become popular in recent years, cf., e.g., [Dzi88, BMN04, Car04, DR04, DDE05, BGN07]. In this chapter we introduce Sobolev spaces on hypersurfaces and finite elements on discretized surfaces and we develop various tools for their analysis such as weakly acute triangulations of surfaces and Helmholtz decompositions of discrete tangential vector fields. In addition, we define orthogonal projections onto surfaces which are not necessarily the boundaries of convex domains and we discuss basic tools from differential geometry required in our analysis below. For readers familiar with these techniques we provide a brief overview over the main results from the chapter in the first section and refer to the corresponding parts of the chapter for references and details.

## 1.1 Overview of provided tools

Given a smooth, connected, compact, orientable hypersurface $M \subset \mathbb{R}^{d+1}$ which is either flat or has no boundary, the definition of Sobolev spaces on $M$ is based on the concept of the tangential gradient on $M$. For a smooth function $\phi : M \to \mathbb{R}$ and an extension $\phi^e$ of $\phi$ to an open neighborhood of $M$, the tangential gradient is the column vector

$$\nabla_M \phi := \nabla \phi^e - \big(\nabla \phi^e \cdot \mu\big)\mu$$

defined with the unit normal vector field $\mu$ on $M$. For a (not necessarily tangential) vector field on $M$, its tangential gradient is the matrix whose rows coincide with the tangential gradients of its components. For an approximation $M_h$ of $M$ consisting of $d$-simplices called elements contained in a set $\mathcal{T}_h$, the discrete tangential gradient $\nabla_{M_h}\phi_h$ is defined on each element separately. A bijective transfer operator $\mathcal{P}_h : M_h \to M$ allows to introduce a lifting operator that associates to a given function $\phi_h$ on $M_h$ a function $\widetilde{\phi}_h$ on $M$ and stability of this operator holds in various norms. The space $\mathcal{S}^1(\mathcal{T}_h)$ consists of all $\mathcal{T}_h$-elementwise affine, globally continuous functions on $M_h$ and $\mathcal{L}^0(\mathcal{T}_h)$ denotes the set of all $\mathcal{T}_h$-elementwise constant functions. The finite element stiffness matrix corresponding to the Laplace-Beltrami operator on $M_h$ is defined through the nodal basis $\big(\varphi_z\colon z \in \mathcal{N}_h\big)$ of $\mathcal{S}^1(\mathcal{T}_h)$ by

$$\mathbf{K}_{z,z'} := \int_{M_h} \nabla_{M_h}\varphi_z \cdot \nabla_{M_h}\varphi_{z'} \, \mathrm{d}s_h$$

for all pairs of nodes $z, z' \in \mathcal{N}_h$. If $d = 2$ then the off-diagonal entries in the matrix $\big(\mathbf{K}_{z,z'}\big)_{z,z'\in\mathcal{N}_h}$ are non-positive if the sum of every pair of angles opposite to an inner edge $E \in \mathcal{E}_h$ does not exceed

$\pi$ and if all angles opposite to boundary edges are bounded by $\pi/2$. We say that the triangulation $\mathcal{T}_h$ is weakly acute if $\mathbf{K}_{z,z'} \le 0$ for all distinct pairs $z, z' \in \mathcal{N}_h$.

Important for the analysis of our numerical schemes is also a discrete product rule which asserts that given two functions $v_h, w_h \in \mathcal{S}^1(\mathcal{T}_h)$ and the nodal interpolant $\mathcal{I}_h\big[v_h w_h\big]$ of their product, it holds that

$$\nabla_{M_h}\big[v_h w_h\big] = \mathbf{A}\big(v_h\big)\nabla_{M_h} w_h + \mathbf{A}\big(w_h\big)\nabla_{M_h} v_h$$

with a matrix $\mathbf{A}(v_h) \in \mathbb{R}^{d+1,d+1}$ that approaches $v_h\mathbf{I}$ as the maximal mesh-size $h$ tends to zero. The matrix $\mathbf{A}(w_h)$ is symmetric for $d = 2$ if and only if every triangle in $\mathcal{T}_h$ has a right-angle. Well-known inverse estimates also hold for finite elements on surfaces and the most important ones are

$$\big\|\nabla_{M_h} v_h\big\|_{L^p(M_h)} \le C h_{min}^{-1} \|v_h\|_{L^p(M_h)}, \qquad \|\psi_h\|_{L^r(M_h)} \le C h_{min}^{-d(1/p-1/r)} \|\psi_h\|_{L^p(M_h)}$$

for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)$, piecewise polynomial functions $\psi_h \in L^\infty(M_h)$ (with uniformly bounded polynomial degree), the minimal mesh-size $h_{min}$, and $1 \le p \le r \le \infty$. For two-dimensional surfaces $M_h$, a Helmholtz decomposition of discrete tangential vector fields in $\mathcal{L}^0(\mathcal{T}_h)^3$, i.e., of mappings $\omega_h \in \mathcal{L}^0(\mathcal{T}_h)^3$ such that $\omega_h \cdot \mu_h = 0$ almost everywhere on $M_h$ with a unit normal vector field $\mu_h$ on $M_h$, is required as a technical tool and guarantees that there exist $a_h \in \mathcal{S}^{1,NC}(\mathcal{T}_h)$, $b_h \in \mathcal{S}^1(\mathcal{T}_h)$, and $H_h \in \mathcal{L}^0(\mathcal{T}_h)^3$ such that

$$\omega_h = \nabla_{M_h} a_h + \mathrm{Curl}_{M_h} b_h + H_h.$$

Here, $\mathcal{S}^{1,NC}(\mathcal{T}_h)$ is the non-conforming finite element space containing all $\mathcal{T}_h$-elementwise affine functions on $M_h$ that are continuous at midpoints of interelement boundaries in $\mathcal{T}_h$. The vector field $H_h$ belongs to a subset of $\mathcal{L}^0(\mathcal{T}_h)^3$ whose dimension is bounded $h$-independently and is called the set of discrete harmonic fields on $M_h$. Results on the well-posedness of the nearest-neighbor projection

$$\pi_N : U_{\delta_N}(N) \to N$$

in a tubular neighborhood $U_{\delta_N}(N)$ of a submanifold $N \subset \mathbb{R}^n$ generalize the well-known fact that for boundaries $N = \partial\mathcal{C}$ of convex sets $\mathcal{C}$ the orthogonal projection $\pi_N : \mathbb{R}^n \setminus \mathcal{C} \to N$ is well-defined and non-expanding, i.e., Lipschitz-continuous with Lipschitz constant 1. We also discuss the concept of parallelizable manifolds $N$, which guarantee existence of continuous orthonormal bases of the tangent bundle $TN$. Parallelizability can always be assumed for $C^4$ surfaces $N$ and leads to another equivalent characterization of harmonic maps. We also analyze weak accumulation points of continuous, discrete vector fields assuming their nodal values in a surface $N$ and show that the limiting object has values in $N$ almost everywhere. Finally, we prove two auxiliary results from measure theory, which essentially state that the set of linear combinations of Dirac measures is a closed subset of $C(M)^*$ with respect to the strong topology.

## 1.2 Sobolev spaces on hypersurfaces

Differential operators and Sobolev spaces on Riemannian manifolds and hypersurfaces have been introduced and analyzed in [Heb96, Aub82, HK03, GT01]. Here, we assume that $M \subset \mathbb{R}^m$ is a compact, smooth, orientable, $(m-1)$-dimensional submanifold in $\mathbb{R}^m$ and provide the definitions necessary for our purposes. For brevity, we set $d := m-1$. In order to avoid technical difficulties in the definition of traces of functions on curved surfaces we assume that either $M$ has no boundary or $M$ is a compact subset of $\mathbb{R}^d \times \{0\}$ with Lipschitz continuous boundary. The adjective "smooth" stands for $C^\infty$ and this property of the submanifold $M$ is assumed for simplicity. We note however that all results discussed below can be established for $C^3$ submanifolds $M$ or for sufficiently regular curved submanifolds with regular boundary as well.

We say that $\phi \in C(M)$ is differentiable along $M$ if for every smooth local parametrization $f$ of $M$ the function $\phi \circ f$ is differentiable. Given a smooth, i.e., $C^\infty$, function $\phi : M \to \mathbb{R}$ we let $\phi^e$ denote a differentiable extension of $\phi$ to an open neighborhood of $M$ (e.g., by extending $\phi$ constantly in normal direction) and define the *surface gradient* or *tangential gradient* $\nabla_M \phi$ of $\phi$ along $M$ as the projection of the gradient of $\phi^e$ onto the tangent space of $M$ by

$$\nabla_M \phi := \nabla \phi^e - \big(\nabla \phi^e \cdot \mu\big)\mu.$$

Here $\nabla \phi^e$ denotes the usual full $m$-dimensional gradient of $\phi^e$ and $\mu$ is a unit normal to $M$. For a vector field $\psi \colon M \to \mathbb{R}^\ell$, $\nabla_M$ is applied to each component of $\psi$ and then $\nabla_M \psi$ denotes the matrix whose rows coincide with the transpose of the surface gradients of the components of $\psi$. One easily verifies that this definition is independent of the employed extension $\phi^e$ and we remark that in coordinates the surface gradient can be computed without an extension of $\phi$ and is given by

$$\nabla_M \phi = \sum_{i,j=1}^{d} g^{ij} \partial_j \big(\phi \circ f\big) \partial_i f,$$

where $f : \widehat{\Omega} \to M$ for $\widehat{\Omega} \subset \mathbb{R}^d$ is a local $C^1$ parametrization of $M$ and $g^{ij}$ are the entries of the inverse of the matrix with entries $g_{ij} := \partial_i f \cdot \partial_j f$ for $1 \le i, j \le d$.

Throughout, we let $\mathrm{d}s$ denote the surface area element on $M$ and $(\cdot;\cdot)$ the scalar product in $L^2(M;\mathbb{R}^n)$, the space of all square integrable, vector valued Lebesgue functions on $M$ with corresponding norm $\|\cdot\|$. For a real number $p \ge 1$ we let $W^{1,p}(M)$ denote the completion of $C^\infty(M)$ under the norm

$$||\phi||_{W^{1,p}(M)} := \big(\|\phi\|^p_{L^p(M)} + \big\|\nabla_M \phi\big\|^p_{L^p(M)}\big)^{1/p}.$$

We denote by $W_0^{1,p}(M)$ the closure of $C_c^\infty(M)$, defined as the set of all functions in $C^\infty(M)$ which have compact support in $M$, under the norm $\|\cdot\|_{W^{1,p}(M)}$. If $\partial M = \emptyset$ then we clearly have $W^{1,p}(M) = W_0^{1,p}(M)$. As a consequence of the compactness of $M$, we notice that classical Sobolev embeddings are valid.

**Theorem 1.2.1.** *[Heb96, Proposition 2.4, Theorems 3.5, 3.6; p. 10–24] (i) For $p > 1$ the space $W^{1,p}(M)$ is reflexive.*
*(ii) For any $1 \le q < d$ such that $1/p = 1/q - 1/d$ and all $\phi \in W^{1,q}(M)$ we have*

$$||\phi||_{L^p(M)} \le C d^{-1} p(d-1) ||\phi||_{W^{1,q}(M)},$$

*in particular, for $p = dq/(d-q)$ we have $p(d-1)/d \le d^2/(d-q)$.*
*(iii) For $1 \le q < d$ and $1/p > 1/q - 1/d$ the embedding $W^{1,q}(M) \hookrightarrow L^p(M)$ is compact.*

As in the Euclidean situation, the following Sobolev-Poincaré estimate is valid.

**Theorem 1.2.2.** *[Heb96, Proposition 3.9, p. 26] For $p, q$ such that $1 \le q < d$ and $1/p = 1/q - 1/d$ there exists a constant $C > 0$ such that*

$$\|\phi - \overline{\phi}\|_{L^p(M)} \le C \big\|\nabla_M \phi\big\|_{L^q(M)}$$

*for all $\phi \in W^{1,q}(M)$ with integral mean $\overline{\phi} = \frac{1}{\mathcal{H}^d(M)} \int_M \phi \,\mathrm{d}s$.*

The theorem implies $\|\phi-\overline{\phi}\|_{L^2(M)} \le C\big\|\nabla_M\phi\big\|_{L^2(M)}$ for all $\phi\in W^{1,2}(M)$ owing to compactness of $M$. If $\partial M\neq\emptyset$ then, by assumption, $M$ is flat and we have that $\|\phi\|_{L^2(M)} \le C\big\|\nabla_M\phi\big\|_{L^2(M)}$ for all $\phi\in W_0^{1,2}(M)$, cf., e.g., [Ada75]. If $M$ is flat and we are given a closed subset $\Gamma_{\mathrm{D}}\subseteq\partial\mathcal{M}$ of positive surface measure then we denote by $W_{\mathrm{D}}^{1,p}(M)$ the set of all functions in $W^{1,p}(M)$ whose traces vanish on $\Gamma_{\mathrm{D}}$.

The following lemma shows that the components of the surface gradient of a product of two functions behave as if they were independent. This is not surprising since $\nabla_M$ coincides with the outer derivative for scalar valued functions.

**Lemma 1.2.3.** *For $a\in H^1(M)$ let $\underline{D}_1a,\underline{D}_2a,...,\underline{D}_ma$ denote the entries of the vector field $\nabla_Ma$, i.e.,*

$$\nabla_Ma=\big[\underline{D}_1a,\underline{D}_2a,...,\underline{D}_ma\big]^{\mathrm{T}}.$$

*For $a,b\in H^1(M)\cap L^\infty(M)$ we have $ab\in H^1(M)\cap L^\infty(M)$ and, for $\gamma=1,2,...,m$,*

$$\underline{D}_\gamma(ab)=a\underline{D}_\gamma b+b\underline{D}_\gamma a$$

*almost everywhere on $M$.*

*Proof.* Let $a^e,b^e$ be differentiable extensions of $a$ and $b$ to an open neighborhood of $M$. Then, by definition of the surface gradient, we have

$$\begin{aligned}\big[\underline{D}_1(ab),\underline{D}_2(ab),...,\underline{D}_m(ab)\big]^{\mathrm{T}} &= \nabla_M(ab)\\ &= \nabla(a^eb^e)-\big(\mu\otimes\mu\big)\nabla(a^eb^e)\\ &= a\nabla b^e+b\nabla a^e-b\,\big(\mu\otimes\mu\big)\nabla a^e-a\,\big(\mu\otimes\mu\big)\nabla b^e\\ &= a\nabla_Mb+b\nabla_Ma\\ &= a\big[\underline{D}_1b,\underline{D}_2b,...,\underline{D}_mb\big]^{\mathrm{T}}+b\big[\underline{D}_1a,\underline{D}_2b,...,\underline{D}_mb\big]^{\mathrm{T}}\end{aligned}$$

almost everywhere on $M$, which proves the asserted identity. ☐

## 1.3 Triangulations of hypersurfaces

Given the hypersurface $M$ as in the previous section, we consider a polyhedral hypersurface defined through a set of non-degenerate $d$-simplices $\mathcal{T}_h$ such that $M_h:=\cup_{K\in\mathcal{T}_h}K$ is a compact, orientable, $d$-dimensional, Lipschitz continuous submanifold in $\mathbb{R}^m$. We require that $M_h$ has either no boundary or is a subset of $\mathbb{R}^d\times\{0\}$ with polyhedral boundary. In the latter case we assume that $M=M_h$. We suppose that $\mathcal{T}_h$ is a regular triangulation of $M_h$, i.e., $K\cap K'$ is either empty or an entire subsimplex of $K$ and $K'$ for all distinct $K,K'\in\mathcal{T}_h$. We also suppose that the set of nodes $\mathcal{N}_h$, which consists of all vertices of elements in $\mathcal{T}_h$, is contained in $M$. Hence, $M_h$ serves as an approximation of $M$. In fact, by employing local parametrizations of $M$ we verify that

$$\operatorname{dist}(M_h,M):=\sup_{x\in M_h}\operatorname{dist}(x,M)\le Ch^2.$$

Here $h:=\max_{K\in\mathcal{T}_h}h_K$ with $h_K:=\operatorname{diam}(K)$ for all $K\in\mathcal{T}_h$, is the maximal diameter of elements in $\mathcal{T}_h$. We assume that $\mathcal{T}_h$ satisfies a minimum angle condition and that each $K\in\mathcal{T}_h$ is the image of the scaled reference element

$$\widehat{K}:=h_K\operatorname{conv}\big\{\widehat{z}_0,\widehat{z}_1,...,\widehat{z}_d\big\},$$

under an affine bijection $\mathcal{F}_K\colon \widehat{K} \to K$. The vertices $\widehat{z}_0, \widehat{z}_1, ..., \widehat{z}_d \in \mathbb{R}^d$ of $\widehat{K}$ are assumed to satisfy $\widehat{z}_0 = 0$ and $\big(\widehat{z}_i\big)_{i=1,2,\dots,d}$ coincide with the canonical basis vectors in $\mathbb{R}^d$. The following assumption provides a tool to establish relations between $M$ and $M_h$ and we will always assume its validity.

**Assumption (T).** *There exists a continuous bijection $\mathcal{P}_h\colon M_h \to M$ such that $\mathcal{P}_h|_K \in C^2(K)$ is a diffeomorphism between each $K \in \mathcal{T}_h$ and $\widetilde{K} := \mathcal{P}_h(K)$ (i.e., $\mathcal{P}_h$ and $\mathcal{P}_h^{-1}$ admits twice and once continuously differentiable extensions to open neighborhoods of $K$ and $\widetilde{K}$, respectively) with*

$$\|D\mathcal{P}_h\|_{L^\infty(K)},\ \|D^2\mathcal{P}_h\|_{L^\infty(K)},\ \|D\mathcal{P}_h^{-1}\|_{L^\infty(\widetilde{K})} \le C.$$

For smooth surfaces without boundary and for sufficiently small $h$ we may use the orthogonal (or nearest-neighbor) projection onto $M$ to define $\mathcal{P}_h$.

**Remark 1.3.1.** *There exists a tubular neighborhood $U_{\delta_M}(M)$ of $M$ such that the orthogonal projection*

$$\pi_M\colon U_{\delta_M}(M) \to M$$

*is well-defined and smooth. If $M_h \subset U_{\delta_M}(M)$ and if $h$ is sufficiently small, $\mathcal{P}_h := \pi_M|_{M_h}$ satisfies the requirements of Assumption (T), cf. Section 1.6 and [Dzi88, DD07] for details.*

We remark that $\big(\widetilde{K}\colon K \in \mathcal{T}_h\big)$ defines a partition of $M$ and each $\widetilde{K}$ is parametrized by the mapping $\mathcal{X}_K := \mathcal{P}_h \circ \mathcal{F}_K : \widehat{K} \to \widetilde{K}$. The operator $\mathcal{P}_h$ provided by Assumption (T) allows to lift functions defined on $M_h$ onto $M$.

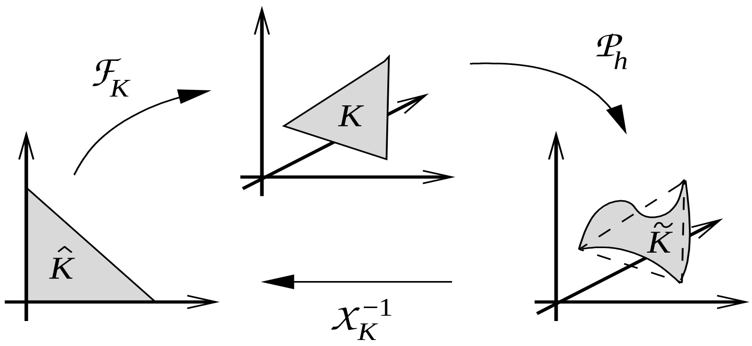


Figure 1.1: Transformations between $\widehat{K}$, $K$, and $\widetilde{K}$.

**Definition 1.3.2.** *The* lifting $\widetilde{v}\colon M \to \mathbb{R}$ *of $v \in L^1(M_h)$ is defined by*

$$\widetilde{v} := v \circ \mathcal{P}_h^{-1}.$$

We let $\mathrm{d}s_h$ denote the surface area element on $M_h$ and for functions $f, g\colon M_h \to \mathbb{R}^\ell$ we set

$$(f; g) := \int_{M_h} f \cdot g \,\mathrm{d}s_h \qquad \text{and} \qquad \|f\| := (f; f)^{1/2}.$$

**Lemma 1.3.3.** *Given $v \in L^1(M_h)$ and $K \in \mathcal{T}_h$ we have*

$$\int_K v \,\mathrm{d}s_h = \int_{\widetilde{K}} \widetilde{v}\,\big((Q_h/Q) \circ \mathcal{X}_K^{-1}\big)\,\mathrm{d}s,$$

*where $Q_h^2 := \det \mathbf{g}_h$ and $Q^2 := \det \mathbf{g}$ for $\mathbf{g}_h := \big(D\mathcal{F}_K\big)^T D\mathcal{F}_K$ and $\mathbf{g} := \big(D\mathcal{X}_K\big)^T D\mathcal{X}_K$, respectively.*

*Proof.* Given $v \in L^1(M_h)$ and $w \in L^1(M)$ we have

$$\int_K v \,\mathrm{d}s_h = \int_{\widehat{K}} v \circ \mathcal{F}_K \, Q_h \,\mathrm{d}\widehat{x}, \qquad \int_{\widetilde{K}} w \,\mathrm{d}s = \int_{\widehat{K}} w \circ \mathcal{X}_K \, Q \,\mathrm{d}\widehat{x}.$$

For $w := \widetilde{v}\,\big((Q_h/Q) \circ \mathcal{X}_K^{-1}\big)$ the second identity yields that

$$\int_{\widetilde{K}} \widetilde{v}\,\big((Q_h/Q) \circ \mathcal{X}_K^{-1}\big) \,\mathrm{d}s = \int_{\widehat{K}} \big(\widetilde{v} \circ \mathcal{X}_K\big)(Q_h/Q)Q \,\mathrm{d}\widehat{x} = \int_{\widehat{K}} v \circ \mathcal{F}_K \, Q_h \,\mathrm{d}\widehat{x}$$

and this implies the assertion. □

The limited regularity of $M_h$ motivates the elementwise definition of the discrete surface gradient on $M_h$.

**Definition 1.3.4.** *Given $\phi \in C(M_h)$ such that $\phi|_K \in C^1(K)$ for all $K \in \mathcal{T}_h$ we set*

$$\nabla_{M_h}\phi|_K := \nabla_{M_h}\phi^e|_K - \big(\nabla_{M_h}\phi^e|_K \cdot \mu_h|_K\big)\mu_h|_K,$$

*for all $K \in \mathcal{T}_h$, where $\mu_h|_K$ is a unit normal to $K \subset M_h$, and $\phi^e$ a differentiable extension of $\phi$ to an open neighborhood of $K$.*

The following lemma is taken from [Mek05]; similar results can be found in [Dzi88, DD07]. We include the proof since some of the estimates therein will be useful in the sequel.

**Lemma 1.3.5** ([Mek05])**.** *Let $v, w \colon M_h \to \mathbb{R}$ be $\mathcal{T}_h$-elementwise differentiable and globally continuous functions with their liftings $\widetilde{v}, \widetilde{w} \colon M \to \mathbb{R}$ from Definition 1.3.2. Then*

$$\int_{M_h} \nabla_{M_h} v \cdot \nabla_{M_h} w \,\mathrm{d}s_h - \int_M \nabla_M \widetilde{v} \cdot \nabla_M \widetilde{w} \,\mathrm{d}s = \int_M \big(\mathbf{F}_h \nabla_M \widetilde{v}\big) \cdot \nabla_M \widetilde{w} \,\mathrm{d}s$$

*with $\mathbf{F}_h \in L^\infty(M;\mathbb{R}^{d+1\times d+1})$ independent of $v, w$ and*

$$\sup_{x \in M} |\mathbf{F}_h(x)| \le C h_K.$$

*Proof.* Let $K \in \mathcal{T}_h$ and define $\widehat{v} \colon \widehat{K} \to \mathbb{R}$ through $\widehat{v} := v \circ \mathcal{F}_K$. Set $\mathbf{G}_h := D\mathcal{F}_K$ and use that the column vectors of $\mathbf{G}_h$ are tangent vectors to $K$, i.e., $\mathbf{G}_h^{\mathrm{T}}\big(\mu_h \circ \mathcal{F}(\widehat{x})\big) = 0$. Then

$$\widehat{\nabla}\widehat{v} = \mathbf{G}_h^{\mathrm{T}}\big((\nabla v^e) \circ \mathcal{F}_K\big) = \mathbf{G}_h^{\mathrm{T}}\big[\big(\nabla v^e - (\nabla v^e \cdot \mu_h)\mu_h\big) \circ \mathcal{F}_K\big] = \mathbf{G}_h^{\mathrm{T}}\big[(\nabla_{M_h} v) \circ \mathcal{F}_K\big].$$

To keep notation short we omit the transformations and simply write

$$\widehat{\nabla}\widehat{v} = \mathbf{G}_h^{\mathrm{T}} \nabla_{M_h} v$$

for the previous identity. The matrix $\mathbf{G}_h^e \in \mathbb{R}^{(d+1)\times(d+1)}$ is defined by appending the unit normal $\mu_h$ as an additional column to $\mathbf{G}_h$, i.e.,

$$\mathbf{G}_h^e := \big[\mathbf{G}_h \ \mu_h\big].$$

Since $\mathcal{F}_K$ is a regular parametrization of $K$, the matrix $\mathbf{G}_h$ has full rank and its columns are tangent vectors to $K$. Thus, the matrix $\mathbf{G}_h^e$ is invertible and we set $\mathbf{D}_h^e := (\mathbf{G}_h^e)^{-1}$. The matrix $\mathbf{D}_h$ is then obtained by deleting the last row of $\mathbf{D}_h^e$. We thereby obtain the identities

$$\nabla_{M_h} v = \left(\mathbf{G}_h^e \mathbf{D}_h^e\right)^{\mathrm{T}} \nabla_{M_h} v = \mathbf{D}_h^{e\,\mathrm{T}} \begin{bmatrix} \widehat{\nabla}\widehat{v} \\ 0 \end{bmatrix} = \mathbf{D}_h^{\mathrm{T}} \widehat{\nabla}\widehat{v}.$$

Set $\mathbf{G} := D\mathcal{X}_K$, $\mathbf{G}^e := \left[\mathbf{G}\ \mu\right]$, $\mathbf{D}^e := (\mathbf{G}^e)^{-1}$, and define $\mathbf{D}$ by deleting the last row of $\mathbf{D}^e$. Then, in fact, $\mathbf{D}^{e\mathrm{T}} = \left[\mathbf{D}^T\ \mu\right]$. Arguing as above, we infer that

$$\widehat{\nabla}\widehat{v} = \mathbf{G}^{\mathrm{T}} \nabla_M v \qquad \text{and} \qquad \nabla_M \widetilde{v} = \mathbf{D}^{\mathrm{T}} \widehat{\nabla}\widehat{v}.$$

A combination of the previous identities reveals that

$$\nabla_{M_h} v = \left(\mathbf{G}\mathbf{D}_h\right)^{\mathrm{T}} \nabla_M \widetilde{v}$$

and, upon incorporating Lemma 1.3.3,

$$\begin{aligned} \int_K \nabla_{M_h} v \cdot \nabla_{M_h} w \, \mathrm{d}s_h &= \int_{\widetilde{K}} \left( \left(\mathbf{G}\mathbf{D}_h\right)\left(\mathbf{G}\mathbf{D}_h\right)^{\mathrm{T}} \nabla_M \widetilde{v} \right) \cdot \nabla_M \widetilde{w} \, (Q_h/Q) \, \mathrm{d}s \\ &= \int_{\widetilde{K}} \nabla_M \widetilde{v} \cdot \nabla_M \widetilde{w} \, \mathrm{d}s + \int_{\widetilde{K}} \left( (Q_h/Q) \mathbf{G}\mathbf{D}_h \mathbf{D}_h^{\mathrm{T}} \mathbf{G}^{\mathrm{T}} - \mathbf{I}_{(d+1)\times(d+1)} \right) \nabla_M \widetilde{v} \cdot \nabla_M \widetilde{w} \, \mathrm{d}s. \end{aligned} \tag{3.1}$$

Since

$$\mathbf{I}_{(d+1)\times(d+1)} = \left(\mathbf{G}^e\right)^{-1} \mathbf{G}^e = \begin{bmatrix} \mathbf{D} \\ \mu^{\mathrm{T}} \end{bmatrix} [\mathbf{G}\ \mu] = \begin{bmatrix} \mathbf{D}\mathbf{G} & \mathbf{D}\mu \\ \mu^{\mathrm{T}}\mathbf{G} & 1 \end{bmatrix}$$

and

$$\mathbf{I}_{(d+1)\times(d+1)} = \mathbf{G}^e \left(\mathbf{G}^e\right)^{-1} = [\mathbf{G}\ \mu] \begin{bmatrix} \mathbf{D} \\ \mu^{\mathrm{T}} \end{bmatrix} = \mathbf{G}\mathbf{D} + \mu \otimes \mu$$

we have that $\mathbf{D}\mathbf{G} = \mathbf{I}_{d\times d}$, $\mathbf{G}\mathbf{D} = \mathbf{I}_{(d+1)\times(d+1)} - \mu \otimes \mu$, and $\mathbf{G}\mathbf{D}$ is symmetric. Therefore

$$\mathbf{G}\mathbf{D}\mathbf{D}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}} = \mathbf{G}\mathbf{D}\left(\mathbf{G}\mathbf{D}\right)^{\mathrm{T}} = \mathbf{G}\left(\mathbf{D}\mathbf{G}\right)\mathbf{D} = \mathbf{G}\mathbf{D} = \mathbf{I}_{(d+1)\times(d+1)} - \mu \otimes \mu.$$

Employing this identity in (3.1) and using that $\left(\mu \otimes \mu\right) \nabla_M w = 0$ we observe that

$$\int_K \nabla_{M_h} v \cdot \nabla_{M_h} w \, \mathrm{d}s_h = \int_{\widetilde{K}} \nabla_M \widetilde{v} \cdot \nabla_M \widetilde{w} \, \mathrm{d}s + \int_{\widetilde{K}} \frac{1}{Q} \mathbf{G}^{\mathrm{T}} \left(Q_h \mathbf{D}_h \mathbf{D}_h^T - Q\mathbf{D}\mathbf{D}^{\mathrm{T}}\right)^{\mathrm{T}} \mathbf{G} \nabla_M \widetilde{v} \cdot \nabla_M w \, \mathrm{d}s.$$

Setting

$$\mathbf{F}_h := \frac{1}{Q} \mathbf{G}^{\mathrm{T}} \left(Q_h \mathbf{D}_h \mathbf{D}_h^{\mathrm{T}} - Q\mathbf{D}\mathbf{D}^{\mathrm{T}}\right) \mathbf{G}$$

it only remains to prove the bound for $\mathbf{F}_h$ in order to complete the proof of the lemma. With the identities proved above, one verifies that $\mathbf{g}^{-1} = (\mathbf{G}^{\mathrm{T}}\mathbf{G})^{-1} = \mathbf{D}\mathbf{D}^{\mathrm{T}}$ and $\mathbf{g}_h^{-1} = (\mathbf{G}_h^{\mathrm{T}}\mathbf{G}_h)^{-1} = \mathbf{D}_h\mathbf{D}_h^{\mathrm{T}}$. Therefore,

$$Q_h \mathbf{D}_h \mathbf{D}_h^{\mathrm{T}} - Q\mathbf{D}\mathbf{D}^{\mathrm{T}} = Q_h \mathbf{g}_h^{-1} - Q\mathbf{g}^{-1} = (Q_h - Q)\mathbf{g}_h^{-1} + Q\mathbf{g}_h^{-1}(\mathbf{g} - \mathbf{g}_h)\mathbf{g}^{-1}.$$

Uniform bounds for $\mathbf{G}_h$ and (through the assumptions on $\mathcal{P}_h$ also) for $\mathbf{G}$ imply

$$|\mathbf{g} - \mathbf{g}_h| = |\mathbf{G}^{\mathrm{T}}\mathbf{G} - \mathbf{G}_h^{\mathrm{T}}\mathbf{G}_h| = |\mathbf{G}^{\mathrm{T}}(\mathbf{G} - \mathbf{G}_h) + (\mathbf{G} - \mathbf{G}_h)^{\mathrm{T}}\mathbf{G}_h| \leq C|\mathbf{G} - \mathbf{G}_h|.$$

Since

$$Q - Q_h = \frac{Q^2 - Q_h^2}{Q + Q_h} = \frac{\det \mathbf{g} - \det \mathbf{g}_h}{Q + Q_h}, \tag{3.2}$$

since the determinant is continuously differentiable, and since the fact that $\mathcal{F}_K$ is the nodal interpolant of $\mathcal{X}_K$ implies

$$\sup_{\widehat{x}\in\widehat{K}} |\mathbf{G}_h - \mathbf{G}| = \sup_{\widehat{x}\in\widehat{K}} |\nabla\mathcal{X}_K - \nabla\mathcal{F}_K| \le Ch\|D^2\mathcal{X}\|_{L^\infty(\widehat{K})},$$

we verify the asserted estimate. □

**Remark 1.3.6.** *Owing to the minimum angle condition guaranteed by the assumed regularity of the triangulations and Assumption (T), the matrices* $\mathbf{D}$ *and* $\mathbf{D}_h$ *are uniformly bounded.*

The following lemma provides stability estimates for the lifting operator in various norms, cf. [Dzi88] for similar results.

**Lemma 1.3.7.** *For* $1 \le p \le \infty$ *and* $v \in L^p(M_h)$ *we have*

$$C^{-1}\|v\|_{L^p(M_h)} \le \|\widetilde{v}\|_{L^p(M)} \le C\|v\|_{L^p(M_h)},$$

*and, if* $v|_K \in C^1(K)$ *for all* $K \in \mathcal{T}_h$*, then*

$$C^{-1}\|\nabla_{M_h} v\|_{L^p(M_h)} \le \|\nabla_M \widetilde{v}\|_{L^p(M)} \le C\|\nabla_{M_h} v\|_{L^p(M_h)}.$$

*If* $K \in \mathcal{T}_h$ *and* $v|_K \in C^2(K)$ *then*

$$\|D^2_{M_h} v\|_{L^2(K)} \le C\big(\|D^2_{M_h}\widetilde{v}\|_{L^2(\widetilde{K})} + \|\nabla_M \widetilde{v}\|_{L^2(\widetilde{K})}\big),$$

*where* $D^2_{M_h} v := \big(\underline{D}_{h,\gamma}\underline{D}_{h,\delta} v\big)_{\gamma,\delta=1,2,\dots,d+1}$ *and* $D^2_M v := \big(\underline{D}_\gamma \underline{D}_\delta v\big)_{\gamma,\delta=1,2,\dots,d+1}$ *with* $\underline{D}_{h,\gamma}$ *and* $\underline{D}_\gamma$ *denoting the components of* $\nabla_{M_h}$ *and* $\nabla_M$*, respectively.*

*Proof.* The first estimate follows directly from Lemma 1.3.3 by replacing $v$ by $|v|^p$ with $1 \le p < \infty$. For $p = \infty$ the estimate is an immediate consequence of the definition of $\widetilde{v}$. For $K \in \mathcal{T}_h$ and $v$ such that $v|_K \in C^1(K)$ we have, as in the proof of Lemma 1.3.5,

$$\nabla_{M_h} v = \big(\mathbf{G}\mathbf{D}_h\big)^{\mathrm{T}}\nabla_M \widetilde{v} \qquad \text{and} \qquad \nabla_M \widetilde{v} = \big(\mathbf{G}_h\mathbf{D}\big)^{\mathrm{T}}\nabla_{M_h} v. \tag{3.3}$$

Uniform bounds for $\mathbf{G}$, $\mathbf{G}_h$, $\mathbf{D}$, and $\mathbf{D}_h$ imply the second estimate of the lemma. The identities in (3.3) also imply that

$$\underline{D}^h_\gamma v = \sum_{j=1}^{d+1} \big(\mathbf{G}\mathbf{D}_h\big)^{\mathrm{T}}_{\gamma\delta}\underline{D}_\delta \widetilde{v}.$$

With Lemma 1.2.3 we thus verify that

$$\begin{aligned}
\nabla_{M_h}\underline{D}^h_\gamma v &= \big(\mathbf{G}\mathbf{D}_h\big)^{\mathrm{T}}\nabla_M\big[\underline{D}^h_\gamma v\big] \\
&= \sum_{\delta=1}^{d+1}\big(\mathbf{G}\mathbf{D}_h\big)^{\mathrm{T}}\nabla_M\big[\big(\mathbf{G}\mathbf{D}_h\big)^{\mathrm{T}}_{\gamma\delta}\underline{D}_\gamma\widetilde{v}\big] \\
&= \sum_{\delta=1}^{d+1}\big(\mathbf{G}\mathbf{D}_h\big)^{\mathrm{T}}\big[\underline{D}_\delta\widetilde{v}\nabla_M\big(\mathbf{G}\mathbf{D}_h\big)^{\mathrm{T}}_{\gamma\delta} + \big(\mathbf{G}\mathbf{D}_h\big)^{\mathrm{T}}_{\gamma\delta}\underline{D}_\delta\nabla_M\widetilde{v}\big].
\end{aligned}$$

Using that $\mathbf{G}$, $\mathbf{D}_h$, and $\nabla_M \mathbf{G}_{ij}$ are uniformly bounded we verify the assertion. □

**Remark 1.3.8.** *For the particular choice $\mathcal{P}_h = \pi_M|_{M_h}$ the third estimate of the previous lemma can be improved in the sense that a factor $h$ can introduced in front of the $L^2$ norm of the gradient, cf. [Dzi88].*

## 1.4 Finite elements on triangulated surfaces

Given a regular triangulation $\mathcal{T}_h$ that defines an approximation $M_h$ of the hypersurface $M$ as in the previous section, we let $\mathcal{N}_h \subset M$ denote the set of all nodes in $\mathcal{T}_h$ (vertices of elements) and $\mathcal{E}_h$ the set of all $(d-1)$–dimensional subsimplices of elements in $\mathcal{T}_h$, i.e., edges of triangles if $d = 2$ or faces of tetrahedra if $d = 3$. We define $h_K := \operatorname{diam}(K)$ for all $K \in \mathcal{T}_h$ and set $h := \max_{K\in\mathcal{T}_h} h_K$. Analogously, we write $h_E := \operatorname{diam}(E)$ for all $E \in \mathcal{E}_h$. The lowest order $C^0$-conforming finite element space $\mathcal{S}^1(\mathcal{T}_h)$ subordinate to the triangulation $\mathcal{T}_h$ consists of all globally continuous, $\mathcal{T}_h$-elementwise affine functions and the space $\mathcal{L}^0(\mathcal{T}_h)$ is the set of $\mathcal{T}_h$–elementwise constant functions on $M_h$, i.e.,

$$\begin{aligned}\mathcal{S}^1(\mathcal{T}_h) &:= \big\{\phi_h \in C(M_h) : \phi_h|_K \text{ affine for all } K \in \mathcal{T}_h\big\},\\ \mathcal{L}^0(\mathcal{T}_h) &:= \big\{v_h \in L^\infty(M_h) : v_h|_K \text{ constant for all } K \in \mathcal{T}_h\big\}.\end{aligned}$$

The nodal basis $\big(\varphi_z \colon z \in \mathcal{N}_h\big)$ of $\mathcal{S}^1(\mathcal{T}_h)$ consists of the hat functions $\varphi_z \in \mathcal{S}^1(\mathcal{T}_h)$ which satisfy $\varphi_z(z) = 1$ and $\varphi_z(z') = 0$ for all distinct $z, z' \in \mathcal{N}_h$. We set $\omega_z := \operatorname{supp}\varphi_z$ and $h_z := \operatorname{diam}(\omega_z)$ for all $z \in \mathcal{N}_h$.

For a function $\phi \in C\big(M_h\big)$ its nodal interpolant $\mathcal{I}_h\phi \in \mathcal{S}^1(\mathcal{T}_h)$ is defined by

$$\mathcal{I}_h\phi := \sum_{z\in\mathcal{N}_h} \phi(z)\varphi_z.$$

We also define an interpolation operator acting on continuous vector fields on $M_h$ by applying $\mathcal{I}_h$ to each component of the vector field. A routine application of the Bramble-Hilbert Lemma [Cia02, BS02] shows that there exists a constant $C > 0$ such that for every $\phi \in C\big(M_h\big)$ with $\phi|_K \in C^2(K)$ for all $K \in \mathcal{T}_h$ the interpolation error satisfies for each $K \in \mathcal{T}_h$

$$h_K^{-2}\big\|\phi - \mathcal{I}_h\phi\big\|_{L^2(K)} + h_K^{-1}\big\|\nabla_{M_h}\big(\phi - \mathcal{I}_h\phi\big)\big\|_{L^2(K)} \le C\big\|D^2_{M_h}\phi\|_{L^2(K)}. \tag{4.4}$$

In the following subsections we provide important tools used below.

### 1.4.1 Weakly acute triangulations

Some triangulations enable discrete monotonicity arguments. The following lemma defines such a class known as weakly acute triangulations [Cia02] introduced here on two-dimensional surfaces.

**Lemma 1.4.1.** *Suppose that $d = 2$ so that $\mathcal{T}_h$ consists of triangles. Assume that for all $K, K' \in \mathcal{T}_h$ such that $K \cap K' = E \in \mathcal{E}_h$ that the sum of the inner angles $\alpha$ and $\alpha'$ of $K$ and $K'$ opposite to $E$ satisfy $\alpha + \alpha' \le \pi$. Assume that for all $E \in \mathcal{E}_h$ and $K \in \mathcal{T}_h$ such that $E = K \cap \partial M_h$ the inner angle $\alpha$ of $K$ opposite to $E$ satisfies $\alpha \le \pi/2$. Then for all distinct $z, z' \in \mathcal{N}_h$ we have*

$$\mathbf{K}_{z,z'} := \int_{M_h} \nabla_{M_h}\varphi_z \cdot \nabla_{M_h}\varphi_{z'}\,\mathrm{d}s_h \le 0. \tag{4.5}$$

*Proof.* Since the intersection of the support of $\varphi_z$ and $\varphi_{z'}$ is non-trivial if and only if $z, z' \in E$ for some $E \in \mathcal{E}_h$ we may restrict to this case. Assume first that $E = K \cap K'$ for distinct $K, K' \in \mathcal{T}_h$. After appropriate translation, rotation, and dilation we may assume that $z = A$, $z' = B$, $K = \operatorname{conv}\{A, B, C\}$, and $K' = \operatorname{conv}\{A, B, D\}$ with $A = (0, 0, 0)$, $B = (1, 0, 0)$, $C = (\xi, -\eta, 0)$, and $D = (a, b, c)$ for $\xi, \eta, a, b, c \in \mathbb{R}$ such that $\xi > 0$ and if $c = 0$ then $b > 0$, cf. Figure 1.2.

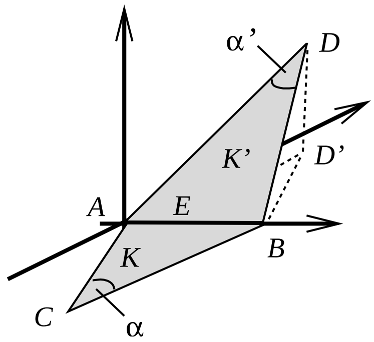


Figure 1.2: Two elements $K, K' \in \mathcal{T}_h$ sharing an inner edge $E = \operatorname{conv}\{z, z'\} \in \mathcal{E}_h$ where $z = A$ and $z' = B$.

One directly verifies that

$$\nabla_{M_h}\varphi_z|_K = \big[-1 \; (1-\xi)/\eta \; 0\big]^{\mathrm{T}}, \qquad \nabla_{M_h}\varphi_{z'}|_K = \big[1 \; \xi/\eta \; 0\big]^{\mathrm{T}}, \qquad \mathcal{H}^2(K) = \eta/2.$$

To compute the surface gradients on $K'$, assume that $b \neq 0$ (the case $b = 0$ is actually simpler, since then $\nabla_{M_h} = \big[\partial_1 \; 0 \; \partial_3\big]$, but leads to the same formula given below), set $D' := (a, b, 0)$, and define the tetrahedron $\widetilde{K}' := \operatorname{conv}\{A, B, D, D'\}$. Then, extending $\varphi_z$ and $\varphi_{z'}$ in such a way to functions $\varphi_z^e$ and $\varphi_{z'}^e$ that they are affine on $\widetilde{K}'$ and vanish at $D'$, we infer

$$\nabla\varphi_z^e|_{\widetilde{K}'} = \big[-1 \; (a-1)/b \; 0\big]^{\mathrm{T}}, \qquad \nabla\varphi_{z'}^e|_{\widetilde{K}'} = \big[1 \; -a/b \; 0\big]^{\mathrm{T}}.$$

Since the unit normal to $K'$ is (up to a sign) given by $\mu_h|_{K'} = \big[0 \; -c \; b\big]^{\mathrm{T}}/\delta$, where $\delta^2 := b^2 + c^2$, the definition of the surface gradient shows

$$\nabla_{M_h}\varphi_z|_{K'} = \big(\mathbf{I}_{3\times 3} - \mu_h \otimes \mu_h\big)\nabla\varphi_z^e = \frac{1}{\delta^2}\begin{bmatrix} \delta^2 & 0 & 0 \\ 0 & b^2 & bc \\ 0 & bc & c^2 \end{bmatrix}\begin{bmatrix} -1 \\ (a-1)/b \\ 0 \end{bmatrix} = \frac{1}{\delta^2}\begin{bmatrix} -\delta^2 \\ b(a-1) \\ c(a-1) \end{bmatrix}$$

and a similar calculation shows $\nabla_{M_h}\varphi_{z'}|_{K'} = \frac{1}{\delta^2}\big[\delta^2 \; -ab \; -ac\big]^{\mathrm{T}}$. Noting also that $\mathcal{H}^2(K') = \delta/2$ we find

$$\mathcal{H}^2(K)\nabla_{M_h}\varphi_z|_K \cdot \nabla_{M_h}\varphi_{z'}|_K = \frac{1}{2}\eta\big(-1 + \xi(1-\xi)/\eta^2\big),$$
$$\mathcal{H}^2(K')\nabla_{M_h}\varphi_z|_{K'} \cdot \nabla_{M_h}\varphi_{z'}|_{K'} = \frac{1}{2}\delta\big(-1 + a(1-a)/\delta^2\big).$$

The trigonometric identity $\cot\alpha = \cot(\alpha_1 + \alpha_2) = (\cot\alpha_1 \cot\alpha_2 - 1)/(\cot\alpha_1 + \cot\alpha_2)$ implies

$$\cot\alpha = \eta\big(1 - \xi(1-\xi)/\eta^2\big), \qquad \cot\alpha' = \delta\big(1 - a(1-a)/\delta^2\big)$$

so that we verify, upon combining the previous identities,

$$\int_{K\cup K'} \nabla_{M_h}\varphi_z \cdot \nabla_{M_h}\varphi_{z'}\,\mathrm{d}s_h = -\frac{1}{2}\big(\cot\alpha + \cot\alpha'\big).$$

Employing that $\cot\alpha+\cot\alpha' = \sin(\alpha+\alpha')/(\sin\alpha\sin\alpha')$ proves the assertion for pairs of nodes that are connected by an inner edge. The case of a boundary edge $E \subset \partial M_h$ follows from the above calculations. □

The previous lemma motivates the following definition.

**Definition 1.4.2.** *We say that a regular triangulation $\mathcal{T}_h$ is* weakly acute *if $\mathbf{K}_{z,z'} \le 0$ for all distinct $z, z' \in \mathcal{N}_h$.*

**Remarks 1.4.3.** *(i) By Lemma 1.4.1 a triangulation of a two-dimensional surface is weakly acute if every sum of angles opposite to an inner edge is bounded by $\pi$ and every angle opposite to an edge on the boundary is bounded by $\pi/2$. The proof of Lemma 1.4.1 shows that this characterization is sharp. For three-dimensional triangulations this is more involved, cf. [KK01, KK03]. A sufficient condition is that all angles between faces of tetrahedra are bounded by $\pi/2$.*
*(ii) A regular triangulation $\mathcal{T}_h$ consisting of triangles can always be refined in such a way that the resulting triangulation is weakly acute either by "edge-flipping" or performing a Delauney refinement, see [She02].*

**Lemma 1.4.4.** *Let $v_h \in \mathcal{S}^1(\mathcal{T}_h)$. Then, with $\mathbf{K}_{z,z'}$ as in the previous lemma (where $z = z' \in \mathcal{N}_h$ is allowed in* (4.5)*) we have*

$$\big\|\nabla_{M_h} v_h\big\|^2 = -\frac{1}{2}\sum_{z,z'\in\mathcal{N}_h} \mathbf{K}_{z,z'}|v_h(z) - v_h(z')|^2.$$

*Proof.* Notice that $\mathbf{K}_{z,z'} = \mathbf{K}_{z',z}$ and $\sum_{z'\in\mathcal{N}_h}\mathbf{K}_{z,z'} = 0$ owing to $\sum_{z'\in\mathcal{N}_h}\varphi_{z'} \equiv 1$. Hence,

$$\begin{aligned}
\big\|\nabla_{M_h} v_h\big\|^2 &= \sum_{z,z'\in\mathcal{N}_h} \mathbf{K}_{z,z'} v_h(z) v_h(z') \\
&= \sum_{z,z'\in\mathcal{N}_h} \mathbf{K}_{z,z'} v_h(z)\big(v_h(z') - v_h(z)\big) \\
&= \frac{1}{2}\sum_{z,z'\in\mathcal{N}_h} \mathbf{K}_{z,z'} v_h(z)\big(v_h(z') - v_h(z)\big) + \frac{1}{2}\sum_{z,z'\in\mathcal{N}_h} \mathbf{K}_{z,z'} v_h(z')\big(v_h(z) - v_h(z')\big) \\
&= -\frac{1}{2}\sum_{z,z'\in\mathcal{N}_h} \mathbf{K}_{z,z'}\big(v_h(z') - v_h(z)\big)^2.
\end{aligned}$$

This proves the statement. □

**Remark 1.4.5.** *More generally, the proof of the lemma shows that we have*

$$\big(\nabla_{M_h} v_h; \nabla_{M_h} w_h\big) = -\frac{1}{2}\sum_{z,z'\in\mathcal{N}_h} \mathbf{K}_{z,z'}\big(v_h(z) - v_h(z')\big)\big(w_h(z) - w_h(z')\big)$$

*for $v_h, w_h \in \mathcal{S}^1(\mathcal{T}_h)$.*

### 1.4.2 Discrete product rule

Below we will need the following discrete product rule for finite elements. We generalize the argumentation of [BN04] to finite elements on non–flat surfaces.

**Lemma 1.4.6.** *There exists a linear operator* $\mathbf{A} : \mathcal{S}^1(\mathcal{T}_h) \to \mathcal{L}^0(\mathcal{T}_h)^{(d+1)\times(d+1)}$ *such that for all* $v_h, w_h \in \mathcal{S}^1(\mathcal{T}_h)$ *the identity*

$$\nabla_{M_h}\mathcal{I}_h\big[v_h w_h\big] = \mathbf{A}\big(v_h\big)\nabla_{M_h} w_h + \mathbf{A}\big(w_h\big)\nabla_{M_h} v_h$$

*holds almost everywhere on* $M_h$*. For all* $K \in \mathcal{T}_h$ *we have*

$$\big\|\mathbf{A}(v_h) - v_h\mathbf{I}_{(d+1)\times(d+1)}\big\|_{L^\infty(K)} \le Ch_K\big\|\nabla v_h\big\|_{L^\infty(K)}.$$

*Proof.* Let $K \in \mathcal{T}_h$ and set $\widehat{v}_h := v_h \circ \mathcal{F}_K$, $\widehat{w}_h := w_h \circ \mathcal{F}_K$, and $\widehat{\mathcal{I}}_h\big[\widehat{v}_h\widehat{w}_h\big] := \mathcal{I}_h\big[v_h w_h\big] \circ \mathcal{F}_K$. Then, for $\gamma = 1, 2, ..., d$, the choice of $\widehat{K}$ (cf. Section 1.3) implies

$$\begin{aligned}\widehat{\partial}_\gamma\widehat{\mathcal{I}}_h\big[\widehat{v}_h\widehat{w}_h\big] &= \big(\widehat{v}_h(\widehat{z}_\gamma)\widehat{w}_h(\widehat{z}_\gamma) - \widehat{v}_h(\widehat{z}_0)\widehat{w}_h(\widehat{z}_0)\big)/h_K\\ &= \frac{1}{2}\big(\widehat{v}_h(\widehat{z}_0) + \widehat{v}_h(\widehat{z}_\gamma)\big)\big(\widehat{w}_h(\widehat{z}_\gamma) - \widehat{w}_h(\widehat{z}_0)\big)/h_K + \frac{1}{2}\big(\widehat{w}_h(\widehat{z}_\gamma) + \widehat{w}_h(\widehat{z}_0)\big)\big(\widehat{v}_h(\widehat{z}_\gamma) - \widehat{v}_h(\widehat{z}_0)\big)/h_K\\ &=: \widehat{\mathbf{A}}^{\gamma\gamma}\big(\widehat{v}_h\big)\widehat{\partial}_\gamma\widehat{w}_h + \widehat{\mathbf{A}}^{\gamma\gamma}\big(\widehat{w}_h\big)\widehat{\partial}_\gamma\widehat{v}_h.\end{aligned}$$

Here we used that $\widehat{K} = h_K \operatorname{conv}\{\widehat{z}_0, \widehat{z}_1, ..., \widehat{z}_d\}$, $\widehat{z}_0 = 0$, and that the vectors $\widehat{z}_1, ..., \widehat{z}_d$ coincide with the canonical basis of $\mathbb{R}^d$. Since

$$\nabla_{M_h}\phi_h = \mathbf{D}_h^{\mathrm{T}}\widehat{\nabla}\widehat{\phi}_h \qquad \text{and} \qquad \widehat{\nabla}\widehat{\phi}_h = \mathbf{G}_h^{\mathrm{T}}\nabla_{M_h}\phi_h$$

(cf. the proof of Lemma 1.3.5) for all $\phi_h \in \mathcal{S}^1(\mathcal{T}_h)$ and $\widehat{\phi}_h = \phi_h \circ \mathcal{F}_K$, the asserted identity follows from

$$\mathbf{A}(v_h) := \mathbf{D}_h^{\mathrm{T}}\widehat{\mathbf{A}}(\widehat{v}_h)\mathbf{G}_h^{\mathrm{T}} + v_h(\mu_h \otimes \mu_h)$$

for the diagonal matrix $\widehat{\mathbf{A}}(\widehat{v}_h) \in \mathbb{R}^{d\times d}$ with entries given by $\big(\widehat{v}_h(\widehat{z}_0) + \widehat{v}_h(\widehat{z}_\gamma)\big)/2$ for $\gamma = 1, 2, ..., d$ and for a unit normal $\mu_h$ of $K$. Arguing as in the proof of Lemma 1.3.5 we infer that

$$\mathbf{I}_{(d+1)\times(d+1)} - \mu_h \otimes \mu_h = \mathbf{G}_h\mathbf{D}_h = \big(\mathbf{G}_h\mathbf{D}_h\big)^{\mathrm{T}} = \mathbf{D}_h^{\mathrm{T}}\mathbf{G}_h^{\mathrm{T}}.$$

Therefore, we have

$$\begin{aligned}\big\|\mathbf{A}(v_h) - v_h\mathbf{I}_{(d+1)\times(d+1)}\big\|_{L^\infty(K)} &= \big\|\mathbf{A}(v_h) - v_h\mathbf{D}_h^{\mathrm{T}}\mathbf{G}_h^{\mathrm{T}} - v_h(\mu_h \otimes \mu_h)\big\|_{L^\infty(K)}\\ &= \big\|\mathbf{D}_h^{\mathrm{T}}\big(\widehat{\mathbf{A}}(\widehat{v}_h) - v_h\mathbf{I}_{d\times d}\big)\mathbf{G}_h^{\mathrm{T}}\big\|_{L^\infty(K)}.\end{aligned}$$

For $\gamma = 1, 2, ..., d$ the function

$$\widehat{x} \mapsto \widehat{\mathbf{A}}^{\gamma\gamma}\big(\widehat{v}_h\big) - \widehat{v}(\widehat{x})$$

vanishes at $\widehat{x} = \frac{1}{2}\big(\widehat{z}_\gamma + \widehat{z}_0\big)$. A discrete Poincaré estimate and a transformation argument thus imply the asserted estimate since $|\mathbf{G}_h|$, $|\mathbf{D}_h| \le C$ uniformly in $h$. $\square$

**Lemma 1.4.7.** *Suppose that $d = 2$ and let $\mathbf{A} : \mathcal{S}^1(\mathcal{T}_h) \to \mathcal{L}^0(\mathcal{T}_h)^{3\times 3}$ be as in the previous lemma. Then, for $v_h \in \mathcal{S}^1(\mathcal{T}_h)$ and $K \in \mathcal{T}_h$ we have*

$$\big\|\mathbf{A}\big(v_h\big) - \mathbf{A}^{\mathrm{T}}\big(v_h\big)\big\|_{L^\infty(K)} \leq C \min_{\gamma=1,2,3} |\cos\alpha_{K,\gamma}| \, \|v_h\|_{L^\infty(K)},$$

*where $\alpha_{K,\gamma}$, $\gamma = 1, 2, ..., d+1$ are the interior angles of the triangle $K$. In particular, $\mathbf{A}(v_h)|_K$ is symmetric if $K$ has a right angle.*

*Proof.* For $K = \mathrm{conv}\{z_0, z_1, z_2\}$ we can scale $\widehat{K}$ so that the transformation $\mathcal{F}_K \colon \widehat{K} \to K$ is given by $\mathcal{F}_K(\widehat{x}) = z_0 + \mathbf{G}_h\widehat{x}$ for $\widehat{x} \in \widehat{K}$ and the matrix $\mathbf{G}_h \in \mathbb{R}^{3\times 2}$ is given by

$$\mathbf{G}_h = \Big[h_{E_1}^{-1}\big(z_1 - z_0\big)\ h_{E_2}^{-1}\big(z_2 - z_0\big)\Big],$$

where $h_{E_\gamma} := \big|z_\gamma - z_0\big|$. Then, if as in the proof of Lemma 1.3.5, $\mu_h$ is a unit normal to $K$, $\mathbf{D}_h^e = \big[\mathbf{G}_h\ \mu_h\big]^{-1}$, and $\mathbf{D}_h$ is obtained by deleting the last row of $\mathbf{D}_h^e$, we have with

$$\mathbf{A}\big(v_h\big) = \mathbf{D}_h^{\mathrm{T}}\widehat{\mathbf{A}}\big(\widehat{v}_h\big)\mathbf{G}_h^{\mathrm{T}} + v_h\mu_h \otimes \mu_h$$

that

$$\begin{aligned}\mathbf{A}\big(v_h\big) - \mathbf{A}^{\mathrm{T}}\big(v_h\big) &= \mathbf{D}_h^{\mathrm{T}}\widehat{\mathbf{A}}\big(\widehat{v}_h\big)\mathbf{G}_h^{\mathrm{T}} - \mathbf{G}_h\widehat{\mathbf{A}}\big(\widehat{v}_h\big)\mathbf{D}_h \\ &= \big(\mathbf{D}_h^{\mathrm{T}} - \mathbf{G}_h\big)\widehat{\mathbf{A}}\big(\widehat{v}_h\big)\mathbf{G}_h^{\mathrm{T}} + \mathbf{G}_h\widehat{\mathbf{A}}\big(\widehat{v}_h\big)\big(\mathbf{G}_h^{\mathrm{T}} - \mathbf{D}_h\big).\end{aligned}$$

We then notice, using the definition of $\mathbf{D}_h^e$ and realizing that it is uniformly bounded, that

$$\begin{aligned}\big|\mathbf{D}_h - \mathbf{G}_h^{\mathrm{T}}\big| &\leq C\big|\mathbf{D}_h^e - \big[\mathbf{G}_h\ \mu_h\big]^{\mathrm{T}}\big| \\ &= C\big|\mathbf{D}_h^e\big(\mathbf{I}_{(d+1)\times(d+1)} - \big[\mathbf{G}_h\ \mu_h\big]\big[\mathbf{G}_h\ \mu_h\big]^{\mathrm{T}}\big)\big| \\ &\leq C\big|\mathbf{I}_{(d+1)\times(d+1)} - \big[\mathbf{G}_h\ \mu_h\big]\big[\mathbf{G}_h\ \mu_h\big]^{\mathrm{T}}\big|.\end{aligned}$$

From the definition of $\mathbf{G}_h$ and the fact that $\mu_h$ is orthogonal to $K$ we find

$$\big[\mathbf{G}_h\ \mu_h\big]\big[\mathbf{G}_h\ \mu_h\big]^{\mathrm{T}} = \begin{bmatrix} 1 & \cos\alpha_K & 0 \\ \cos\alpha_K & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\alpha_K$ is the inner angle of $K$ between the vectors $z_1 - z_0$ and $z_2 - z_0$. Since we can interchange the role of the nodes we verify the assertion. ☐

**Definition 1.4.8.** *A sequence of triangulations $\big(\mathcal{T}_h\big)_{h>0}$ consisting of triangles is called* logarithmically right-angled *if*

$$\lim_{h\to 0} \log h_{min}^{-1} \sup_{K\in\mathcal{T}_h} \inf_{\gamma=1,2,3} |\cos\alpha_{K,\gamma}| = 0,$$

*where $\alpha_{K,\gamma}$, $\gamma = 1, 2, 3$, are the interior angles of the triangle $K \in \mathcal{T}_h$.*

### 1.4.3 Inverse inequalities and reduced integration

We will frequently employ the following inverse inequalities.

**Lemma 1.4.9.** *Suppose that $d \geq 2$ and $h_{min} \leq 1$. For $v_h \in \mathcal{S}^1(\mathcal{T}_h)$, a $\mathcal{T}_h$–elementwise polynomial function $\psi_h \in L^p(M_h)$, and $1 \leq p \leq r \leq \infty$ we have*

$$\big\|\nabla_{M_h} v_h\big\|_{L^p(K)} \leq C h_K^{-1} \|v_h\|_{L^p(K)}$$

*and*

$$\|\psi_h\|_{L^r(M_h)} \leq C h_{min}^{-d(1/p-1/r)} \|\psi_h\|_{L^p(M_h)}$$

*and*

$$\|v_h\|_{L^\infty(M_h)} \leq C h_{min}^{1-d/2} \log h_{min}^{-1} \big\|\nabla_{M_h} v_h\big\|_{L^2(M_h)},$$

*where $C$ depends on the polynomial degree of $\psi_h$.*

*Proof.* The first estimate follows directly from a compactness and a scaling argument. For the proof of the second estimate we also employ a local scaling argument and the fact that $\|\psi_h\|_{L^p(K)} \leq \|\psi_h\|_{L^p(M_h)}$ for $K \subset M_h$ to verify that if $r < \infty$ we have

$$\begin{aligned}\|\psi_h\|_{L^r(M_h)}^r &= \sum_{K\in\mathcal{T}_h} \|\psi_h\|_{L^r(K)}^r \leq C \sum_{K\in\mathcal{T}_h} h_K^{-rd(1/p-1/r)} \|\psi_h\|_{L^p(K)}^r \\ &\leq C h_{min}^{-rd(1/p-1/r)} \max_{K\in\mathcal{T}_h} \|\psi_h\|_{L^p(K)}^{r-p} \sum_{K\in\mathcal{T}_h} \|\psi_h\|_{L^p(K)}^p \leq C h_{min}^{-rd(1/p-1/r)} \|\psi_h\|_{L^p(M_h)}^r.\end{aligned}$$

The case $r = \infty$ follows from obvious modifications. The Sobolev estimate of Theorem 1.2.1 guarantees that for $1 \leq q < d$ and $p = dq/(d-q)$ we have

$$\|v_h\|_{L^p(M_h)} \leq C\|\widetilde{v}_h\|_{L^p(M)} \leq C(d-q)^{-1}\|\nabla_M \widetilde{v}_h\|_{L^q(M)} \leq C(d-q)^{-1}\|\nabla_{M_h} v_h\|_{L^d(M_h)}$$

where we also used stability properties of the lifting operator provided by Lemma 1.3.7. For $q = d - \big|\log h_{min}\big|^{-1}$ we have $p = |\log h_{min}| dq$ and by the second estimate of the lemma with $r = \infty$

$$\|v_h\|_{L^\infty(M_h)} \leq C h_{min}^{-(|\log h_{min}|q)^{-1}} \|v_h\|_{L^p(M_h)} \leq C h_{min}^{-|\log h_{min}|^{-1}} \|v_h\|_{L^p(M_h)},$$

where we used $q \geq 1$ in the last inequality. We notice that if $h_{min} \leq 1$ then $\log h_{min} = -|\log h_{min}|$ and hence

$$h_{min}^{-|\log h_{min}|^{-1}} = \exp\big(|\log h_{min}|^{-1}|\log h_{min}|\big) = \exp(1).$$

The combination of the previous estimates with the inverse estimate

$$\big\|\nabla_{M_h} v_h\big\|_{L^d(M_h)} \leq C h_{min}^{1-d/2} \big\|\nabla_{M_h} v_h\big\|_{L^2(M_h)}$$

as above proves the third estimate and completes the proof of the lemma. □

The following definition introduces reduced or numerical integration which enables control over nodal values of finite element functions.

**Definition 1.4.10.** *For functions or vector fields* $\phi, \psi \in C\big(M_h;\mathbb{R}^\ell\big)$*,* $\ell \in \mathbb{N}$*, we define the* discrete inner product *on* $\mathcal{T}_h$ *by setting*

$$\big(\phi;\psi\big)_h := \int_{M_h} \mathcal{I}_h\big[\phi\cdot\psi\big]\,\mathrm{d}s_h = \sum_{z\in\mathcal{N}_h} \beta_z \phi(z)\cdot\psi(z),$$

*where* $\beta_z := \int_{M_h} \varphi_z\,\mathrm{d}s_h$*. We also define*

$$\|\phi\|_h := \big(\phi;\phi\big)_h^{1/2}.$$

One directly shows that the estimates

$$\|\phi_h\| \le \|\phi_h\|_h \le C\|\phi_h\|$$

are satisfied for all $\phi_h \in \mathcal{S}^1(\mathcal{T}_h)$. Results on nodal interpolation imply that

$$\big|\big(\phi_h;\psi_h\big)_h - \big(\phi_h;\psi_h\big)\big| \le Ch\,\|\phi_h\|\,\|\nabla_{M_h}\psi_h\|$$

for all $\phi_h, \psi_h \in \mathcal{S}^1(\mathcal{T}_h)^\ell$. We remark that owing to the assumed minimum angle condition we have for $\phi_h \in \mathcal{S}^1(\mathcal{T}_h)^\ell$ and $1 \le p < \infty$ that

$$C^{-1}\|\phi_h\|_{L^p(M_h)}^p \le \sum_{z\in\mathcal{N}_h} h_z^d|\phi_h(z)|^p \le C\|\phi_h\|_{L^p(M_h)}^p. \tag{4.6}$$

For proofs of the estimates we refer to [CT85, GR92].

**Remark 1.4.11.** *Reduced integration has a stabilizing effect immediately illustrated in one space dimension [BCPP04]: For a partition of the interval* $(0,1)$ *into intervals* $K_j = [z_{j-1}, z_j]$ *of length* $h_j := z_j - z_{j-1}$ *for* $j = 1,\dots,J$ *and* $0 = z_0 < z_1 < \dots < z_J = 1$ *we have*

$$\big(v_h;w_h\big)_h = \big(v_h;w_h\big) + \frac{1}{6}\int_{(0,1)} h_{\mathcal{T}_h}^2 v_h' w_h'\,\mathrm{d}x,$$

*for all* $v_h, w_h \in \mathcal{S}^1(\mathcal{T}_h)$ *where* $h_{\mathcal{T}_h}|_{K_j} = h_j$*.*

## 1.5 Decomposition of discrete tangential vector fields

As a technical tool we will need to decompose discrete tangential vector fields on surfaces into a (discrete) divergence-free and a rotation-free part. This requires the use of non-standard, non-conforming finite element spaces. Related results are given in [AFW97] but since the author is unaware of a reference for the assertions in a periodic or non-flat setting required here, short proofs of the results are included.

### 1.5.1 Discrete Helmholtz decomposition on the two-dimensional torus

We assume that $d = 2$ and consider a flat, periodic setting first, i.e., we let

$$M = M_h = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2. \tag{5.7}$$

In this case we write $\nabla_M = \nabla = \big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\big) = (\partial_1, \partial_2)$ and we use $\nabla_{M_h}$ to denote the elementwise application of $\nabla$ to an elementwise differentiable function. Moreover, we will consider $[0,1]^2$ as the fundamental domain of $\mathbb{T}^2$.

**Definition 1.5.1.** *A* triangulation $\mathcal{T}_h$ of $\mathbb{T}^2$ *is a regular triangulation of* $[0,1]^2$ *such that for each node* $z \in [0,1]\times\{0\} \cup \{0\}\times[0,1]$ *there exists a node* $z' \in [0,1]\times\{1\} \cup \{1\}\times[0,1]$ *satisfying either* $z = z' + (1,0)$ *or* $z = z' + (0,1)$*. Such two nodes are identified and we set*

$$\mathcal{S}^1_{\#}(\mathcal{T}_h) := \big\{ v_h \in \mathcal{S}^1(\mathcal{T}_h) : v_h|_{(0,1)\times\{0\}} = v_h|_{(0,1)\times\{1\}} \text{ and } v_h|_{\{0\}\times(0,1)} = v_h|_{\{1\}\times(0,1)} \big\}.$$

*Two edges* $E, E' \in \mathcal{E}_h$ *such that* $E = E' + (1,0)$ *or* $E = E' + (0,1)$ *are identified and all edges* $E \in \mathcal{E}_h$ *are called* interior edges*.*

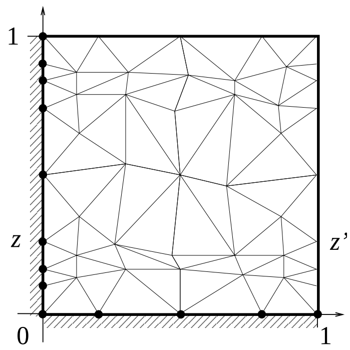


Figure 1.3: Example of an admissible triangulation of $\mathbb{T}^2$. Nodes and edges along the shaded sides can be eliminated via identification with nodes and edges on the opposite sides.

In the following we assume that $\mathcal{T}_h$ is a regular triangulation of $\mathbb{T}^2$ in the sense of the previous definition.

**Definition 1.5.2.** *(i) For* $v \in C^1(\mathbb{T}^2;\mathbb{R})$ *the* vectorial curl *of* $v$*, denoted* $\operatorname{Curl} v$*, is defined by*

$$\operatorname{Curl} v := \big(-\partial_2 v, \partial_1 v\big)^{\mathrm{T}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla v.$$

*(ii) For* $\psi = \big(\psi_1, \psi_2\big) \in C^1(\mathbb{T}^2;\mathbb{R}^2)$ *the* scalar curl *of* $\psi$*, denoted* $\operatorname{curl}\psi$*, is defined by*

$$\operatorname{curl}\psi := \partial_1\psi_2 - \partial_2\psi_1 = \operatorname{div} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{\mathrm{T}} \psi.$$

*(iii) For* $w \in L^1(\mathbb{T}^2;\mathbb{R})$ *such that* $w|_K \in C^1(K)$ *for all* $K \in \mathcal{T}_h$ *the* $\mathcal{T}_h$-elementwise vectorial curl *of* $w$*, denoted* $\operatorname{Curl}_{\mathcal{T}_h} w$*, is defined by elementwise application of* Curl*, i.e., for each* $K \in \mathcal{T}_h$ *we have*

$$\big(\operatorname{Curl}_{\mathcal{T}_h} w\big)|_K := \operatorname{Curl} w|_K.$$

The finite element space introduced in the following definition is known as the (non-conforming) Crouzeix-Raviart finite element space, cf. [CR73].

**Definition 1.5.3.** *For each edge* $E \in \mathcal{E}_h$ *let* $z_E$ *denote the midpoint of* $E$*. Then, define*

$$\mathcal{S}^{1,NC}_{\#}(\mathcal{T}_h) := \big\{ v_h \in L^\infty(\mathbb{T}^2) \colon v_h|_K \text{ is affine for all } K \in \mathcal{T}_h \text{ and } v_h \text{ is continuous at } z_E \text{ for all } E \in \mathcal{E}_h \big\}.$$

**Lemma 1.5.4.** *Given* $a_h \in \mathcal{S}^1_\#(\mathcal{T}_h)$ *and* $b_h \in \mathcal{S}^{1,NC}_\#(\mathcal{T}_h)$ *we have*

$$\big(\nabla a_h; \mathrm{Curl}_{\mathcal{T}_h} b_h\big) = 0.$$

*Proof.* A $\mathcal{T}_h$-elementwise application of Stokes' theorem, the identity $\mathrm{curl}\,\nabla a_h \equiv 0$, and the fact that the tangential component of $\nabla a_h$ is continuous across $E$ for each $E \in \mathcal{E}_h$ show

$$\begin{aligned}\big(\nabla a_h; \mathrm{Curl}_{\mathcal{T}_h} b_h\big) &= \sum_{K\in\mathcal{T}_h} \Big\{ \int_K \big(\mathrm{curl}\,\nabla a_h\big) b_h \,\mathrm{d}x + \int_{\partial K} \big(\nabla a_h \cdot \tau_K\big) b_h \,\mathrm{d}t \Big\} \\ &= \sum_{E\in\mathcal{E}_h} \int_E \big(\nabla a_h \cdot \tau_E\big) \big[b_h\big] \,\mathrm{d}t,\end{aligned}$$

where $\tau_K, \tau_E$ are unit tangent vectors to $\partial K$ for $K \in \mathcal{T}_h$ and to $E \in \mathcal{E}_h$, respectively, and $\big[b_h\big]$ denotes the jump of $b_h$ across an edge $E \in \mathcal{E}_h$. Since $b_h$ is continuous at $z_E$ we have $\int_E \big[b_h\big] \,\mathrm{d}t = 0$ for each $E \in \mathcal{E}_h$. Upon noting that $\nabla a_h \cdot \tau_E$ is constant on each edge $E \in \mathcal{E}_h$ we verify the assertion of the lemma. □

**Definition 1.5.5.** *The space of* discrete harmonic fields *is defined by*

$$\mathcal{H}_\#(\mathcal{T}_h;\mathbb{R}^2) := \Big\{ H_h \in \mathcal{L}^0(\mathcal{T}_h)^2 : \big(H_h; \mathrm{Curl}_{\mathcal{T}_h} w_h\big) = 0 \textit{ for all } w_h \in \mathcal{S}^{1,NC}_\#(\mathcal{T}_h) \textit{ and } \big(H_h; \nabla v_h\big) = 0 \textit{ for all } v_h \in \mathcal{S}^1_\#(\mathcal{T}_h) \Big\}.$$

**Lemma 1.5.6.** *It holds* $\dim \mathcal{H}_\#(\mathcal{T}_h;\mathbb{R}^2) = 2$.

*Proof.* In this proof we do not identify edges or nodes that could be eliminated through periodicity. Obviously, the dimension of $\mathcal{L}^0(\mathcal{T}_h)^2$ equals $2\,\mathrm{card}\,\mathcal{T}_h$. We let $\mathcal{E}_{h,\#}$ and $\mathcal{N}_{h,\#}$ be the sets containing the edges and nodes on the sides $[0,1]\times\{0\}$ and $\{0\}\times[0,1]$ that can be eliminated via identification with edges and nodes on the opposite side of $(0,1)^2$ according to periodicity. Since each edge belongs to two nodes we have that

$$\mathrm{card}\,\mathcal{E}_{h,\#} = \mathrm{card}\,\mathcal{N}_{h,\#} - 1.$$

Moreover, we have

$$\dim \mathcal{S}^1_\#(\mathcal{T}_h) = \mathrm{card}\,\mathcal{N}_h - \mathrm{card}\,\mathcal{N}_{h,\#}$$

and

$$\dim \mathcal{S}^{1,NC}_\#(\mathcal{T}_h) = \mathrm{card}\,\mathcal{E}_h - \mathrm{card}\,\mathcal{E}_{h,\#}.$$

The condition that

$$\big(H_h; \nabla v_h\big) = 0$$

is satisfied for all $v_h \in \mathcal{S}^1_\#(\mathcal{T}_h)$ imposes $\dim \mathcal{S}^1_\#(\mathcal{T}_h) - 1$ linearly independent conditions on a function $H_h \in \mathcal{L}^0(\mathcal{T}_h)^2$. Indeed, suppose for a contradiction that there exists $v_h^* \in \mathcal{S}^1_\#(\mathcal{T}_h)$ with

$$\big(\psi_h; \nabla v_h^*\big) = 0$$

for all $\psi_h \in \mathcal{L}^0(\mathcal{T}_h)^2$. In particular, for $\psi_h = \nabla v_h^*$ it follows $\big\|\nabla v_h^*\big\| = 0$, which implies that $v_h^*$ is constant and thus the assertion on the number of linearly independent conditions. Similarly, using that $\mathrm{Curl}_{\mathcal{T}_h} w_h \in \mathcal{L}^0(\mathcal{T}_h)^2$ for all $w_h \in \mathcal{S}^{1,NC}_\#(\mathcal{T}_h)$, we find that the condition

$$\big(H_h; \mathrm{Curl}_{\mathcal{T}_h} w_h\big) = 0$$

for all $w_h \in \mathcal{S}^{1,NC}_{\#}(\mathcal{T}_h)$ imposes $\dim \mathcal{S}^{1,NC}_{\#}(\mathcal{T}_h) - 1$ linearly independent conditions. It remains to show that the two conditions are mutually linearly independent. If the second condition depends on the first one then there exists $w_h^* \in \mathcal{S}^{1,NC}_{\#}(\mathcal{T}_h)$ such that

$$\big(\psi_h; \operatorname{Curl}_{\mathcal{T}_h} w_h^*\big) = 0$$

is satisfied for all $\psi_h \in \mathcal{L}^0(\mathcal{T}_h)^2$ with $\big(\psi_h; \nabla v_h\big) = 0$ for all $v_h \in \mathcal{S}^1_{\#}(\mathcal{T}_h)$. Since $\operatorname{Curl}_{\mathcal{T}_h} w_h^*$ satisfies $\big(\operatorname{Curl}_{\mathcal{T}_h} w_h^*; \nabla v_h\big) = 0$ for all $v_h \in \mathcal{S}^1_{\#}(\mathcal{T}_h)$ we deduce that $w_h^*$ is constant and this implies the linear independence. The converse statement is analogous.

On combining the above identities and using Euler's identity

$$\operatorname{card} \mathcal{N}_h - \operatorname{card} \mathcal{E}_h + \operatorname{card} \mathcal{T}_h = 1$$

we find that

$$\begin{aligned}
\dim \mathcal{H}_{\#}(\mathcal{T}_h; \mathbb{R}^2) &= 2 \operatorname{card} \mathcal{T}_h - \big(\dim \mathcal{S}^1_{\#}(\mathcal{T}_h) - 1\big) - \big(\dim \mathcal{S}^{1,NC}_{\#}(\mathcal{T}_h) - 1\big) \\
&= 2 \operatorname{card} \mathcal{T}_h - \operatorname{card} \mathcal{N}_h + \operatorname{card} \mathcal{N}_{h,\#} - \operatorname{card} \mathcal{E}_h + \operatorname{card} \mathcal{E}_{h,\#} + 2 \\
&= 2 \operatorname{card} \mathcal{T}_h - \operatorname{card} \mathcal{N}_h - \operatorname{card} \mathcal{E}_h + 2 \operatorname{card} \mathcal{E}_{h,\#} + 3 \\
&= 3 \operatorname{card} \mathcal{T}_h - 2 \operatorname{card} \mathcal{E}_h + 2 \operatorname{card} \mathcal{E}_{h,\#} + 2 \\
&= 2.
\end{aligned}$$

Here we used

$$3 \operatorname{card} \mathcal{T}_h - \big(\operatorname{card} \mathcal{E}_h - \operatorname{card} \mathcal{E}_{h,\#}\big) = \operatorname{card} \mathcal{E}_h - \operatorname{card} \mathcal{E}_{h,\#}$$

in the last identity. For a proof of the latter equation notice that both sides equal the dimension of $\mathcal{S}^{1,NC}_{\#}(\mathcal{T}_h)$. ☐

With the preparations of the previous lemmas we are able to provide the following orthogonal decomposition of two-dimensional, $\mathcal{T}_h$-elementwise constant, periodic vector fields.

**Proposition 1.5.7.** *Let $\omega_h \in \mathcal{L}^0(\mathcal{T}_h)^2$. Then there exist uniquely defined $a_h \in \mathcal{S}^1_{\#}(\mathcal{T}_h)$, $b_h \in \mathcal{S}^{1,NC}_{\#}(\mathcal{T}_h)$, and $H_h \in \mathcal{H}_{\#}(\mathcal{T}_h; \mathbb{R}^2)$ such that $\int_{M_h} a_h \,\mathrm{d}x = 0$, $\int_{M_h} b_h \,\mathrm{d}x = 0$, and*

$$\omega_h = \nabla a_h + \operatorname{Curl}_{\mathcal{T}_h} b_h + H_h.$$

*Moreover,*

$$\|\omega_h\|^2 = \big\|\nabla a_h\big\|^2 + \big\|\operatorname{Curl}_{\mathcal{T}_h} b_h\big\|^2 + \big\|H_h\big\|^2$$

*and*

$$\big(\omega_h; \nabla v_h\big) = \big(\nabla a_h; \nabla v_h\big)$$

*for all $v_h \in \mathcal{S}^1_{\#}(\mathcal{T}_h)$.*

*Proof.* Let $a_h \in \mathcal{S}^1_{\#}(\mathcal{T}_h)$ be the uniquely defined function that satisfies $\int_{M_h} a_h \,\mathrm{d}x = 0$ and

$$\big(\nabla a_h; \nabla v_h\big) = \big(\omega_h; \nabla v_h\big)$$

for all $v_h \in \mathcal{S}^1_{\#}(\mathcal{T}_h)$. Let $b_h$ be the uniquely defined function in $\mathcal{S}^{1,NC}_{\#}(\mathcal{T}_h)$ such that $\int_{M_h} b_h \,\mathrm{d}x = 0$ and

$$\big(\operatorname{Curl}_{\mathcal{T}_h} b_h; \operatorname{Curl}_{\mathcal{T}_h} w_h\big) = \big(\omega_h; \operatorname{Curl}_{\mathcal{T}_h} w_h\big)$$

for all $w_h \in \mathcal{S}^{1,NC}_{\#}(\mathcal{T}_h)$. Define $H_h := \omega_h - \nabla a_h - \operatorname{Curl}_{\mathcal{T}_h} b_h$. Then, using Lemma 1.5.4 and the definition of $b_h$ we find

$$\big(H_h; \operatorname{Curl}_{\mathcal{T}_h} w_h\big) = \big(\omega_h - \operatorname{Curl}_{\mathcal{T}_h} b_h; \operatorname{Curl}_{\mathcal{T}_h} w_h\big) - \big(\nabla a_h; \operatorname{Curl}_{\mathcal{T}_h} w_h\big) = 0$$

for all $w_h \in \mathcal{S}^{1,NC}_{\#}(\mathcal{T}_h)$. Similarly, we deduce that

$$\big(H_h; \nabla v_h\big) = \big(\omega_h - \nabla a_h; \nabla v_h\big) - \big(\operatorname{Curl}_{\mathcal{T}_h} b_h; \nabla v_h\big) = 0$$

for all $v_h \in \mathcal{S}^1_{\#}(\mathcal{T}_h)$. This proves that $H_h \in \mathcal{H}_{\#}(\mathcal{T}_h;\mathbb{R}^2)$. With the above identities we verify that

$$\begin{aligned}\|\omega_h\|^2 &= \big(\omega_h; \nabla a_h\big) + \big(\omega_h; \operatorname{Curl}_{\mathcal{T}_h} b_h\big) + \big(\omega_h; H_h\big)\\ &= \big\|\nabla a_h\big\|^2 + \big\|\operatorname{Curl}_{\mathcal{T}_h} b_h\big\|^2 + \big\|H_h\big\|^2,\end{aligned}$$

which finishes the proof of the lemma. ☐

A so-called averaging or recovery operator $\mathcal{A}_h : L^1(\mathbb{T}^2) \to \mathcal{S}^1_{\#}(\mathcal{T}_h)$ is defined in the following lemma and allows to approximate a discontinuous function $\varrho_h \in L^1(\mathbb{T}^2)$ by a continuous function $\mathcal{A}_h \varrho_h \in \mathcal{S}^1_{\#}(\mathcal{T}_h)$. We refer to [Car99, CB02] for related assertions and estimates.

**Lemma 1.5.8.** *Given $f \in L^1(\mathbb{T}^2)$ define $\mathcal{A}_h f \in \mathcal{S}^1_{\#}(\mathcal{T}_h)$ by*

$$\mathcal{A}_h f := \sum_{z \in \mathcal{N}_h} f_z \varphi_z, \qquad f_z := \mathcal{H}^2(\omega_z)^{-1} \int_{\omega_z} f \,\mathrm{d}x.$$

*There exists $C > 0$ such that for all $w_h \in \mathcal{S}^{1,NC}_{\#}(\mathcal{T}_h)$ we have*

$$\big\|w_h - \mathcal{A}_h w_h\big\|_{L^2(\omega_z)} \le C h_z \big\|\nabla_{M_h} w_h\big\|_{L^2(\widehat{\omega}_z)}$$

*for all $z \in \mathcal{N}_h$ and $\widehat{\omega}_z := \cup_{y \in \mathcal{N}_h \cap \overline{\omega}_z} \omega_y$, and*

$$\big\|h_{\mathcal{T}_h}^{-1}\big(w_h - \mathcal{A}_h w_h\big)\big\| \le C \big\|\nabla_{M_h} w_h\big\|,$$

*where $h_{\mathcal{T}_h} \in L^\infty(M_h)$ satisfies $h_{\mathcal{T}_h}|_K = h_K$ for all $K \in \mathcal{T}_h$. Here, $\nabla_{M_h} w_h$ is defined elementwise.*

*Proof.* The first estimate follows from a compactness argument: Let $z \in \mathcal{N}_h$ and suppose that $w_h \in \mathcal{S}^{1,NC}_{\#}(\mathcal{T}_h)$ is such that $\nabla_{M_h} w_h|_{\widehat{\omega}_z} = 0$. Then $w_h$ is constant on $\widehat{\omega}_z$. By definition of $\widehat{\omega}_z$ we deduce that $\mathcal{A}_h w_h$ is constant on the smaller set $\omega_z$ and in particular

$$\big\|w_h - \mathcal{A}_h w_h\big\|_{L^2(\omega_z)} = 0.$$

A compactness and a scaling argument then provide the first estimate. The second estimate is an immediate consequence of the first one upon noting that the coverings $\big(\omega_z \colon z \in \mathcal{N}_h\big)$ and $\big(\widehat{\omega}_z \colon z \in \mathcal{N}_h\big)$ have finite overlaps and that $C^{-1} h_K \le h_z \le C h_K$ for $K \in \mathcal{T}_h$ and $z \in \mathcal{N}_h$ such that $K \subset \overline{\omega}_z$. ☐

### 1.5.2 Decomposition of tangential vector fields on triangulated two-dimensional hypersurfaces

We briefly indicate necessary changes in the above discussion to define a discrete Helmholtz decomposition on triangulated hypersurfaces without boundary. Throughout this subsection $\mathcal{T}_h$ denotes a triangulation that defines the hypersurface $M_h$ which serves as an approximation of the submanifold $M$ as in Section 1.3.

**Definition 1.5.9.** *For each edge $E \in \mathcal{E}_h$ let $z_E$ denote the midpoint of $E$ and define*

$$\mathcal{S}^{1,NC}(\mathcal{T}_h) := \big\{ v_h \in L^\infty(M_h) \colon v_h|_K \text{ is affine for all } K \in \mathcal{T}_h \\ \text{and } v_h \text{ is continuous at } z_E \text{ for all } E \in \mathcal{E}_h \big\}.$$

*For $w_h \in \mathcal{S}^{1,NC}(\mathcal{T}_h)$ we define the $\mathcal{T}_h$–elementwise constant, tangential vector field* $\operatorname{Curl}_{M_h} w_h$ *on $M_h$ by*

$$\operatorname{Curl}_{M_h} w_h|_K := \mu_h|_K \times \nabla_{M_h} w_h|_K,$$

*for each $K \in \mathcal{T}_h$ and where $\mu_h$ is a unit normal to $M_h$ defined on each $K \in \mathcal{T}_h$. Moreover, we define the set of* discrete harmonic fields *on $M_h$ by*

$$\mathcal{H}(\mathcal{T}_h;\mathbb{R}^3) := \Big\{ H_h \in \mathcal{L}^0(\mathcal{T}_h)^3 \colon H_h \cdot \mu_h = 0 \text{ almost everywhere on } M_h, \\ \big(H_h; \operatorname{Curl}_{M_h} w_h\big) = 0 \text{ for all } w_h \in \mathcal{S}^{1,NC}(\mathcal{T}_h), \\ \big(H_h; \nabla_{M_h} v_h\big) = 0 \text{ for all } v_h \in \mathcal{S}^1(\mathcal{T}_h) \Big\}.$$

As in the flat situation we have the following orthogonality.

**Lemma 1.5.10.** *For $a_h \in \mathcal{S}^1(\mathcal{T}_h)$ and $b_h \in \mathcal{S}^{1,NC}(\mathcal{T}_h)$ we have*

$$\big(\nabla_{M_h} a_h; \operatorname{Curl}_{M_h} b_h\big) = 0.$$

*Proof.* The identity follows from a $\mathcal{T}_h$-elementwise integration by parts on $M_h$; we refer the reader to [DDE05, CDD⁺04, GT01] for details on integration by parts on surfaces with boundary. Using that $\nabla_{M_h} a_h$ and $\operatorname{Curl}_{M_h} b_h$ are tangential vector fields and that the derivative of $a_h$ along an edge $E \in \mathcal{E}_h$ is continuous we have

$$\begin{aligned}
\big(\nabla_{M_h} a_h; \operatorname{Curl}_{M_h} b_h\big) &= -\big(\mu_h \times \nabla_{M_h} a_h; \nabla_{M_h} b_h\big) \\
&= \sum_{K \in \mathcal{T}_h} \int_K \operatorname{div}_{M_h}\big(\mu_h \times \nabla_{M_h} a_h\big) b_h \,\mathrm{d}s_h - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \big(\mu_h \times \nabla_{M_h} a_h\big) \cdot \mu_h^{co} b_h \,\mathrm{d}t \\
&= \sum_{K \in \mathcal{T}_h} \int_{\partial K} \Big(\nabla_{M_h} a_h \cdot \big(\mu_h \times \mu_h^{co}\big)\Big) b_h \,\mathrm{d}t \\
&= \sum_{K \in \mathcal{T}_h} \int_{\partial K} \big(\nabla_{M_h} a_h \cdot \tau_K\big) b_h \,\mathrm{d}t \\
&= \sum_{E \in \mathcal{E}_h} \int_E \big(\nabla_{M_h} a_h \cdot \tau_E\big) \big[b_h\big] \,\mathrm{d}t,
\end{aligned}$$

where $\mu_h^{co} = \tau_K \times \mu_h|_K$ denotes the co-normal to $K$ on $\partial K$ and $\big[b_h\big]$ the jump of $b_h$ across an edge $E \in \mathcal{E}_h$. Since $\big[b_h\big]$ has vanishing integral mean we verify the assertion. □

The dimension of the discrete harmonic fields on $M_h$ depends on the topology of $M_h$ through the Euler characteristic of $M_h$. In the following proof it is important to notice that $\|\nabla_{M_h} a_h\| = 0$ and $\|\mathrm{Curl}_{M_h} b_h\| = 0$ for $a_h \in \mathcal{S}^1(\mathcal{T}_h)$ and $b_h \in \mathcal{S}^{1,NC}(\mathcal{T}_h)$ imply that $a_h$ and $b_h$ are constant on $M_h$.

**Lemma 1.5.11.** *Suppose that $M$ and $M_h$ are topologically equivalent in the sense that their Euler characteristics $X_M$ and $X_{M_h}$ coincide. Then, the dimension of $\mathcal{H}(\mathcal{T}_h; \mathbb{R}^3)$ equals $2 - X_M$.*

*Proof.* The proof follows the lines of the proof of Lemma 1.5.6 with the difference that

$$\dim \mathcal{S}^1(\mathcal{T}_h) = \mathrm{card}\, \mathcal{N}_h \qquad \text{and} \qquad \dim \mathcal{S}^{1,NC}(\mathcal{T}_h) = \mathrm{card}\, \mathcal{E}_h$$

and that Euler's identity reads [Sta86]

$$\mathrm{card}\, \mathcal{N}_h - \mathrm{card}\, \mathcal{E}_h + \mathrm{card}\, \mathcal{T}_h = X_{M_h} = X_M$$

Therefore,

$$\begin{aligned}
\dim \mathcal{H}(\mathcal{T}_h; \mathbb{R}^2) &= 2\, \mathrm{card}\, \mathcal{T}_h - \big(\dim \mathcal{S}^1(\mathcal{T}_h) - 1\big) - \big(\dim \mathcal{S}^{1,NC}(\mathcal{T}_h) - 1\big) \\
&= 2\, \mathrm{card}\, \mathcal{T}_h - \mathrm{card}\, \mathcal{N}_h - \mathrm{card}\, \mathcal{E}_h + 2 \\
&= 3\, \mathrm{card}\, \mathcal{T}_h - 2\, \mathrm{card}\, \mathcal{E}_h + 2 - X_M \\
&= 2 - X_M.
\end{aligned}$$

For the last identity we used that the identity

$$3\, \mathrm{card}\, \mathcal{T}_h - \mathrm{card}\, \mathcal{E}_h = \mathrm{card}\, \mathcal{E}_h.$$

holds since both sides of the equation equal the number of degrees of freedom in $\mathcal{S}^{1,NC}(\mathcal{T}_h)$. □

**Remark 1.5.12.** *For the two-dimensional sphere we have $X_M = 2$ while for a two-dimensional torus we have $X_M = 0$.*

The following proposition then follows as Proposition 1.5.7.

**Proposition 1.5.13.** *Let $\omega_h \in \mathcal{L}^0(\mathcal{T}_h)^3$ such that $\omega_h \cdot \mu_h = 0$ almost everywhere on $M_h$. Then there exist uniquely defined $a_h \in \mathcal{S}^1(\mathcal{T}_h)$, $b_h \in \mathcal{S}^{1,NC}(\mathcal{T}_h)$, and $H_h \in \mathcal{H}(\mathcal{T}_h; \mathbb{R}^3)$ such that $\int_{M_h} a_h \,\mathrm{d}x = 0$, $\int_{M_h} b_h \,\mathrm{d}x = 0$, and*

$$\omega_h = \nabla_{M_h} a_h + \mathrm{Curl}_{M_h} b_h + H_h.$$

*Moreover,*

$$\|\omega_h\|^2 = \|\nabla_{M_h} a_h\|^2 + \|\mathrm{Curl}_{M_h} b_h\|^2 + \|H_h\|^2$$

*and*

$$\big(\omega_h; \nabla_{M_h} v_h\big) = \big(\nabla_{M_h} a_h; \nabla_{M_h} v_h\big)$$

*for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)$.*

*Proof.* The proof follows the lines of the proof of Proposition 1.5.7. □

**Remark 1.5.14.** *We remark that other discrete decompositions of tangential vector fields are possible, e.g., if $\omega_h$ belongs to the lowest order Raviart-Thomas finite element space then $\omega_h$ can be decomposed into the discrete gradient of a $\mathcal{T}_h$-elementwise constant function, the vectorial Curl of function in $\mathcal{S}^1(\mathcal{T}_h)$, and a remainder which belongs to a finite dimensional set, cf. [AFW98].*

As in the flat, periodic case we can define a recovery operator.

**Lemma 1.5.15.** *Given $f \in L^1(M_h)$ define $\mathcal{A}_h f \in \mathcal{S}^1(\mathcal{T}_h)$ by*

$$\mathcal{A}_h f := \sum_{z\in\mathcal{N}_h} f_z \varphi_z, \qquad f_z := \mathcal{H}^2(\omega_z)^{-1} \int_{\omega_z} f \,\mathrm{d}x.$$

*There exists $C > 0$ such that for all $w_h \in \mathcal{S}^{1,NC}(\mathcal{T}_h)$ we have*

$$\big\|w_h - \mathcal{A}_h w_h\big\|_{L^2(\omega_z)} \le C h_z \big\|\nabla_{M_h} w_h\big\|_{L^2(\widehat{\omega}_z)}$$

*for all $z \in \mathcal{N}_h$ and $\widehat{\omega}_z := \cup_{y\in\mathcal{N}_h\cap\overline{\omega}_z} \omega_y$, and*

$$\big\|h_{\mathcal{T}_h}^{-1}\big(w_h - \mathcal{A}_h w_h\big)\big\| \le C \big\|\nabla_{M_h} w_h\big\|,$$

*where $h_{\mathcal{T}_h} \in L^\infty(M_h)$ satisfies $h_{\mathcal{T}_h}|_K = h_K$ for all $K \in \mathcal{T}_h$.*

*Proof.* The proof follows the lines of the proof of Lemma 1.5.8. □

## 1.6 Projections onto surfaces and elementary differential geometry

For a convex set $\mathcal{C} \subset \mathbb{R}^n$ it is well known that the orthogonal projection onto $\mathcal{C}$ is well-defined in the entire space $\mathbb{R}^n$ and Lipschitz continuous with constant less than or equal to 1. In particular, if $N = \partial\mathcal{C}$ then the projection defines an operator $\pi_N : \mathbb{R}^n \setminus \mathcal{C} \to N$. If $N$ is not the boundary of a convex set then it is still possible to define an orthogonal (or nearest-neighbor) projection in a small tubular neighborhood of $N$ provided that $N$ is sufficiently regular. We include a proof for a compact, $k$–dimensional $C^\ell$ submanifold without boundary, $\ell \ge 2$, in $\mathbb{R}^n$ which guarantees that the projection $\pi_N$ is $C^{\ell-1}$.

**Theorem 1.6.1.** *Let $\ell \ge 2$ and suppose that $N \subset \mathbb{R}^n$ is a compact, k-dimensional $C^\ell$ submanifold in $\mathbb{R}^n$ without boundary. There exists $\delta_N > 0$ such that for all $u \in U_{\delta_N}(N) := \{q \in \mathbb{R}^n : \mathrm{dist}(q,N) < \delta_N\}$ there exists a uniquely defined element $\pi_N(u) \in N$ such that*

$$|u - \pi_N(u)| = \mathrm{dist}(u, N).$$

*The mapping $\pi_N : U_\delta(N) \to N$ is $C^{\ell-1}$ regular, satisfies $D\pi_N(p)|_{T_pN} = \mathrm{id}_{T_pN}$ for all $p \in N$, and $D\pi_N(p)\nu = 0$ for $\nu \in \mathbb{R}^n$ such that $\nu \perp T_pN$.*

*Proof.* By compactness and continuity of $N$ there exists for every $u_0 \in \mathbb{R}^n$ a $p_0 \in N$ such that $|u_0 - p_0| = \mathrm{dist}(u_0, N)$. We aim at deriving conditions that ensure that $p_0$ is uniquely defined and depends in a differntiable way on $u_0$. For a local parametrization $f\colon \widehat{\Omega} \to N$ such that $f(\xi_0) = p_0$ for some $\xi_0 \in \widehat{\Omega}$ we then have that $\xi_0$ solves the minimization problem

$$\min_{\xi\in\widehat{\Omega}} \frac{1}{2}|u_0 - f(\xi)|^2$$

and the functions

$$F_j(u_0, \xi) := -\big(u_0 - f(\xi)\big) \cdot \partial_j f(\xi)$$

vanish for $j = 1, 2, ..., k$. If, in addition, the Hessian of $G(\xi) := \frac{1}{2}|u_0 - f(\xi)|^2$, given for $i, j = 1, 2, ..., k$ by

$$\partial_i F_j(u, \xi) = \partial_i f(\xi) \cdot \partial_j f(\xi) - \big(u - f(\xi)\big) \cdot \partial_i \partial_j f(\xi),$$

is positive definite, then $\xi_0$ is unique and the projection onto $N$ is well-defined. Fix $\xi_0 \in \widehat{\Omega}$, set $p_0 := f(\xi_0)$, and let $\nu_0$ denote a unit normal to $N$ at $p$. Then, for $s \in \mathbb{R}$ and $u_0 := p_0 + s\nu_0$ we have $F_j(u_0, \xi_0) = 0$. Moreover,

$$\partial_i F_j(u_0, \xi_0) = \partial_i f(\xi_0) \cdot \partial_j f(\xi_0) - s\nu_0 \cdot \partial_i \partial_j f(\xi_0)$$

and the first term on the right-hand side is (an entry of) the first fundamental form which is positive definite while the second term on the right-hand side is the second fundamental form which is uniformly bounded by some $\kappa_N > 0$. Therefore, we have

$$\partial_i F_j(u_0, \xi_0) \geq (C - s\kappa_N)\delta_{ij}$$

in the sense of bilinear forms and where $\kappa$ is the largest absolute eigenvalue of the second fundamental form. Hence, the matrix $\big(\partial_i F_j(u_0, \xi_0)\big)_{i,j=1,2,...,k}$ is positive definite for $s$ sufficiently small, i.e., $s \leq \delta_N = C/\kappa_N$. The implicit function theorem guarantees that the function $g : u_0 \mapsto \xi_0$ satisfies $g \in C^{\ell-1}$. The projection $\pi_N$ is then defined by $\pi_N := f \circ g$. Since $g(p) = p$ for all $p \in N$ we immediately observe that $D\pi_N(p)|_{T_pN} = \mathrm{id}_{T_pN}$ for all $p \in N$. Moreover, since $u - f(g(u))$ is orthogonal to $T_{f(g(u))}N$, we find that $D\pi_N(p)\nu = 0$ for $\nu \in \mathbb{R}^n$ such that $\nu \perp T_pN$. Finally, to guarantee global well-posedness of the projection, it may be necessary to decrease $\delta_N$ appropriately, depending on the ratio of the geodesic and the Euclidean distances, cf. the left plot of Figure 1.4. □

**Remark 1.6.2.** *The proof of the theorem shows that $\delta_N$ is given (up to constants depending on the global geometry of $N$) by $\delta_N = \big(\max_{i=1,2,...,k} \max_{p \in N} |\kappa_i(p)|\big)^{-1}$, where $\kappa_i(p)$, $i = 1, 2, ..., k$, denote the principal curvatures of $N$ at $p$. This estimate is optimal in the sense that $\pi_N(0)$ is not defined if $N = \{p \in \mathbb{R}^2 : |p| = 1\}$ is the unit circle in $\mathbb{R}^2$, cf. the right plot of Figure 1.4.*

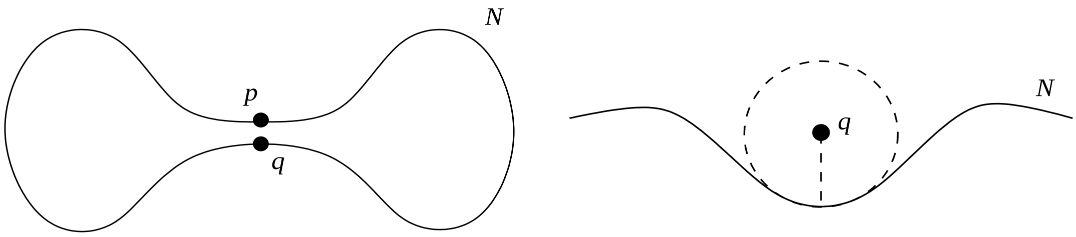


Figure 1.4: Small Euclidean but large geodesic distance between points $p, q \in N$ (left) and undefined projection onto $N$ at $q$ (right).

**Remark 1.6.3.** *The proof of the theorem shows that if the distance function* $\mathrm{dist}(\cdot, N)$ *is known explicitly then the operator $\pi_N$ can be evaluated through the identity*

$$\pi_N(u) = u - \mathrm{dist}(u, N)\nabla \mathrm{dist}(u, N)$$

*for all $u \in U_{\delta_N}(N)$.*

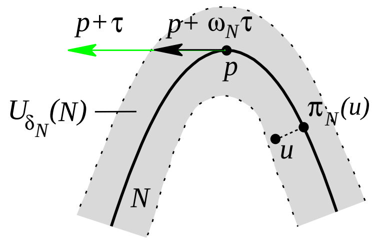


Figure 1.5: Choice of the damping parameter $\omega_N$ in Lemma 1.6.4.

**Lemma 1.6.4.** *Let $N$ and $\delta_N$ be as in Theorem 1.6.1. There exists $\omega_N > 0$ such that for all $p \in N$ and all $\tau \in T_pN$, $s \in \mathbb{R}$ with $|\tau| \le 1$ and $|s| \le \omega_N$ we have $p + s\tau \in U_{\delta_N}(N)$. If $N = \partial\mathcal{C}$ for a convex set $\mathcal{C} \subset \mathbb{R}^n$ then $\pi_N\big(p + s\tau\big)$ is well-defined for all $s \in \mathbb{R}$ and we set $\omega_N := \infty$ in this case.*

*Proof.* The proof follows from a Taylor expansion of a local parametrization $f\colon \widehat{\Omega} \to N$. For $\xi_0 \in \widehat{\Omega}$ such that $f(\xi_0) = p$ and $e \in \mathbb{R}^k$ such that $Df(\xi_0)e = \tau$ we have $|e| \le C$ and

$$f(\xi_0 + se) = f(\xi_0) + sDf(\xi_0)e + \mathcal{O}(s^2).$$

Since $f(\xi_0 + se) \in N$ we verify that

$$\operatorname{dist}\big(f(\xi_0) + sDf(\xi_0)e, N\big) \le \big|f(\xi_0) + sDf(\xi_0)e - f(\xi_0 + se)\big| = Cs^2.$$

Hence, for $s$ sufficiently small we verify the first part of the lemma. The statement for $N = \partial\mathcal{C}$ is obvious. □

**Remark 1.6.5.** *A careful inspection of the proof of Lemma 1.6.4 shows that a maximal $\omega_N$ is up to generic constants given by $\omega_N \approx \delta_N \approx \big(\max_{i=1,2,\dots,k}\max_{p\in N}|\kappa_i(p)|\big)^{-1}$.*

The following definition guarantees that there exist continuous, unit vector fields that define an orthonormal basis of $T_pN$ for all points $p$ on the submanifold $N$.

**Definition 1.6.6.** *We say that the compact, $k$-dimensional $C^\ell$ submanifold $N \subset \mathbb{R}^n$, $\ell \ge 2$, is* parallelizable *if there exist continuously differentiable unit vector fields $e_1, e_2, \dots, e_k\colon N \to \mathbb{R}^n$ such that for all $p \in N$ the vectors $\big(e_1(p), e_2(p), \dots, e_k(p)\big)$ form an orthonormal basis for $T_pN$.*

**Remark 1.6.7.** *Not every compact, $C^2$ submanifold $N$ is parallelizable, e.g., the two-dimensional unit sphere $S^2$ is not parallelizable. A construction in [Hél02] shows however that every compact $C^4$ submanifold $N$ without boundary can be isometrically embedded into a parallelizable $C^3$ submanifold $\widehat{N}$.*

**Lemma 1.6.8.** *(i) Let $N$ be as in Theorem 1.6.1 and suppose that $N$ is parallelizable. Then there exist compactly supported, continuously differentiable vector fields $\overline{e}_1, \overline{e}_2, \dots, \overline{e}_k\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $e_i(p) = \overline{e}_i(p)$ for all $p \in N$, $1 \le i \le k$, and $e_i$ as in Definition 1.6.6.*
*(ii) If $N$ is also orientable then there exist compactly supported, continuously differentiable vector fields $\overline{\nu}_{k+1}, \dots, \overline{\nu}_n\colon \mathbb{R}^n \to \mathbb{R}^n$ such that the vectors*

$$\overline{e}_1(p),\, \overline{e}_2(p), \dots, \overline{e}_k(p),\, \overline{\nu}_{k+1}(p), \dots, \overline{\nu}_n(p)$$

*form an orthonormal basis for $\mathbb{R}^n$.*

*Proof.* Owing to the assumptions in (i) on $N$ there exist vector fields $e_1, e_2, ..., e_k : N \to \mathbb{R}^n$ so that the desired properties are satisfied in an open neighborhood of $N$. Choose $p_0 \in N$ and define the non-smooth extension $\widetilde{\pi}_N : \mathbb{R}^n \to N$ of the projection $\pi_N : U_{\delta_0} \to N$ by setting $\widetilde{\pi}_N(u) := p_0$ for $u \in \mathbb{R}^n \setminus U_{\delta_0}$. Then, let $\chi : \mathbb{R}_{\geq 0} \to [0,1]$ be a continuously differentiable function such that $\chi(s) = 1$ for $s \leq \delta_0/2$ and $\chi(s) = 0$ for $s \geq 3\delta_0/4$, and define

$$\overline{e}_j(u) := \chi\big(\mathrm{dist}(u, N)\big) e_j\big(\widetilde{\pi}_N(u)\big).$$

This function is continuously differentiable in $\mathbb{R}^n$ and has compact support. We proceed analogously for the functions $\nu_\ell$, $\ell = k+1, ..., n$, in the second statement to complete the proof of the lemma. $\square$

**Lemma 1.6.9.** *Suppose that $N$ is $C^2$. There exists a constant $C > 0$ such that for all $p, q \in N$ and $\nu \in \mathbb{R}^n$ satisfying $\nu \perp T_p N$ and $|\nu| = 1$ we have*

$$(p - q) \cdot \nu \leq C|p - q|^2.$$

*Proof.* We may suppose that $p, q$ belong to a parametrized neighborhood in $N$, since otherwise they have a positive distance greater than some $\varepsilon_0$ in which case the statement is immediate with $C = \varepsilon_0^{-1}$. For a local parametrization $f : \widehat{\Omega} \to N$ and $\xi_0, \xi_1 \in \widehat{\Omega}$ such that $f(\xi_0) = p$ and $f(\xi_1) = q$, a Taylor expansion provides

$$\begin{aligned} q &= f(\xi_1) \\ &= f(\xi_0) + Df(\xi_0)(\xi_1 - \xi_0) + \mathcal{O}(|\xi_1 - \xi_0|^2) \\ &= p + Df(\xi_0)(\xi_1 - \xi_0) + \mathcal{O}(|\xi_1 - \xi_0|^2). \end{aligned}$$

Notice that $Df(\xi_0)(\xi_1 - \xi_0) \in T_p N$ and hence the result follows upon testing the identity with $\nu$ and noting that $|\xi_1 - \xi_0| = |f^{-1}(p) - f^{-1}(q)| \leq C|p - q|$. $\square$

## 1.7 Equivalent characterizations of harmonic maps

We recall the definition of a harmonic map into a submanifold $N$ and discuss equivalent weak formulations. For questions concerning existence of harmonic maps we refer the reader to [Hél02, Jos84, EL95, EF01].

**Definition 1.7.1.** *A vector field $u \in W^{1,2}(M; \mathbb{R}^n)$ is called a* (weakly) harmonic map into *$N$ if $u(x) \in N$ for almost every $x \in M$ and if it is stationary for the functional $E : W^{1,2}(M; \mathbb{R}^n) \to \mathbb{R}$,*

$$v \mapsto \frac{1}{2} \int_M |\nabla_M v|^2 \,\mathrm{d}s,$$

*with respect to perturbations of the form $\pi_N(u + \phi)$ for vector fields $\phi \in L^\infty(M; \mathbb{R}^n) \cap W^{1,2}(M; \mathbb{R}^n)$ that are compactly supported in $M$.*

We include the following proposition from [FMS98] which is adopted to harmonic maps on hypersurfaces.

**Proposition 1.7.2.** *Suppose that $N$ is a compact, parallelizable, $k$-dimensional $C^2$ submanifold in $\mathbb{R}^n$ without boundary. A vector field $u \in W^{1,2}(M;\mathbb{R}^n)$ such that $u(x) \in N$ for almost every $x \in M$ is a harmonic map into $N$ if and only if one of the following equivalent conditions is satisfied:*
*(i) for all $v \in W_0^{1,2}(M;\mathbb{R}^n)$ such that $v(x) \in T_{u(x)}N$ for almost every $x \in M$ we have*

$$\big(\nabla_M u; \nabla_M v\big) = 0;$$

*(ii) if $\big(e^i\big)_{i=1,2,...,k} \subset W^{1,2}(M;\mathbb{R}^n)$ are such that the vectors $e^1(x), e^2(x), ..., e^k(x)$ form an orthonormal basis for $T_{u(x)}N$ for almost every $x \in M$ and if $\vartheta^i := \sum_{\alpha=1}^n e^{i,\alpha}\nabla_M u^\alpha$ and $\omega^{ij} := \sum_{\alpha=1}^n e^{j,\alpha}\nabla_M e^{i,a}$ then we have*

$$\big(\vartheta^i; \nabla_M \eta\big) + \sum_{j=1}^k \big(\omega^{ij} \cdot \vartheta^j; \eta\big) = 0$$

*for all $\eta \in W_0^{1,2}(M) \cap L^\infty(M)$ and $i = 1, 2, ..., k$;*
*(iii) for all $w \in W_0^{1,2}(M;\mathbb{R}^n) \cap L^\infty(M;\mathbb{R}^n)$ we have*

$$\big(\nabla_M u; \nabla_M w\big) = \big(A_N(u)\big[\nabla_M u; \nabla_M u\big]; w\big),$$

*where $A_N$ denotes the second fundamental form on $N$ given by*

$$A_N(u)[\nabla_M u; \nabla_M u] = \sum_{\ell=k+1}^n \sum_{\gamma=1}^m A_{N,\ell}(\underline{D}_\gamma u; \underline{D}_\gamma u)\overline{\nu}^\ell \circ u$$

*with $A_{N,\ell}(X,Y) = X \cdot D_Y \overline{\nu}^\ell$ and $\overline{\nu}^\ell$, $\ell = k+1, ..., n$ as in Lemma 1.6.8.*

*Proof.* Suppose that $u$ is a harmonic map into $N$. Then, considering perturbations $u_t := \pi_N(u + t\phi)$ for $\phi \in C_c^\infty(M;\mathbb{R}^n)$ with $t$ sufficiently small it follows that

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0}\big(\nabla_M u; \nabla_M \pi_N(u + t\phi)\big) \\ &= \frac{d}{dt}\Big|_{t=0}\Big\{\big(\nabla_M u; \nabla_M u\big) + t\big(\nabla_M u; \nabla_M\big[D\pi_N(u)\phi\big]\big) + o(t)\Big\} \\ &= \big(\nabla_M u; \nabla_M\big[D\pi_N(u)\phi\big]\big), \end{aligned}$$

where we used that $\pi_N(u + t\phi) = u + tD\pi_N(u)\phi + o(t)$. If $\phi(x) \in T_{u(x)}N$ then $D\pi_N(u)\phi = \phi$ and the identity in (i) follows from a density argument as in [FMS98]. The implication that if (i) is satisfied then $u$ is a harmonic map follows from the same identity since $D\pi_N\big(u(x)\big)\phi(x) \in T_{u(x)}N$ for almost every $x \in M$.
Equivalence of (i) and (ii) can be seen as follows: for $v = \eta e^i$ with $\eta \in W_0^{1,2}(M) \cap L^\infty(M)$ we have,

using $\underline{D}_\gamma u(x) \in T_{u(x)}N$ for almost every $x \in M$ and employing Lemma 1.2.3,

$$\begin{aligned}
\big(\nabla_M u; \nabla_M v\big) &= \sum_{\gamma=1}^{m} \big(\underline{D}_\gamma u; \underline{D}_\gamma v\big) = \sum_{\gamma=1}^{m}\sum_{j=1}^{k} \big((e^j \cdot \underline{D}_\gamma u)e^j; \underline{D}_\gamma v\big) \\
&= \sum_{\gamma=1}^{m}\sum_{j=1}^{k} \big(e^j \cdot \underline{D}_\gamma u; e^j \cdot \underline{D}_\gamma v\big) = \sum_{\gamma=1}^{m}\sum_{j=1}^{k}\sum_{\alpha,\beta=1}^{n} \big(e^{j,\alpha}\underline{D}_\gamma u^\alpha; e^{j,\beta}\underline{D}_\gamma v^\beta\big) \\
&= \sum_{j=1}^{k}\sum_{\alpha,\beta=1}^{n} \big(e^{j,\alpha}\nabla_M u^\alpha; e^{j,\beta}\nabla_M v^\beta\big) = \sum_{j=1}^{k}\sum_{\alpha,\beta=1}^{n} \big(e^{j,\alpha}\nabla_M u^\alpha; e^{j,\beta}\nabla_M(\eta e^{i,\beta})\big) \\
&= \sum_{j=1}^{k}\sum_{\alpha,\beta=1}^{n} \big(e^{j,\alpha}\nabla_M u^\alpha; \eta e^{j,\beta}\nabla_M e^{i,\beta}\big) + \sum_{j=1}^{k}\sum_{\alpha,\beta=1}^{n} \big(e^{j,\alpha}\nabla_M u^\alpha; e^{j,\beta}e^{i,\beta}\nabla_M \eta\big) \\
&= \sum_{j=1}^{k} \big(\vartheta^j; \eta\omega^{ij}\big) + \big(\vartheta^i; \nabla_M \eta\big).
\end{aligned}$$

To establish equivalence of (i) and (iii), choose $w \in W_0^{1,2}(M;\mathbb{R}^n) \cap L^\infty(M;\mathbb{R}^n)$ and let $\eta_j$ and $\mu_\ell$ be such that

$$w = \sum_{j=1}^{k} \eta_j e^j + \sum_{\ell=k+1}^{n} \mu_\ell \nu^\ell,$$

where $\widehat{\nu}^\ell = \overline{\nu}^\ell \circ u$. Writing $w^{||} := \sum_{j=1}^{k} \eta_j e^j$ we have, using $\underline{D}_\gamma u \cdot \widehat{\nu}^\ell = 0$, that

$$\begin{aligned}
\big(\nabla_M u; \nabla_M w\big) &= \big(\nabla_M u; \nabla_M w^{||}\big) + \sum_{\ell=k+1}^{n}\sum_{\gamma=1}^{m} \big(\underline{D}_\gamma u; \underline{D}_\gamma\big[\mu_\ell \widehat{\nu}^\ell\big]\big) \\
&= \big(\nabla_M u; \nabla_M w^{||}\big) + \sum_{\ell=k+1}^{n}\sum_{\gamma=1}^{m} \big(\underline{D}_\gamma u; \mu_\ell \underline{D}_\gamma \widehat{\nu}^\ell\big) \\
&= \big(\nabla_M u; \nabla_M w^{||}\big) + \sum_{\ell=k+1}^{n}\sum_{\gamma=1}^{m} \big(\underline{D}_\gamma u; \mu_\ell D\overline{\nu}^\ell \cdot \underline{D}_\gamma u\big) \\
&= \big(\nabla_M u; \nabla_M w^{||}\big) + \sum_{\ell=k+1}^{n}\sum_{\gamma=1}^{m} \big(A_{N,\ell}\big[\underline{D}_\gamma u; \underline{D}_\gamma u\big]; \mu_\ell\big) \\
&= \big(\nabla_M u; \nabla_M w^{||}\big) + \big(A_N(u)\big[\nabla_M u; \nabla_M u\big]; w\big),
\end{aligned}$$

where we used that $e^i \cdot e^j = \delta_{ij}$ almost everywhere on $M$. If (i) is satisfied then the first term on the right-hand side vanishes which establishes (ii). Conversely, if (ii) holds true then the identity reduces to $\big(\nabla_M u; \nabla_M w^{||}\big) = 0$ which is the statement given in (i). □

**Remark 1.7.3.** *The condition of item (ii) in the proposition is satisfied if $N$ is parallelizable. As noted in Remark 1.6.7 if $N$ is $C^4$ then there exists an isometric isomorphism $J\colon N \to \widehat{N}$ with a parallelizable $C^3$ submanifold $\widehat{N}$. It is proved in [Hél02, Lemma 4.1.2] that $u\colon M \to N$ is weakly harmonic if and only if $J \circ u\colon M \to \widehat{N}$ is weakly harmonic, see also [Hél91, CTZ93]. Therefore, we may assume that $N$ is parallelizable provided it is $C^4$.*

## 1.8 Weak limits of discrete vector fields into surfaces

The following lemma shows that if a given sequence of finite element functions $\big(u_h\big)_{h>0}$ attains its nodal values in a surface and if the sequence converges weakly in $W^{1,2}$ then also the weak limit attains its values in the surface almost everywhere. We let $M$ and $M_h$ be as in Section 1.3 and adapt arguments from [MSŠ97] to the current setting.

**Lemma 1.8.1.** *Let $N \subset \mathbb{R}^n$ be a compact, continuous submanifold. Suppose that $\big(\widetilde{u}_h\big)_{h>0}$ is a bounded sequence in $W^{1,2}(M;\mathbb{R}^n)$ such that for each $h > 0$ the function $\widetilde{u}_h$ is the lifting of a function $u_h \in \mathcal{S}^1(\mathcal{T}_h)^n$ which satisfies $u_h(z) \in N$ for all $z \in \mathcal{N}_h$. Then, every weak accumulation point $u \in W^{1,2}(M;\mathbb{R}^n)$ of the sequence satisfies $u(x) \in N$ for almost every $x \in M$.*

*Proof.* For $h > 0$ and $K \in \mathcal{T}_h$ fix $z_K \in \mathcal{N}_h$ such that $z_K \in K$. Given $\delta > 0$ define

$$\Sigma_{h,\delta} := \big\{K \in \mathcal{T}_h : \|u_h - u_h(z_K)\|_{L^\infty(K)} \geq \delta\big\}.$$

Then, the inclusion

$$A_{h,\delta} := \big\{x \in M_h : \operatorname{dist}\big(u_h(x), N\big) \geq \delta\big\} \subseteq \bigcup_{K\in\Sigma_{h,\delta}} K$$

implies

$$\mathcal{H}^d(A_{h,\delta}) \leq \sum_{K\in\Sigma_{h,\delta}} \mathcal{H}^d(K).$$

Using the Poincaré estimate $||v_h||_{L^\infty(K)} \leq Ch_K||\nabla v_h||_{L^\infty(K)}$, which holds for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)$ such that $v_h(z_K) = 0$, we verify that

$$\begin{aligned}\sum_{K\in\Sigma_{h,\delta}} \mathcal{H}^d(K) &\leq \sum_{K\in\mathcal{T}_h} \mathcal{H}^d(K)\delta^{-2}||u_h - u_h(z_K)||^2_{L^\infty(K)} \\ &\leq C \sum_{K\in\mathcal{T}_h} \delta^{-2}h_K^2\big\|\nabla_{M_h} u_h\big\|^2_{L^2(K)} \\ &\leq Ch^2\delta^{-2}\big\|\nabla_{M_h} u_h\big\|^2_{L^2(M_h)}.\end{aligned}$$

Hence, as $h \to 0$ we have that $\mathcal{H}^d(A_{h,\delta}) \to 0$. Let

$$\widetilde{A}_{h,\delta} := \big\{x \in M : \operatorname{dist}\big(\widetilde{u}_h(x), N\big) \geq \delta\big\}.$$

Then, by Lemma 1.3.3 we have

$$\begin{aligned}\mathcal{H}^d(A_{h,\delta}) &= \int_{M_h} \chi_{\{x\in M_h:\operatorname{dist}(u_h(x),N)\geq\delta\}}\,\mathrm{d}s_h = \int_M \chi_{\{x\in M_h:\operatorname{dist}(u_h(x),N)\geq\delta\}} \circ \mathcal{P}_h^{-1} Q_h Q^{-1}\,\mathrm{d}s \\ &= \int_M \chi_{\{x\in M:\operatorname{dist}(\widetilde{u}_h(x),N)\geq\delta\}} Q_h Q^{-1}\,\mathrm{d}s \geq \mathcal{H}^d(\widetilde{A}_{h,\delta}) \min_{x\in M} Q_h Q^{-1}\end{aligned}$$

and for $h$ sufficiently small we have $Q_h Q^{-1} \geq \frac{1}{2}$, cf. the proof of Lemma 1.3.5. This implies that also

$$\mathcal{H}^d(\widetilde{A}_{h,\delta}) \to 0$$

as $h \to 0$. In other words, the non-negative function $f_h(x) := \operatorname{dist}\big(\widetilde{u}_h(x), N\big)$, $x \in M$, converges to 0 in measure. This implies, see, e.g., [Rou05, Prop. 1.13], that there exists a subsequence (which is not relabeled) such that for almost every $x \in M$ we have

$$f_h(x) = \operatorname{dist}\big(\widetilde{u}_h(x), N\big) \to 0.$$

Since a subsequence of $\big(\widetilde{u}_h\big)$ converges weakly in $W^{1,2}(M;\mathbb{R}^n)$, hence strongly in $L^2(M;\mathbb{R}^n)$, and in particular pointwise almost everywhere we have by continuity of the distance function that

$$\operatorname{dist}\big(\widetilde{u}_h(x), N\big) \to \operatorname{dist}\big(u(x), N\big)$$

for almost every $x \in M$. This shows that $u(x) \in N$ for almost every $x \in M$. □

**Remark 1.8.2.** *If $N = S^{n-1}$ then the assertion of the previous lemma follows from the nodal interpolation estimate*

$$\big\| |u_h|^2 - 1 \big\| = \big\| |u_h|^2 - \mathcal{I}_h\big[|u_h|^2\big] \big\| \leq Ch \big\| \nabla_{M_h} |u_h|^2 \big\| \leq 2Ch \big\| \nabla_{M_h} u_h \big\|.$$

*A similar estimate can be derived if $N$ is given as the zero level set of a function $F \in W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^{n-k})$.*

The following lemma shows that if $u_h \in \mathcal{S}^1(\mathcal{T}_h)^n$ satisfies $u_h(z) \in N$ for all $z \in \mathcal{N}_h$ then the partial derivatives of $u_h$ are almost tangent vectors to $N$ provided that $N$ is $C^2$.

**Lemma 1.8.3.** *Suppose that $N$ is a compact $C^2$ submanifold in $\mathbb{R}^n$ and let $u_h \in \mathcal{S}^1(\mathcal{T}_h)^n$ be such that $u_h(z) \in N$ for all $z \in \mathcal{N}_h$. Let $z \in \mathcal{N}_h$ and $K \in \mathcal{T}_h$ such that $z \in K$. For $\nu \in \mathbb{R}^n$ such that $\nu \perp T_{u_h(z)}N$ and $|\nu| = 1$ we have*

$$\nu \cdot \underline{D}_{h,\gamma} u_h \leq C h_K \big| \nabla_{M_h} u_h \big|^2$$

*on $K$ for $\gamma = 1, 2, ..., d+1$ with a constant $C > 0$ that only depends on $N$ and the geometry of $\mathcal{T}_h$ but not on $h_K$ or $u_h$.*

*Proof.* For $K = \mathcal{F}_K(\widehat{K})$ set $\widehat{u}_h := u_h \circ \mathcal{F}_K$. We may and will assume that $\mathcal{F}_K(\widehat{z}_0) = z_0$. With the notation of the proof of Lemma 1.3.5 we have for $\alpha = 1, 2, ..., n$

$$\nabla_{M_h} u_h^\alpha = \mathbf{D}_h^{\mathrm{T}} \widehat{\nabla} \widehat{u}_h^\alpha$$

and, if $e_\gamma \in \mathbb{R}^m$ denotes the $\gamma$-th canonical basis vector in $\mathbb{R}^m$,

$$\underline{D}_{h,\gamma} u_h^\alpha = \nabla_{M_h} u_h^\alpha \cdot e_\gamma = \big(\mathbf{D}_h e_\gamma\big)^{\mathrm{T}} \widehat{\nabla} \widehat{u}_h^\alpha.$$

This yields that

$$\underline{D}_{h,\gamma} u_h \cdot \nu = \sum_{\alpha=1}^{n} \underline{D}_{h,\gamma} u_h^\alpha \, \nu^\alpha = \big(\mathbf{D}_h e_\gamma\big)^{\mathrm{T}} \sum_{\alpha=1}^{n} \widehat{\nabla} \widehat{u}_h^\alpha \, \nu^\alpha.$$

Now, for each component $\widehat{\partial}_\delta \widehat{u}_h^\alpha$ of $\widehat{\nabla} \widehat{u}_h^\alpha$ for $\delta = 1, 2, ..., d$ we deduce

$$\sum_{\alpha=1}^{n} \widehat{\partial}_\delta \widehat{u}_h^\alpha \nu^\alpha = \sum_{\alpha=1}^{n} h_K^{-1} \big(\widehat{u}_h^\alpha(\widehat{z}_\delta) - \widehat{u}_h^\alpha(\widehat{z}_0)\big) \nu^\alpha = h_K^{-1} \big(\widehat{u}_h(\widehat{z}_\delta) - \widehat{u}_h(\widehat{z}_0)\big) \cdot \nu.$$

Lemma 1.6.9 guarantees that $(p-q)\cdot\nu \le C|p-q|^2$ for $p,q\in N$ and $\nu\in\mathbb{R}^n$ satisfying $\nu \perp T_pN$ and $|\nu|=1$. Since $\nu \perp T_{\widehat{u}_h(\widehat{z}_0)}N$ we thus infer that

$$h_K^{-1}\big(\widehat{u}_h(\widehat{z}_\delta)-\widehat{u}_h(\widehat{z}_0)\big)\cdot\nu \le Ch_K^{-1}\big|\widehat{u}_h(\widehat{z}_\delta)-\widehat{u}_h(\widehat{z}_0)\big|^2 = Ch_K\big|\widehat{\partial}_\delta\widehat{u}_h\big|^2 \le Ch_K\big|\widehat{\nabla}\widehat{u}_h\big|^2.$$

On combining the previous estimates, using that $\mathbf{D}_h$ is bounded $h$-independently, and incorporating the identity $\widehat{\nabla}\widehat{u}_h = \mathbf{G}_h^{\mathrm{T}}\nabla_{M_h}u_h$ with a uniformly bounded matrix $\mathbf{G}_h$, we verify that

$$\underline{D}_{h,\gamma}u_h\cdot\nu \le Ch_K\big|\nabla_{M_h}u_h\big|^2$$

which finishes the proof. □

Occasionally we will impose boundary conditions. The following lemma is a consequence of the fact that every bounded linear operator is weakly continuous.

**Lemma 1.8.4.** *Suppose that $M\subset\mathbb{R}^d\times\{0\}$ is a bounded Lipschitz domain with polyhedral boundary, let $\Gamma_{\mathrm{D}}\subseteq\partial M$ be closed with $\mathcal{H}^{d-1}(\Gamma_{\mathrm{D}})>0$, and let $u_{\mathrm{D}}\in C(\Gamma_{\mathrm{D}};\mathbb{R}^n)$. Suppose that $\big(\mathcal{T}_h\big)_{h>0}$ is a sequence of triangulations such that $\Gamma_{\mathrm{D}}$ is matched exactly by edges in $\mathcal{E}_h\cap\Gamma_{\mathrm{D}}$ for each $h>0$. Assume that $\overline{u}_{\mathrm{D},h}\in\mathcal{S}^1(\mathcal{T}_h)^n$ is such that $u_{\mathrm{D},h}:=\overline{u}_{\mathrm{D},h}|_{\Gamma_{\mathrm{D}}}$ and satisfies*

$$u_{\mathrm{D},h}\to u_{\mathrm{D}} \quad \text{in } L^2\big(\Gamma_{\mathrm{D}};\mathbb{R}^n\big)$$

*as $h\to 0$. If $w_h\in\mathcal{S}^1(\mathcal{T}_h)^n$ satisfies $w_h|_{\Gamma_{\mathrm{D}}}=u_{\mathrm{D},h}$ and the sequence $\big(w_h\big)_{h>0}$ is bounded in $W^{1,2}(M;\mathbb{R}^n)$ then every accumulation point $w\in W^{1,2}(M;\mathbb{R}^n)$ satisfies $w|_{\Gamma_{\mathrm{D}}}=u_{\mathrm{D}}$.*

*Proof.* Since the trace operator $T: W^{1,2}(M;\mathbb{R}^n)\to L^2(\partial M;\mathbb{R}^n)$ composed with the restriction to $\Gamma_{\mathrm{D}}$ is bounded and linear, it is weakly continuous and hence we have

$$u_{\mathrm{D}} = \lim_{h'\to 0} u_{\mathrm{D},h} = \lim_{h'\to 0} T(w_h)|_{\Gamma_{\mathrm{D}}} = T(w)|_{\Gamma_{\mathrm{D}}},$$

where $(h')$ is a subsequence such that $w_{h'}\rightharpoonup w$ in $W^{1,2}(M;\mathbb{R}^n)$. □

## 1.9 Auxiliary results from measure theory

We conclude the chapter with two elementary results from measure theory. The first result states that the space of linear combinations of Dirac measures is a closed subset of $C(M)^*$ with respect to the strong topology while the second one allows to identify the supports of the limits of certain sequences in $C(M)^*$.

**Lemma 1.9.1.** *Let $M\subset\mathbb{R}^d$ be compact and let $\big(F_\ell\big)_{\ell\in\mathbb{N}}$ be a bounded sequence in $C(M)^*$. If for each $\ell\in\mathbb{N}$ the support of $F_\ell$ is finite, i.e., $F_\ell=\sum_{j=1}^{L_\ell}a_j^\ell\delta_{x_j^\ell}$ for $L_\ell\in\mathbb{N}$ and $a_j^\ell\in\mathbb{R}$, $x_j^\ell\in M$, $j=1,2,...,L_\ell$, and if $F_\ell\to F$ strongly as $\ell\to\infty$ for some $F\in C(K)^*$, i.e.,*

$$\sup_{\eta\in C(M):\,\|\eta\|_{L^\infty(M)}\le 1}\langle F_\ell - F,\eta\rangle\to 0$$

*as $\ell\to\infty$, then there exist $\big(a_j\big)_{j\in\mathbb{N}}\subset\mathbb{R}$ and $\big(x_j\big)_{j\in\mathbb{N}}\subset M$ such that $F=\sum_{j=1}^\infty a_j\delta_{x_j}$. If $\sum_{j=1}^{L_\ell}|a_j^\ell|^s\le C_1$ for some $s>0$ and all $\ell\in\mathbb{N}$ then $\sum_{j=1}^\infty|a_j|^s\le C_1$.*

*Proof.* Riesz' representation theorem, see, e.g., [Rud87, Theorem 6.19] for details, provides an isometric isomorphism between $C(M)^*$ and the set of regular Borel measures on $M$ such that for every $G \in C(M)^*$ we have

$$\sup_{\eta\in C(M):\ \|\eta\|_{L^\infty(M)}\le 1} \langle G,\eta\rangle = \sup_{\{E_k\,:\,k\in\mathbb{N}\}\in\mathcal{P}(M)} \sum_{k=1}^{\infty} \big|G(E_k)\big|, \tag{9.8}$$

where we identified $G$ with the measure provided by the isomorphism and where $\mathcal{P}(M)$ denotes the set of all countable, measurable partitions of $M$. The set

$$\Gamma := \bigcup_{\ell\in\mathbb{N}} \{x_1^\ell, x_2^\ell, ..., x_{L_\ell}^\ell\}$$

is countable and we enumerate its elements as $\Gamma = \big\{x_1, x_2, x_3, ...\big\}$. We set $a_j := F\big(\{x_j\}\big)$ for $j \in \mathbb{N}$ and define $F' := \sum_{j=1}^\infty a_j\delta_{x_j} \in C(K)^*$. To finish the proof of the first statement it suffices to show that $F$ is supported on $\Gamma$ since this implies $F' = F$. Each $F_\ell$ is supported on $\Gamma$ and thus for every measurable set $A \subset M\setminus\Gamma$ we have by considering the partition $\{A, M\setminus A\}$ in (9.8) that

$$\big|F(A)\big| = \big|F(A) - F_\ell(A)\big| \le \sup_{\eta\in C(M):\ \|\eta\|_{L^\infty(M)}\le 1} \langle F_\ell - F,\eta\rangle$$

and the right-hand side can be made arbitrarily small, i.e., $F(A) = 0$. To prove the second part of the lemma we first notice that for every $x \in M$ we have

$$\big|F_\ell\big(\{x\}\big) - F\big(\{x\}\big)\big| \to 0$$

as $h \to 0$. With Fatou's lemma we then deduce that

$$\begin{aligned} C_1 &\ge \liminf_{\ell\to\infty} \sum_{j=1}^{L_\ell} |a_j^\ell|^s = \liminf_{\ell\to\infty} \sum_{j=1}^{\infty} \big|F_\ell\big(\{x_j\}\big)\big|^s \\ &\ge \sum_{j=1}^{\infty} \liminf_{\ell\to\infty} \big|F_\ell\big(\{x_j\}\big)\big|^s = \sum_{j=1}^{\infty} \big|F\big(\{x_j\}\big)\big|^s = \sum_{j=1}^{\infty} |a_j|^s \end{aligned}$$

which finishes the proof of the lemma. □

**Remarks 1.9.2.** *(i) Strong convergence in $C(K)^*$ is a selective notion of convergence as, e.g., the sequence of functionals $\delta_{1/N}$ does not converge to $\delta_0$ strongly in $C([0,1])^*$.*
*(ii) The statement of the lemma is still true if the support of $F_\ell$ is countable for every $\ell \in \mathbb{N}$.*

**Lemma 1.9.3.** *Let $M \subset \mathbb{R}^d$ be compact and let $\big(F_h\big)_{h>0}$ be a bounded sequence in $C(M)^*$. Suppose that there exist $C > 0$ and $L \in \mathbb{N}$ such that for each $h > 0$ and all $\eta \in C^1(M)$ we have*

$$\big|F_h(\eta)\big| \le Ch\|\nabla\eta\| + \sum_{j=1}^{L} \varrho_j^h \big|\eta(x_j^h)\big|$$

*for $\varrho_j^h \in \mathbb{R}$ and $x_j^h \in M$ for $j = 1, 2, ..., L$. Then there exist $L' \le L$ and $\varrho_j \in \mathbb{R}$, $y_j \in M$, $j = 1, 2, ..., L'$ such that for a subsequence which is not relabeled we have*

$$F_h \rightharpoonup^* \sum_{j=1}^{L'} \varrho_j\delta_{y_j}$$

*as $h \to 0$. If $s \in (0,1]$ and $\sum_{j=1}^{L} |\sigma_j^h|^s \le C_1$ for all $h > 0$ then $\sum_{j=1}^{L'} |\varrho_j|^s \le C_1$.*

*Proof.* Since $F_h$ is a bounded sequence in $C(M)^*$ there exists a weak limit $F \in C(M)^*$ of a subsequence which we do not relabel in the following. Passing to another subsequence we may assume that the $L$-tupels $\big(x_1^h, ..., x_L^h\big)$ converge strongly to $\big(x_1, ..., x_L\big) \in M^L$ as $h \to 0$. For $\eta \in C^1(M)$ with $\operatorname{supp}\eta \subset M \setminus \big\{x_1, ..., x_L\big\}$ we have, owing to the assumptions on $F_h$, that

$$\begin{aligned}\big|F(\eta)\big| &\le \big|F(\eta) - F_h(\eta)\big| + \big|F_h(\eta)\big| \\ &\le \big|F(\eta) - F_h(\eta)\big| + Ch\|\nabla\eta\| + \sum_{j=1}^{L} \varrho_j^h \big|\eta(x_j^h)\big|.\end{aligned}$$

The right-hand side vanishes as $h \to 0$ owing to $F_h \rightharpoonup^* F$, convergence of $\big(x_1^h, ..., x_L^h\big)$ to $\big(x_1, ..., x_L\big)$, and the fact that $\eta$ vanishes in an open neighborhood of $\big\{x_1, ..., x_L\big\}$. Therefore, we deduce that $F$ is supported on $\big\{x_1, ..., x_L\big\}$, i.e.,

$$F = \sum_{j=1}^{L'} \varrho_j \delta_{y_j},$$

for appropriate $L' \le L$, $\varrho_j \in \mathbb{R}$, $j = 1, 2, ..., L'$, and $\big\{y_1, ..., y_{L'}\big\} \subseteq \big\{x_1, ..., x_L\big\}$. We set $\varepsilon := \min_{i,j=1,...,L'} |y_i - y_j|/2$. For $i \in \{1, ..., L'\}$ we choose $\eta_i \in C^1(M)$ such that $\big|\eta_i(x)\big| \le 1$ for all $x \in M$, $\eta_i(y_i) = 1$ and $\eta_i(y_j) = 0$ for $j \ne i$, $\operatorname{supp}\eta_i \subseteq B_\varepsilon(y_i) \cap M$, and $\big\|\nabla\eta_i\big\| \le C\varepsilon^{-1}$. Then, for each $h > 0$ we have

$$\begin{aligned}|\varrho_i| = |F(\eta_i)| &\le \big|F(\eta_i) - F_h(\eta_i)\big| + \big|F_h(\eta_i)\big| \le \big|F(\eta_i) - F_h(\eta_i)\big| + Ch\varepsilon^{-1} + \sum_{j=1,...,L,\, |y_i - x_j^h| \le \varepsilon} \varrho_j^h \\ &\le \Big(\big|F(\eta_i) - F_h(\eta_i)\big|^s + \big(Ch\varepsilon^{-1}\big)^s + \sum_{j=1,...,L:\, |y_i - x_j^h| \le \varepsilon} \big|\varrho_j^h\big|^s\Big)^{1/s},\end{aligned}$$

where we used $[z]_{\ell^1} \le [z]_{\ell^s}$ for $\big(z_j\big)_{j\in\mathbb{N}} \subset \mathbb{R}$ and $[z]_{\ell^s} := \big(\sum_{j\in\mathbb{N}} |z_j|^s\big)^{1/s}$. For each $x_j^h$ there is at most one $i$ such that $x_j^h \in B_\varepsilon(y_i)$. Therefore, we deduce that

$$\sum_{i=1}^{L'} |\varrho_i|^s \le \sum_{i=1}^{L'} \big|F(\eta_i) - F_h(\eta_i)\big|^s + L'\big(Ch\varepsilon^{-1}\big)^s + \sum_{j=1}^{L} \big|\varrho_j^h\big|^s.$$

Since the first two terms on the right-hand side vanish as $h \to 0$ and the third one is bounded by $C_1$ we verify the assertion of the lemma. $\square$

# Chapter 2

# Convergence of discrete harmonic maps

Stability of solutions of nonlinear partial differential equations is often expressed in terms of weak compactness results: Given a bounded sequence of vector fields that satisfy the equation up to a compact perturbation the question is whether weak accumulation points are exact solutions of the problem under consideration. Such results are naturally linked to convergence of numerical approximations but owing to the limited choice of discrete test functions, the results are usually not directly applicable. In this chapter we prove weak convergence of sequences of (almost) discrete harmonic maps into a given surface $N$ when certain discretization parameters tend to zero by appropriately modifying existing weak compactness results for harmonic maps. While this task is relatively straightforward for harmonic maps into spheres it is essentially more involved when less symmetry is available. In fact, convergence for a large class of target manifolds will only be shown in two space dimensions.

## 2.1 Weak compactness results for harmonic maps

We briefly describe in this section weak compactness results on a continuous level which will be adapted for the analysis of numerical approximations in the subsequent sections. For ease of presentation we restrict in this section to the case that $M \subset \mathbb{R}^d \times \{0\}$ and write $\nabla$ instead of $\nabla_M$, omitting the trivial last component.

### 2.1.1 Harmonic maps into spheres

Suppose that $\big(u_\ell\big)_{\ell\in\mathbb{N}} \subset W^{1,2}(M;\mathbb{R}^n)$ is a bounded sequence of harmonic maps into $S^{n-1} \subset \mathbb{R}^n$, the $(n-1)$-dimensional unit sphere. Then, owing to Proposition 1.7.2 we have

$$\big(\nabla u_\ell; \nabla v\big) = 0$$

for all $v \in W_0^{1,2}(M;\mathbb{R}^n)$ such that $v(x) \in T_{u_\ell(x)}S^{n-1}$ for almost every $x \in M$. We assume for the time being that $n = 3$. Then, it is well known that for almost every $x \in M$ we have $v(x) \cdot u_\ell(x) = 0$, i.e., $v(x) \in T_{u_\ell(x)}S^2$, if and only if there exists $w(x)$ such that $v(x) = w(x) \times u_\ell(x)$ and this defines a vector field $w \in W_0^{1,2}(M;\mathbb{R}^3)$, provided that $v$ is essentially bounded. For $\gamma = 1, 2, ..., d$ we have

$$\partial_\gamma\big[u_\ell \times w\big] = \big(\partial_\gamma u_\ell\big) \times w + u_\ell \times \big(\partial_\gamma w\big)$$

almost everywhere in $M$ and the properties of the cross product imply

$$\partial_\gamma u_\ell \cdot \partial_\gamma \big[u_\ell \times w\big] = \partial_\gamma u_\ell \cdot \big[u_\ell \times \partial_\gamma w\big] = -\big[u_\ell \times \partial_\gamma u_\ell\big] \cdot \partial_\gamma w.$$

We thus deduce that the identity

$$\sum_{\gamma=1}^{d} \big(u_\ell \times \partial_\gamma u_\ell; \partial_\gamma w\big) = 0$$

is satisfied for all $w \in W_0^{1,2}(M;\mathbb{R}^3)$ and this equation provides an equivalent characterization of harmonic maps into the two-dimensional sphere. Now, owing to the compact embedding of $W^{1,2}(M;\mathbb{R}^3)$ into $L^2(M;\mathbb{R}^3)$, one can perform a limit passage $\ell' \to \infty$ in the previous identity, namely, we verify that

$$\sum_{\gamma=1}^{d} \big(u \times \partial_\gamma u; \partial_\gamma w\big) = 0$$

holds for all $w \in W_0^{1,2}(M;\mathbb{R}^3)$ if $u \in W^{1,2}(M;\mathbb{R}^3)$ is the weak limit of a subsequence $\big(u_{\ell'}\big)_{\ell' \in \mathbb{N}}$. Reversing the above argumentation with the cross product we find

$$\big(\nabla u; \nabla v\big) = -\sum_{\gamma=1}^{d} \big(u \times \partial_\gamma u; \partial_\gamma w\big) = 0.$$

Pointwise convergence almost everywhere in $M$ of $\big(u_{\ell'}\big)$ shows that $u(x) \in S^2$ for almost every $x \in M$ and hence $u$ is a harmonic map into $S^2$.

To understand the less symmetric situation for $n \neq 3$ we notice that $b \cdot a = 0$ for $a, b \in \mathbb{R}^n$ with $|a| = 1$ is satisfied if and only if $b = \mathbf{X}a$ holds with a skew-symmetric matrix $\mathbf{X} \in so(n)$. For a proof of this fact, let $a, b \in \mathbb{R}^n$ be two such vectors and choose $\mathbf{O} \in SO(n)$ such that $\mathbf{O}a = e_1$ is the first canonical basis vector in $\mathbb{R}^n$, set $b' := \mathbf{O}b$, and notice that owing to $\mathbf{O}^{\mathrm{T}}\mathbf{O} = \mathbf{I}_{n\times n}$ we have $b' \cdot e_1 = 0$, i.e., the first component of $b'$ is zero. Let $\mathbf{X}'$ be the matrix whose first column coincides with $b'$, whose first row equals $-b'^{\mathrm{T}}$, and which has vanishing entries otherwise. Then, $\mathbf{X}' \in so(n)$ and $b' = \mathbf{X}'e_1$. Setting $\mathbf{X} := \mathbf{O}^{\mathrm{T}}\mathbf{X}'\mathbf{O}$ we verify

$$b = \mathbf{O}^{\mathrm{T}} b' = \mathbf{O}^{\mathrm{T}}\mathbf{X}' e_1 = \mathbf{O}^{\mathrm{T}}\mathbf{X}'\mathbf{O}a = \mathbf{X}a$$

and $\mathbf{X}^{\mathrm{T}} = \big(\mathbf{O}^{\mathrm{T}}\mathbf{X}'\mathbf{O}\big)^{\mathrm{T}} = \mathbf{O}^{\mathrm{T}}\mathbf{X}'^{\mathrm{T}}\mathbf{O} = -\mathbf{O}^{\mathrm{T}}\mathbf{X}'\mathbf{O} = -\mathbf{X}$, i.e., $\mathbf{X} \in so(n)$, which finishes the proof.

Therefore, if $v \in W_0^{1,2}(M;\mathbb{R}^n) \cap L^\infty(M;\mathbb{R}^n)$ satisfies $v(x) \in T_{u_\ell(x)}S^{n-1}$ for almost every $x \in M$ then

$$v(x) = \sum_{s=1}^{n(n-1)/2} \eta_s(x)\mathbf{X}_s u_\ell(x)$$

for $\eta_s \in W_0^{1,2}(M) \cap L^\infty(M)$, $s = 1, 2, ..., n(n-1)/2$, and a basis $\big(\mathbf{X}_s : s = 1, 2, ..., n(n-1)/2\big)$ of $so(n)$. Noting that $\partial_\gamma u_\ell \cdot \big(\mathbf{X}_s \partial_\gamma u_\ell\big) = 0$ almost everywhere in $M$ we deduce that

$$0 = \sum_{\gamma=1}^{d} \big(\partial_\gamma u_\ell; \partial_\gamma v\big) = \sum_{\gamma=1}^{d} \sum_{s=1}^{n(n-1)/2} \big(\partial_\gamma u_\ell; \mathbf{X}_s u_\ell \partial_\gamma \eta_s\big).$$

The passage to a limit in this equation is possible since no products of partial derivatives of $u_\ell$ occur. Arguing as above, we deduce that every weak accumulation point of the sequence $\big(u_\ell\big)_{\ell\in\mathbb{N}}$ is a harmonic map into $S^{n-1}$.

We remark that in the language of differential forms the existence of $\mathbf{X} \in so(n)$ such that $b = \mathbf{X}a$ is equivalent to the existence of $c \in \Lambda^2(\mathbb{R}^n)$, the space of alternating bilinear forms on $\mathbb{R}^n \times \mathbb{R}^n$, such that

$$b = *^{-1}\big( *c \wedge a\big),$$

where $*$ and $\wedge$ denote the Hodge duality operator and the wedge product, respectively. Employing the facts that $a \cdot b = a \wedge *b$ for $a, b \in \Lambda^1(\mathbb{R}^n)$ and that $a \wedge (*c \wedge a) = -\big(a \wedge a\big) \wedge *c = 0$ for $a \in \Lambda^1(\mathbb{R}^n)$ and $c \in \Lambda^2(\mathbb{R}^n)$, an equivalent characterization of harmonic maps into spheres is the validity of the identity

$$\sum_{\gamma=1}^{d} \big(\partial_\gamma u; \partial_\gamma \left[\phi \wedge u\right]\big) = \sum_{\gamma=1}^{d} \big(u \wedge \partial_\gamma u; \partial_\gamma \phi\big) = 0$$

for all $\phi \in W_0^{1,2}\big(M; \Lambda^2(\mathbb{R}^n)\big)$, where we omitted the Hodge duality operator.

The discussion shows that weak compactness results for harmonic maps into spheres are consequences of the identity

$$u \wedge \Delta u = 0$$

or, equivalently, of the conservation law

$$\operatorname{div}\big(u \wedge \nabla u\big) = 0. \tag{1.1}$$

Such identities have been employed in [RSK89], [Che89], and [Sha88] for the analysis of harmonic map heat flow into spheres and wave map equations. The important fact about the identity (1.1) is that derivatives of $u$ enter linearly and not quadratically in the equation. The main reason for validity of (1.1) is the symmetry of the target manifold $S^{n-1}$ and Noether's theorem provides an interesting generalization of (1.1) to targets with certain symmetries: If $X$ is a Lipschitz continuous tangent vector field on $N$ which is an infinitesimal symmetry of $\Delta_M$ and $u : M \to N$ is critical for the Dirichlet energy on $M$ among such vector fields then

$$\operatorname{div}\big(X(u)^{\mathrm{T}} \nabla u\big) = 0.$$

We remark that $X$ is an infinitesimal symmetry for $\Delta_M$ if it is a Killing vector field, i.e., $L_X h = 0$, where $h$ denotes the metric on $N$ and $L_X$ the Lie-derivative defined by $X$; we refer to [Hél02, Raw84] for details. In case of the unit sphere such vector fields are given by $X\colon S^{n-1} \to TS^{n-1}$, $p \mapsto \mathbf{A}p$ for $\mathbf{A} \in so(n)$. The skew-symmetric matrices generate $SO(n)$ which is a symmetry for $\Delta_M$ in the sense that $\Delta_M \mathbf{R}u = \mathbf{R}\Delta_M u$ for $\mathbf{R} \in SO(n)$ and $u \in W^{1,2}(M; \mathbb{R}^n)$ such that $u \in S^{n-1}$ almost everywhere on $M$.

### 2.1.2 Harmonic maps into general targets

We next discuss the weak compactness result of [FMS98] for Lipschitz domains $M \subset \mathbb{R}^2 \times \{0\}$ and $k$-dimensional target manifolds $N \subset \mathbb{R}^n$ that are smooth, compact, and without boundary but not necessarily orientable. This will serve as a guideline for the generalization of the weak convergence result in [MSŠ97] for a finite difference method to finite element schemes for harmonic maps introduced below. As above, suppose that $\big(u_\ell\big)_{\ell\in\mathbb{N}} \subset W^{1,2}(M; \mathbb{R}^n)$ is a bounded sequence of harmonic maps into $N$. Thus, for each $\ell \in \mathbb{N}$ we have $u_\ell(x) \in N$ for almost every $x \in M$, $\|\nabla u_\ell\| \le C$, and

$$\big(\nabla u_\ell; \nabla v\big) = 0$$

for all $v \in W_0^{1,2}(M;\mathbb{R}^n)$ such that $v(x) \in T_{u_\ell(x)}N$ for almost every $x \in M$. By restricting to a cube $Q \subset M$ and reflecting $u$ across the sides of $Q$ we may and will assume that each $u_\ell$ is periodic, see [FMS98] and Section 2.3 for details about this argumentation. We choose an orthonormal frame $\big(e_\ell^i\big)_{i=1,2,\dots,k} \subset W^{1,2}(M;\mathbb{R}^n)$ for $u_\ell^{-1}TN$, i.e., we select vector fields $e_\ell^i \in W^{1,2}(M;\mathbb{R}^n)$, $i = 1, 2, ..., k$ such that for almost every $x \in M$ the family $\big(e_\ell^i(x)\big)_{i=1,2,\dots,k}$ is an orthonormal basis for $T_{u_\ell(x)}N$. This is possible if $N$ is parallelizable which can always be assumed by a construction in [Hél02, Lemma 4.1.2] if $N$ is $C^4$, see also [Hél91, CTZ93] and Remark 1.7.3. We then employ $v = \eta e_\ell^i$, expand the rows of $\nabla u$ in the basis $\big(e_\ell^i\big)_{i=1,2,\dots,k}$, and argue as in the proof of Proposition 1.7.2 to deduce that for $\omega_\ell^{ij} := e_\ell^{j,\mathrm{T}} \nabla e_\ell^i$ and $\vartheta_\ell^j := e_\ell^{j,\mathrm{T}} \nabla u_\ell$ the identity

$$\sum_{j=1}^{k} \big(\omega_\ell^{ij} \cdot \vartheta_\ell^j; \eta\big) + \big(\vartheta_\ell^i; \nabla\eta\big) = 0 \tag{1.2}$$

is satisfied for all $\eta \in C_c^\infty(M)$ and each $\ell \in \mathbb{N}$. With a good choice of the frame $\big(e_\ell^i\big)$ (i.e., using "Coulomb gauge", see Lemma 2.2.3 below) we have $\operatorname{div} \omega_\ell^{ij} = 0$ so that $\omega_\ell^{ij} = \operatorname{Curl} b_\ell^{ij} + H_\ell^{ij}$ with periodic functions $b_\ell^{ij} \in W^{1,2}(M)$ and harmonic fields $H_\ell^{ij} \in L^2(M;\mathbb{R}^2)$ satisfying $\|\operatorname{Curl} b_\ell^{ij}\|^2 + \|H_\ell^{ij}\|^2 = \|\omega_\ell^{ij}\|^2$. Then, (1.2) can be written as

$$\sum_{j=1}^{k} \Big\{ \big(\operatorname{Curl} b_\ell^{ij} \cdot \vartheta_\ell^j; \eta\big) + \big(H_\ell^{ij} \cdot \vartheta_\ell^j; \eta\big) \Big\} + \big(\vartheta_\ell^i; \nabla\eta\big) = 0. \tag{1.3}$$

Let $u \in W^{1,2}(M;\mathbb{R}^n)$ be a weak accumulation point of the sequence $\big(u_\ell\big)$ so that for a subsequence, which we do not relabel in the following, we have $u_\ell \rightharpoonup u$ in $W^{1,2}(M;\mathbb{R}^n)$. By the compact embedding of $W^{1,2}(M;\mathbb{R}^n)$ into $L^2(M;\mathbb{R}^n)$ we deduce $u_\ell \to u$ in $L^2(M;\mathbb{R}^n)$ and in particular that pointwise convergence holds almost everywhere in $M$. Therefore, we have $u(x) \in N$ for almost every $x \in M$ since $N$ is continuous. Since $\big(e_\ell^i\big)_{\ell\in\mathbb{N}}$ is bounded in $W^{1,2}(M;\mathbb{R}^n)$ we have, if necessary after extraction of another subsequence, $e_\ell^i \rightharpoonup e^i$ in $W^{1,2}(M;\mathbb{R}^n)$ for $i = 1, 2, ..., k$ and using pointwise convergence almost everywhere as above we verify that the family $\big(e^i\big)_{i=1,2,\dots,k}$ is an orthonormal frame for $u^{-1}TN$. As another consequence of the compact embedding of $W^{1,2}(M;\mathbb{R}^n)$ into $L^2(M;\mathbb{R}^n)$ and the fact that $u, e^i \in L^\infty(M;\mathbb{R}^n)$ we deduce that, as $\ell \to \infty$,

$$\omega_\ell^{ij} \rightharpoonup \omega^{ij} = e^{i,\mathrm{T}} \nabla e^j \quad \text{in } L^2(M;\mathbb{R}^2), \qquad \vartheta_\ell^j \rightharpoonup \vartheta^j = e^{j,\mathrm{T}} \nabla u \quad \text{in } L^2(M;\mathbb{R}^2).$$

We also have that

$$b_\ell^{ij} \rightharpoonup b^{ij} \quad \text{in } W^{1,2}(M), \qquad H_\ell^{ij} \to H^{ij} \quad \text{in } L^2(M;\mathbb{R}^2),$$

where strong convergence of the sequence of harmonic fields $\big(H_\ell^{ij}\big)_{\ell\in\mathbb{N}}$ follows from the fact that it belongs to a finite-dimensional subspace of $L^2(M;\mathbb{R}^n)$. Therefore, we deduce that

$$\omega^{ij} = \operatorname{Curl} b^{ij} + H^{ij}.$$

We want to pass to the limit in (1.3) as $\ell \to \infty$ to show that $u$ is a harmonic map into $N$. This is straightforward for the second and third term on the left-hand side of (1.3). To identify the limit of the first term, we notice that by definition of $\vartheta_\ell^{ij}$, we have

$$\mathrm{Curl}\, b_\ell^{ij} \cdot \vartheta_\ell^j = \sum_{\alpha=1}^{n} e^{j,\alpha} \mathrm{Curl}\, b_\ell^{ij} \cdot \nabla u_\ell^\alpha$$

and the right-hand side has a Jacobian structure so that a result from concentration and compensation compactness, see [Lio85], based on Wente's inequality in a periodic setting implies that

$$\mathrm{Curl}\, b_\ell^{ij} \cdot \vartheta_\ell^j \to \mathrm{Curl}\, b^{ij} \cdot \vartheta^j + \sum_{\iota \in \mathbb{N}} s_\iota \delta_{x_\iota}$$

in the sense of distributions; we refer the reader to [FMS98] for a detailed discussion. Here, $\big(x_\iota\big)_{\iota \in \mathbb{N}} \subset M$ and $\big(s_\iota\big)_{\iota \in \mathbb{N}} \subseteq \mathbb{R}$ satisfies $\sum_{\iota \in \mathbb{N}} |s_\iota| < \infty$. A combination of the limits identified above implies that

$$\sum_{j=1}^{k} \big(\omega^{ij} \cdot \vartheta^j; \eta\big) + \big(\vartheta^j; \nabla \eta\big) = \sum_{\iota \in \mathbb{N}} s_\iota \eta(x_\iota) \tag{1.4}$$

for all $\eta \in C_c^\infty(M)$. Since the left-hand side of (1.4) belongs to $L^1(M) + H^{-1}(M)$ which does not contain Dirac measures, one can show that the right-hand side of (1.4) has to vanish identically, i.e., $s_\iota = 0$ for all $\iota \in \mathbb{N}$. Again we refer the reader to [FMS98] for details. Proposition 1.7.2 implies that $u$ is a harmonic map into $N$ and concludes this outline of the compactness result into general targets due to [FMS98].

The moving frame technique employed in the weak compactness proof is an elegant tool for the analysis of harmonic maps into a large class of targets but seems restricted to two-dimensional submanifolds $M$ and is rather indirect. A new compactness result has recently been established in [Riv07] and avoids the choice of a moving frame. Assuming here that for ease of presentation, $N$ is of codimension one with unit normal $\nu$, the result uses the fact that $u \in W^{1,2}(M; \mathbb{R}^n)$ is a harmonic map into the hypersurface $N$ if and only if $u(x) \in N$ for almost every $x \in M$ and

$$-\Delta u^i = \sum_{j=1}^{n} \Omega^{ij} \nabla u^j \tag{1.5}$$

with $\Omega^{ij} \in \mathbb{R}^2$ given for $i, j = 1, 2, ..., n$ and $\widehat{\nu} := \nu \circ u$ by

$$\Omega^{ij} = \widehat{\nu}^i \nabla \widehat{\nu}^j - \widehat{\nu}^j \nabla \widehat{\nu}^i.$$

The vectors $\Omega^{ij}$ are skew-symmetric in the sense that $\Omega^{ij} = -\Omega^{ji}$. This skew-symmetry is the key to proving regularity and weak compactness results for harmonic maps from two-dimensional submanifolds into compact $C^2$ submanifolds $N \subset \mathbb{R}^n$ without boundary. This is an improvement since the result by [FMS98] summarized above generally requires $C^4$ regularity of the target manifold. It is interesting to note that the regularity result of [Riv07] based on the skew-symmetry of $\Omega^{ij}$ is optimal in the sense that weak solutions of (1.5) with symmetric $\Omega^{ij}$ can have singularities, see [Fre73] for an explicit example.

Though weak compactness results for harmonic maps into general targets are only known for two-dimensional domains $M$ and $C^2$ targets $N$, no counterexamples for failure of weak compactness of harmonic maps are known.

## 2.2 Weak accumulation of periodic discrete harmonic maps in 2D

We aim at adapting the weak convergence result for harmonic maps into general targets outlined in Section 2.1 to a finite element setting following ideas in [MSŠ97] for the analysis of a finite difference scheme on planar lattices. We assume in this section that

$$M = M_h = \mathbb{T}^2$$

is the two-dimensional torus with fundamental domain $[0,1]^2$. This assumption allows to employ Wente's inequality [Wen69] which is needed for a critical limit passage. A reduction of the general case $M \subset \mathbb{R}^2 \times \{0\}$ to the periodic setting will be investigated below in Section 2.3; validity of the result on two-dimensional curved surfaces is discussed in Section 2.4. We recall that the subscript # indicates periodicity of discrete functions.

### 2.2.1 Discrete Hodge system

We begin with an equivalent characterization of discrete harmonic maps similar to the one given in Proposition 1.7.2.

**Definition 2.2.1.** *Let $u_h \in \mathcal{S}^1_\#(\mathcal{T}_h)^n$ be such that $u_h(z) \in N$ for all $z \in \mathcal{N}_h$. Suppose that $\big(e_h^i\big)_{i=1,2,...,k} \subset \mathcal{S}^1_\#(\mathcal{T}_h)^n$ is an* orthonormal frame *for*

$$u_h^{-1}TN := \Big\{ w_h \in \mathcal{S}^1_\#(\mathcal{T}_h)^n : w_h(z) \in T_{u_h(z)}N \text{ for all } z \in \mathcal{N}_h \Big\},$$

*i.e., for all $z \in \mathcal{N}_h$ the vectors $e_h^i(z)$, $i = 1,2,...,k$, form an orthonormal basis for $T_{u_h(z)}N$. For $i,j = 1,2,...,k$ define $\omega_h^{ij}, \overline{\omega}_h^{ij}, \vartheta_h^i, \overline{\vartheta}_h^i \in L^2(M;\mathbb{R}^2)$ by*

$$\omega_h^{ij} := \sum_{\alpha=1}^n \mathbf{A}^{\mathrm{T}}\big(e_h^{j,\alpha}\big)\nabla e_h^{i,\alpha}, \qquad \overline{\omega}_h^{ij} := \sum_{\alpha=1}^n e_h^{j,\alpha}\nabla e_h^{i,\alpha},$$

*and*

$$\vartheta_h^i := \sum_{\alpha=1}^n \mathbf{A}^{\mathrm{T}}\big(e_h^{i,\alpha}\big)\nabla u_h^\alpha, \qquad \overline{\vartheta}_h^i := \sum_{\alpha=1}^n e_h^{i,\alpha}\nabla u_h^\alpha,$$

*where $\mathbf{A}$ is as in Lemma 1.4.6.*

Up to error terms, the characterization of harmonic maps given in Proposition 1.7.2 holds also in the discrete setting. Notice that we do not assume that $N$ is orientable in the following lemma; continuous unit normals are only required to exist locally.

**Lemma 2.2.2.** *Suppose that $u_h \in \mathcal{S}^1_\#(\mathcal{T}_h)^n$ satisfies $u_h(z) \in N$ for all $z \in \mathcal{N}_h$ and let $\big(e_h^i\big)_{i=1,2,...,k} \subset \mathcal{S}^1_\#(\mathcal{T}_h)^n$ be an orthonormal frame for $u_h^{-1}TN$. Then, for $i = 1,2,...,k$ and all $\eta_h \in \mathcal{S}^1_\#(\mathcal{T}_h)$ we have*

$$\begin{aligned}&\big(\nabla u_h; \nabla \mathcal{I}_h\big[\eta_h e_h^i\big]\big)\\ &\qquad = \sum_{j=1}^k \big(\overline{\vartheta}_h^j \cdot \overline{\omega}_h^{ij}; \eta_h\big) + \big(\vartheta_h^i; \nabla\eta_h\big) + \Lambda_1\big(u_h, e_h^i, \eta_h\big) + \Lambda_2\big(u_h, e_h^i, \eta_h\big) + \Lambda_3\big(u_h, e_h^i, \eta_h\big),\end{aligned}$$

*where the error terms $\Lambda_1, \Lambda_2, \Lambda_3$ are defined by*

$$\Lambda_1\big(u_h, e_h^i, \eta_h\big) := \sum_{\alpha=1}^{n} \big(\nabla u_h^\alpha; \big[\mathbf{A}(\eta_h) - \eta_h \mathbf{I}\big] \nabla e_h^{i,\alpha}\big),$$

$$\Lambda_2\big(u_h, e_h^i, \eta_h\big) := \sum_{\gamma=1}^{2} \sum_{j=1}^{k} \big(\big[\mathcal{I}_h\big(e_h^j \otimes e_h^j\big) - e_h^j \otimes e_h^j\big] \partial_\gamma u_h; \eta_h \partial_\gamma e_h^i\big),$$

$$\Lambda_3\big(u_h, e_h^i, \eta_h\big) := \sum_{\gamma=1}^{2} \sum_{\ell=k+1}^{n} \big(\mathcal{I}_h\big(\nu_h^\ell \otimes \nu_h^\ell\big) \partial_\gamma u_h; \eta_h \partial_\gamma e_h^i\big),$$

*and where $\big(\nu_h^\ell\big)_{\ell=k+1,\ldots,n} \subset \mathcal{S}^1_\#(\mathcal{T}_h)^n$ is such that for all $z \in \mathcal{N}_h$ the vectors*

$$e_h^1(z), \ldots, e_h^k(z),\ \nu_h^{k+1}(z), \ldots, \nu_h^n(z)$$

*form an orthonormal basis of $\mathbb{R}^n$.*

*Proof.* Fix $1 \le i \le k$. Owing to Lemma 1.4.6 we have for $\alpha = 1, 2, \ldots, n$ that

$$\nabla \mathcal{I}_h\big[\eta_h e_h^{i,\alpha}\big] = \mathbf{A}(\eta_h) \nabla e_h^{i,\alpha} + \mathbf{A}\big(e_h^{i,\alpha}\big) \nabla \eta_h.$$

Hence, it follows that

$$\begin{aligned} \big(\nabla u_h; \nabla \mathcal{I}_h\big[\eta_h e_h^i\big]\big) &= \sum_{\alpha=1}^{n} \big(\nabla u_h^\alpha; \nabla \mathcal{I}_h\big[\eta_h e_h^{i,\alpha}\big]\big) \\ &= \sum_{\alpha=1}^{n} \big(\nabla u_h^\alpha; \eta_h \nabla e_h^{i,\alpha}\big) + \sum_{\alpha=1}^{n} \big(\nabla u_h^\alpha; \big[\mathbf{A}(\eta_h) - \eta_h \mathbf{I}\big] \nabla e_h^{i,\alpha}\big) + \sum_{\alpha=1}^{n} \big(\nabla u_h^\alpha; \mathbf{A}\big(e_h^{i,\alpha}\big) \nabla \eta_h\big) \\ &= \sum_{\alpha=1}^{n} \big(\nabla u_h^\alpha; \eta_h \nabla e_h^{i,\alpha}\big) + \Lambda_1\big(u_h, e_h^i, \eta_h\big) + \big(\vartheta_h^i; \nabla \eta_h\big). \end{aligned}$$

The choice of the vector fields $\big(e_h^i\big)_{i=1,2,\ldots,k}$ and $\big(\nu_h^\ell\big)_{\ell=k+1,\ldots,n}$ and properties of nodal interpolation yield that

$$\partial_\gamma u_h = \sum_{j=1}^{k} \mathcal{I}_h\big[e_h^j \otimes e_h^j\big] \partial_\gamma u_h + \sum_{\ell=k+1}^{n} \mathcal{I}_h\big[\nu_h^\ell \otimes \nu_h^\ell\big] \partial_\gamma u_h$$

almost everywhere in $M$. Thus, we deduce that almost everywhere in $M$ we have

$$\partial_\gamma u_h \cdot \partial_\gamma e_h^i = \sum_{j=1}^{k} \Big(\mathcal{I}_h\big[e_h^j \otimes e_h^j\big] \partial_\gamma u_h\Big) \cdot \partial_\gamma e_h^i + \sum_{\ell=k+1}^{n} \Big(\mathcal{I}_h\big[\nu_h^\ell \otimes \nu_h^\ell\big] \partial_\gamma u_h\Big) \cdot \partial_\gamma e_h^i.$$

For each $j = 1, 2, \ldots, k$ we rewrite the corresponding contribution to the first sum on the right-hand side of the last equation as

$$\begin{aligned} \Big(\mathcal{I}_h\big[e_h^j \otimes e_h^j\big] \partial_\gamma u_h\Big) \cdot \partial_\gamma e_h^i &= \Big(\big[e_h^j \otimes e_h^j\big] \partial_\gamma u_h\Big) \cdot \partial_\gamma e_h^i + \Big(\Big\{\mathcal{I}_h\big[e_h^j \otimes e_h^j\big] - e_h^j \otimes e_h^j\Big\} \partial_\gamma u_h\Big) \cdot \partial_\gamma e_h^i \\ &= \big(\partial_\gamma u_h \cdot e_h^j\big)\big(\partial_\gamma e_h^i \cdot e_h^j\big) + \Big(\Big\{\mathcal{I}_h\big[e_h^j \otimes e_h^j\big] - e_h^j \otimes e_h^j\Big\} \partial_\gamma u_h\Big) \cdot \partial_\gamma e_h^i. \end{aligned}$$

This implies that

$$
\begin{aligned}
\sum_{\alpha=1}^{n} \big(\nabla u_h^{\alpha}; \eta_h \nabla e_h^{i,\alpha}\big) &= \sum_{\gamma=1}^{2} \big(\partial_\gamma u_h; \eta_h \partial_\gamma e_h^i\big) \\
&= \sum_{j=1}^{k} \sum_{\gamma=1}^{2} \big(\partial_\gamma u_h \cdot e_h^j; \eta_h \partial_\gamma e_h^i \cdot e_h^j\big) + \Lambda_2\big(u_h, e_h^i, \eta_h\big) + \Lambda_3\big(u_h, e_h^i, \eta_h\big) \\
&= \sum_{j=1}^{k} \big(e_h^{j,\mathrm{T}} \nabla u_h; \eta_h e_h^{j,\mathrm{T}} \nabla e_h^i\big) + \Lambda_2\big(u_h, e_h^i, \eta_h\big) + \Lambda_3\big(u_h, e_h^i, \eta_h\big) \\
&= \sum_{j=1}^{k} \big(\overline{\vartheta}_h^j \cdot \overline{\omega}_h^{ij}; \eta_h\big) + \Lambda_2\big(u_h, e_h^i, \eta_h\big) + \Lambda_3\big(u_h, e_h^i, \eta_h\big)
\end{aligned}
$$

and proves the lemma. □

### 2.2.2 Coulomb gauge for the orthonormal frame

The next lemma shows that an optimal choice of the frame $\big(e_h^i\big)_{i=1,2,\dots,k}$ guarantees that $\omega_h^{ij}$ is discrete divergence-free for $1 \le i, j \le k$ up to perturbations. These perturbations vanish identically if each $K \in \mathcal{T}_h$ has a right-angle, cf. Lemma 1.4.7.

**Lemma 2.2.3.** *Let $\big(e_h^i\big)_{i=1,2,\dots,k}$ be minimal for*

$$
\big(\widehat{e}_h^i\big)_{i=1,2,\dots,k} \mapsto \frac{1}{2} \sum_{i=1}^{k} \int_{\mathbb{T}^2} \big|\nabla \widetilde{e}_h^i\big|^2 \,\mathrm{d}x
$$

*among all orthonormal frames $\big(\widehat{e}_h^i\big)_{i=1,2,\dots,k}$ for $u_h^{-1}TN$. Then,*

$$
\max_{i=1,2,\dots,k} \big\|\nabla e_h^i\big\| \le C \big\|\nabla u_h\big\|.
$$

*and if $\omega_h^{ij}$ from Definition 2.2.1 is defined with such an optimal orthonormal frame we have for all $\phi_h \in \mathcal{S}^1_{\#}(\mathcal{T}_h)$ that*

$$
\big(\omega_h^{ij}; \nabla \phi_h\big) = \Lambda_4\big(e_h^i, e_h^j, \phi_h\big) + \Lambda_5\big(e_h^i, e_h^j, \phi_h\big),
$$

*where*

$$
\Lambda_4\big(e_h^i, e_h^j, \phi_h\big) := \frac{1}{4} \sum_{\alpha=1}^{n} \Big\{ \big(\nabla e_h^{i,\alpha}; \big[\mathbf{A}(\phi_h) - \mathbf{A}^{\mathrm{T}}(\phi_h)\big] \nabla e_h^{j,\alpha}\big) - \big(\nabla e_h^{j,\alpha}; \big[\mathbf{A}(\phi_h) - \mathbf{A}^{\mathrm{T}}(\phi_h)\big] \nabla e_h^{i,\alpha}\big) \Big\},
$$

*and*

$$
\Lambda_5\big(e_h^i, e_h^j, \phi_h\big) := \frac{1}{2} \sum_{\alpha=1}^{n} \Big\{ \big(\big[\mathbf{A}^{\mathrm{T}}\big(e_h^{i,\alpha}\big) - \mathbf{A}\big(e_h^{i,\alpha}\big)\big] \nabla e_h^{j,\alpha}; \nabla \phi_h\big) + \big(\big[\mathbf{A}^{\mathrm{T}}\big(e_h^{j,\alpha}\big) - \mathbf{A}\big(e_h^{j,\alpha}\big)\big] \nabla e_h^{i,\alpha}; \nabla \phi_h\big) \Big\}.
$$

*Proof.* For the continuously differentiable vector fields $\big(\overline{e}^i\big)_{i=1,2,...,k}$ from Lemma 1.6.8 the family $\big(\mathcal{I}_h\big[\overline{e}^i\circ u_h\big]\big)_{i=1,2,...,k}$ is an orthonormal frame for $u_h^{-1}TN$ satisfying

$$\big\|\nabla\mathcal{I}_h\big[\overline{e}^i\circ u_h\big]\big\| \le C\big\|\nabla\big[\overline{e}^i\circ u_h\big]\big\| \le C\big\|\nabla u_h\big\|\,\big\|D\overline{e}^i\big\|_{L^\infty(\mathbb{R}^n)} \le C\big\|\nabla u_h\big\|.$$

Since the optimal frame is minimal among all possible frames, we verify that $\big\|\nabla e_h^i\big\|\le C\big\|\nabla u_h\big\|$ for $1\le i\le k$. Given any $\mathbf{S}_h\in\mathcal{S}^1_\#(\mathcal{T}_h)^{k\times k}$ satisfying $\mathbf{S}_h(z)\in SO(k)$, the family $\big(\widetilde{e}_h^i\big)_{i=1,2,...,k}$ defined by $\widetilde{e}_h^i:=\mathcal{I}_h\big[\sum_{j=1}^k\mathbf{S}_h^{ij}e_h^j\big]$ is again an orthonormal frame for $u_h^{-1}TN$. Hence, since $\big(e_h^i\big)_{i=1,2,...,k}$ is minimal, the constant mapping $\mathbf{I}_{k\times k}\in\mathcal{S}^1_\#(\mathcal{T}_h)^{k\times k}$ is minimal for

$$\mathbf{S}_h\mapsto\frac{1}{2}\sum_{i=1}^k\int_\Omega\Big|\nabla\mathcal{I}_h\Big[\sum_{j=1}^k\mathbf{S}_h^{ij}\widetilde{e}_h^j\Big]\Big|^2\,\mathrm{d}x \tag{2.6}$$

among all $\mathbf{S}_h\in\mathcal{S}^1_\#(\mathcal{T}_h)^{k\times k}$ satisfying $\mathbf{S}_h(z)\in SO(k)$ for all $z\in\mathcal{N}_h$. Noting that $T_{\mathbf{I}_{k\times k}}SO(k)=so(k)=\{\mathbf{r}\in\mathbb{R}^{k\times k}:\mathbf{r}^{ij}=-\mathbf{r}^{ij}\,\text{for}\;i,j=1,2,...,k\}$, we have for all $\mathbf{r}_h\in\mathcal{S}^1_\#(\mathcal{T}_h)^{k\times k}$ satisfying $\mathbf{r}_h(z)\in so(k)$ for all $z\in\mathcal{N}_h$ that

$$\begin{aligned}
0&=\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\frac{1}{2}\sum_{i=1}^k\int_M\Big|\nabla\mathcal{I}_h\Big[\sum_{j=1}^k\big(\mathbf{I}_{k\times k}^{ij}+\varepsilon\mathbf{r}_h^{ij}\big)e_h^j\Big]\Big|^2\,\mathrm{d}x\\
&=\sum_{i=1}^k\Big(\nabla e_h^i;\nabla\mathcal{I}_h\Big[\sum_{j=1}^k\mathbf{r}_h^{ij}e_h^j\Big]\Big)\\
&=\sum_{i,j=1}^k\sum_{\alpha=1}^n\big(\nabla e_h^{i,\alpha};\nabla\mathcal{I}_h\big[\mathbf{r}_h^{ij}e_h^{j,\alpha}\big]\big)\\
&=\sum_{i,j=1}^k\sum_{\alpha=1}^n\Big\{\big(\nabla e_h^{i,\alpha};\mathbf{A}\big(\mathbf{r}_h^{ij}\big)\nabla e_h^{j,\alpha}\big)+\big(\nabla e_h^{i,\alpha};\mathbf{A}\big(e_h^{j,\alpha}\big)\nabla\mathbf{r}_h^{ij}\big)\Big\}\\
&=\sum_{i,j=1}^k\sum_{\alpha=1}^n\Big\{\big(\nabla e_h^{i,\alpha};\mathbf{A}^{sym}\big(\mathbf{r}_h^{ij}\big)\nabla e_h^{j,\alpha}\big)\\
&\qquad+\frac{1}{2}\big(\nabla e_h^{i,\alpha};\big[\mathbf{A}\big(r_h^{ij}\big)-\mathbf{A}^{\mathrm{T}}\big(\mathbf{r}_h^{ij}\big)\big]\nabla e_h^{j,\alpha}\big)+\big(\mathbf{A}^{\mathrm{T}}\big(e_h^{j,\alpha}\big)\nabla e_h^{i,\alpha};\nabla\mathbf{r}_h^{ij}\big)\Big\},
\end{aligned}$$

where $\mathbf{A}^{sym}\big(\mathbf{r}_h^{ij}\big):=\big\{\mathbf{A}\big(\mathbf{r}_h^{ij}\big)+\mathbf{A}^{\mathrm{T}}\big(\mathbf{r}_h^{ij}\big)\big\}/2$. Upon noting that $\mathbf{r}_h$ is skew-symmetric almost everywhere in $M$ and that $\mathbf{A}\big(\mathbf{r}_h^{ij}\big)$ depends linearly on $\mathbf{r}_h^{ij}$, we infer that

$$\sum_{i,j=1}^k\sum_{\alpha=1}^n\big(\nabla e_h^{i,\alpha};\mathbf{A}^{sym}\big(\mathbf{r}_h^{ij}\big)\nabla e_h^{j,\alpha}\big)=0.$$

Recalling the definition of $\omega_h^{ij}$ we thus deduce that

$$\sum_{i,j=1}^k\Big\{\big(\omega_h^{ij};\nabla\mathbf{r}_h^{ij}\big)+\frac{1}{2}\sum_{\alpha=1}^n\big(\nabla e_h^{i,\alpha};\big[\mathbf{A}\big(\mathbf{r}_h^{ij}\big)-\mathbf{A}^{\mathrm{T}}\big(\mathbf{r}_h^{ij}\big)\big]\nabla e_h^{j,\alpha}\big)\Big\}=0. \tag{2.7}$$

Given $\phi_h \in \mathcal{S}^1_\#(\mathcal{T}_h)$ and $1 \le i,j \le k$, we define $\mathbf{r}_h \in \mathcal{S}^1_\#(\mathcal{T}_h)^{k\times k}$ by setting $\mathbf{r}_h^{ij} := \phi_h$, $\mathbf{r}_h^{ji} := -\phi_h$, and $\mathbf{r}_h^{i'j'} := 0$ for $(i',j') \notin \{(i,j),\,(j,i)\}$. Then, $\mathbf{r}_h(z) \in so(k)$ for all $z \in \mathcal{N}_h$ and (2.7) yields that

$$\big(\omega_h^{ij};\nabla\phi_h\big) - \big(\omega_h^{ji};\nabla\phi_h\big) + 2\Lambda_4\big(e_h^i, e_h^j, \phi_h\big) = 0,$$

or equivalently,

$$\big(\omega_h^{ij};\nabla\phi_h\big) = \Lambda_4\big(e_h^i, e_h^j, \phi_h\big) + \frac{1}{2}\big(\omega_h^{ji} + \omega_h^{ji};\nabla\phi_h\big).$$

Notice that almost everywhere in $M$ we have that

$$\begin{aligned}
0 = \nabla\mathcal{I}_h\big[e_h^i\cdot e_h^j\big] &= \nabla\sum_{\alpha=1}^n \mathcal{I}_h\big[e_h^{i,\alpha}e_h^{j,\alpha}\big]\\
&= \sum_{\alpha=1}^n\Big\{\mathbf{A}\big(e_h^{i,\alpha}\big)\nabla e_h^{j,\alpha} + \mathbf{A}\big(e_h^{j,\alpha}\big)\nabla e_h^{i,\alpha}\Big\}\\
&= \sum_{\alpha=1}^n\Big\{\mathbf{A}^{\mathrm{T}}\big(e_h^{i,\alpha}\big)\nabla e_h^{j,\alpha} + \mathbf{A}^{\mathrm{T}}\big(e_h^{j,\alpha}\big)\nabla e_h^{i,\alpha}\Big\}\\
&\quad+\sum_{\alpha=1}^n\Big\{\Big[\mathbf{A}\big(e_h^{i,\alpha}\big) - \mathbf{A}^{\mathrm{T}}\big(e_h^{i,\alpha}\big)\Big]\nabla e_h^{j,\alpha} + \Big[\mathbf{A}\big(e_h^{j,\alpha}\big) - \mathbf{A}^{\mathrm{T}}\big(e_h^{j,\alpha}\big)\Big]\nabla e_h^{i,\alpha}\Big\}\\
&= \omega_h^{ji} + \omega_h^{ij} - \sum_{\alpha=1}^n\Big\{\Big[\mathbf{A}^{\mathrm{T}}\big(e_h^{i,\alpha}\big) - \mathbf{A}\big(e_h^{i,\alpha}\big)\Big]\nabla e_h^{j,\alpha} + \Big[\mathbf{A}^{\mathrm{T}}\big(e_h^{j,\alpha}\big) - \mathbf{A}\big(e_h^{j,\alpha}\big)\Big]\nabla e_h^{i,\alpha}\Big\}.
\end{aligned}$$

The combination of the last two identities implies the lemma. □

### 2.2.3 Bounds on the error terms

We next incorporate discrete Hodge decompositions of the connection forms $\omega_h^{ij}$ in Coulomb gauge and provide bounds on various error terms.

**Lemma 2.2.4.** *For $1 \le i,j \le k$ let $a_h^{ij} \in \mathcal{S}^1_\#(\mathcal{T}_h)$, $b_h^{ij} \in \mathcal{S}^{1,NC}_\#(\mathcal{T}_h)$, and $H_h^{ij} \in \mathcal{H}_\#(\mathcal{T}_h;\mathbb{R}^2)$ be the components of the discrete Helmholtz decomposition of $\omega_h^{ij}$ according to Proposition 1.5.7. Define $\widehat{b}_h^{ij} := \mathcal{A}_h b_h^{ij} \in \mathcal{S}^1_\#(\mathcal{T}_h)$. Then, for all $\eta_h \in \mathcal{S}^1_\#(\mathcal{T}_h)$ we have*

$$\begin{aligned}
\big(\nabla u_h;\nabla\mathcal{I}_h\big[\eta_h e_h^i\big]\big) &= \sum_{j=1}^k\Big\{\big(\mathrm{Curl}\,\widehat{b}_h^{ij}\cdot\overline{\vartheta}_h^j;\eta_h\big) + \big(H_h^{ij}\cdot\overline{\vartheta}_h^j;\eta_h\big)\Big\} + \big(\vartheta_h^i;\nabla\eta_h\big)\\
&\quad+\Lambda_{1,2,3}\big(u_h,e_h^i,\eta_h\big) + \sum_{j=1}^k\Lambda_{4,5}\big(e_h^i,e_h^j,\psi_h^j\big) + \sum_{j=1}^k\Big\{\Theta_1\big(u_h,e_h^i,e_h^j,\eta_h\big) + \Theta_2\big(u_h,e_h^i,e_h^j,\eta_h\big)\Big\}
\end{aligned}$$

*where $\psi_h^j := \mathcal{G}_h\big[\eta_h\overline{\vartheta}_h^j\big] \in \mathcal{S}^1_\#(\mathcal{T}_h)^n$ satisfies $\int_{\mathbb{T}^2}\psi_h^j\,\mathrm{d}x = 0$ and*

$$\big(\nabla\psi_h;\nabla v_h\big) = \big(\vartheta_h^j\eta_h;\nabla v_h\big)$$

*for all* $v_h \in \mathcal{S}^1_\#(\mathcal{T}_h)^n$*. We abbreviated* $\Lambda_{1,2,3} := \Lambda_1 + \Lambda_2 + \Lambda_3$ *and* $\Lambda_{4,5} := \Lambda_4 + \Lambda_5$ *with the error terms from Lemma 2.2.2 and Lemma 2.2.3, respectively, and the additional terms* $\Theta_1$ *and* $\Theta_2$ *are given by*

$$\Theta_1\big(u_h, e_h^i, e_h^j, \eta_h\big) := \big(\big[\overline{\omega}_h^{ij} - \omega_h^{ij}\big] \cdot \overline{\vartheta}_h^j; \eta_h\big)$$
$$\Theta_2\big(u_h, e_h^i, e_h^j, \eta_h\big) := \big(\big[\mathrm{Curl}_{\mathcal{T}_h} b_h^{ij} - \mathrm{Curl}\, \widehat{b}_h^{ij}\big] \cdot \overline{\vartheta}_h^j; \eta_h\big).$$

*Proof.* Owing to the definition of $a_h^{ij}$ and $\psi_h^j$ we have

$$\big(\nabla a_h^{ij} \cdot \overline{\vartheta}_h^j; \eta_h\big) = \big(\nabla a_h^{ij}; \nabla \psi_h^j\big) = \big(\omega_h^{ij}; \nabla \psi_h^j\big) = \Lambda_{4,5}\big(e_h^i, e_h^j, \psi_h^j\big).$$

According to Lemma 2.2.2 we thus have

$$\begin{aligned}
\big(\nabla u_h; \nabla \mathcal{I}_h\big[\eta_h e_h^i\big]\big) &= \sum_{j=1}^k \big(\overline{\omega}_h^{ij} \cdot \overline{\vartheta}_h^j; \eta_h\big) + \big(\vartheta_h^i; \nabla \eta_h\big) + \Lambda_{1,2,3}\big(u_h, e_h^i, \eta_h\big) \\
&= \sum_{j=1}^k \big(\omega_h^{ij} \cdot \overline{\vartheta}_h^j; \eta_h\big) + \big(\vartheta_h^i; \nabla \eta_h\big) + \Lambda_{1,2,3}\big(u_h, e_h^i, \eta_h\big) + \sum_{j=1}^k \Theta_1\big(u_h, e_h^i, e_h^j, \eta_h\big) \\
&= \sum_{j=1}^k \Big\{\big(\mathrm{Curl}\, \widehat{b}_h^{ij} \cdot \overline{\vartheta}_h^j; \eta_h\big) + \big(H_h^{ij} \cdot \overline{\vartheta}_h^j; \eta_h\big)\Big\} + \big(\vartheta_h^i; \nabla \eta_h\big) \\
&\quad + \Lambda_{1,2,3}\big(u_h, e_h^i, \eta_h\big) + \Lambda_{4,5}\big(e_h^i, e_h^j, \psi_h^j\big) \\
&\quad + \sum_{j=1}^k \Big\{\Theta_1\big(u_h, e_h^i, e_h^j, \eta_h\big) + \Theta_2\big(u_h, e_h^i, e_h^j, \eta_h\big)\Big\},
\end{aligned}$$

which proves the assertion. ☐

**Lemma 2.2.5.** *Suppose that there exists* $C > 0$ *such that for all* $h > 0$ *we have* $||\nabla u_h|| \leq C$ *and* $\max_{i=1,2,\ldots,k} ||\nabla e_h^i|| \leq C$*. For* $1 \leq i \leq k$ *there exists* $f_h^i \in \mathcal{S}^1_\#(\mathcal{T}_h)$ *such that for all* $\eta_h \in \mathcal{S}^1_\#(\mathcal{T}_h)$ *we have*

$$\big(f_h^i; \eta_h\big)_h = \Lambda_{1,2,3}\big(u_h, e_h^i, \eta_h\big) + \sum_{j=1}^k \Big\{\Theta_1\big(u_h, e_h^i, e_h^j, \eta_h\big) + \Theta_2\big(u_h, e_h^i, e_h^j, \eta_h\big)\Big\} \tag{2.8}$$

*and*

$$\big(f_h^i; \eta_h\big)_h \leq C \|\eta_h\|_{L^\infty(\mathbb{T}^2)}. \tag{2.9}$$

*Moreover, for all* $\eta_h \in \mathcal{S}^1_\#(\mathcal{T}_h)$ *we have*

$$\big(f_h^i; \eta_h\big)_h \leq Ch \big\|\nabla \eta_h\big\|_{L^\infty(\mathbb{T}^2)} + C \sum_{z \in \mathcal{N}_h} h_z \max_{j=1,2,\ldots,k} \big\|\nabla e_h^i\big\|_{L^\infty(\omega_z)} \gamma_{h,z}^2 \,|\eta_h(z)|, \tag{2.10}$$

*where for each* $z \in \mathcal{N}_h$

$$\gamma_{h,z} := \max\big\{\big\|\nabla u_h\big\|_{L^2(\omega_z)}, \big\|\nabla e_h^1\big\|_{L^2(\omega_z)}, \ldots, \big\|\nabla e_h^k\big\|_{L^2(\omega_z)}, \big\|\mathrm{Curl}_{\mathcal{T}_h} b_h^{ij}\big\|_{L^2(\widehat{\omega}_z)}\big\}.$$

*Proof.* One directly verifies that the right-hand side of (2.8) defines a linear functional on $\mathcal{S}^1_\#(\mathcal{T}_h)$ (in $\eta_h$). Hence there exists $f_h^i \in \mathcal{S}^1_\#(\mathcal{T}_h)$ such that (2.8) holds for all $\eta_h \in \mathcal{S}^1_\#(\mathcal{T}_h)$. To prove the asserted bounds for $f_h^i$ we estimate each contribution to the right-hand side of (2.8) separately. Without further notice we will use the assumed bound for $\|\nabla u_h\|$ and $\|\nabla e_h^i\|$.

(i) According to the definition and properties of the matrix $\mathbf{A}(\eta_h)$ discussed in Lemma 1.4.6 we have

$$\Lambda_1(u_h, e_h^i, \eta_h) \le C\|\eta_h\|_{L^\infty(\mathbb{T}^2)},$$

as well as

$$\Lambda_1(u_h, e_h^i, \eta_h) \le Ch\big\|\nabla\eta_h\big\|_{L^\infty(\mathbb{T}^2)}.$$

(ii) Since $|e_h^j| \le 1$ almost everywhere in $\mathbb{T}^2$ we have

$$\Lambda_2(u_h, e_h^i, \eta_h) \le C\|\eta_h\|_{L^\infty(\mathbb{T}^2)}.$$

The interpolation estimate $\big\|\mathcal{I}_h\big(e_h^j \otimes e_h^j\big) - e_h^j \otimes e_h^j\big\|_{L^2(\omega_z)} \le Ch_z\|e_h^j\|_{L^\infty(\omega_z)}\big\|\nabla e_h^j\big\|_{L^2(\omega_z)}$ yields that

$$\begin{aligned}\Lambda_2(u_h, e_h^i, \eta_h) &= \sum_{z\in\mathcal{N}_h}\eta_h(z)\sum_{\gamma=1}^{2}\sum_{j=1}^{k}\big(\big[\mathcal{I}_h\big(e_h^j\otimes e_h^j\big) - e_h^j\otimes e_h^j\big]\partial_\gamma u_h;\varphi_z\partial_\gamma e_h^i\big)\\ &\le C\sum_{z\in\mathcal{N}_h} h_z\big\|\nabla e_h^i\big\|_{L^\infty(\omega_z)}\gamma_{h,z}^2\,|\eta_h(z)|.\end{aligned}$$

(iii) Since $|\nu_h^\ell| \le 1$ almost everywhere in $\mathbb{T}^2$ we have

$$\Lambda_3(u_h, e_h^i, \eta_h) \le C\|\eta_h\|_{L^\infty(\mathbb{T}^2)}.$$

Lemma 1.6.9 implies that $\big|\nu_h^\ell\cdot\partial_\gamma u_h\big| \le Ch_K\big|\nabla u_h\big|^2$ on each $K\in\mathcal{T}_h$ so that

$$\Lambda_3(u_h, e_h^i, \eta_h) \le C\sum_{z\in\mathcal{N}_h} h_z\big\|\nabla e_h^i\big\|_{L^\infty(\omega_z)}\gamma_{h,z}^2\,|\eta_h(z)|.$$

(iv) With the definitions of $\overline{\omega}_h^{ij}$, $\omega_h^{ij}$, and $\overline{\vartheta}_h^j$ and Lemma 1.4.6 we verify that

$$\begin{aligned}\Theta_1\big(u_h, e_h^i, e_h^j, \eta_h\big) &= \big(\overline{\omega}_h^{ij} - \omega_h^{ij};\eta_h\overline{\vartheta}_h^j\big)\\ &= \sum_{\alpha=1}^{n}\Big(e_h^{i,\alpha}\nabla e_h^{j,\alpha} - \mathbf{A}^{\mathrm{T}}\big(e_h^{i,\alpha}\big)\nabla e_h^{j,\alpha};\eta_h\overline{\vartheta}_h^j\Big)\\ &\le C\|\eta_h\|_{L^\infty(\mathbb{T}^2)}.\end{aligned}$$

Incorporating the estimate of Lemma 1.4.6 we find that

$$\begin{aligned}\Theta_1\big(u_h, e_h^i, e_h^j, \eta_h\big) &= \sum_{z\in\mathcal{N}_h}\eta_h(z)\big(\overline{\omega}_h^{ij} - \omega_h^{ij};\varphi_z\overline{\vartheta}_h^j\big)\\ &\le C\sum_{z\in\mathcal{N}_h}\max_{\alpha=1,2,\dots,n}\|\mathbf{A}\big(e_h^{i,\alpha}\big) - e_h^{i,\alpha}\mathbf{I}\|_{L^\infty(\omega_z)}\big\|\nabla e_h^j\big\|_{L^2(\omega_z)}\|e_h^j\|_{L^\infty(\omega_z)}\big\|\nabla u_h\big\|_{L^2(\omega_z)}|\eta_h(z)|\\ &\le C\sum_{z\in\mathcal{N}_h} h_z\big\|\nabla e_h^i\big\|_{L^\infty(\omega_z)}\gamma_{h,z}^2\,|\eta_h(z)|.\end{aligned}$$

(v) Owing to inverse estimates, Lemma 1.5.8, and the definition of $b_h^{ij}$ we have that

$$\|\mathrm{Curl}_{\mathcal{T}_h} b_h^{ij} - \mathrm{Curl}\,\widehat{b}_h^{ij}\| \le C\|\mathrm{Curl}_{\mathcal{T}_h} b_h^{ij}\| \le C\|\omega_h^{ij}\| \le C.$$

Therefore, we deduce that

$$\Theta_2\big(u_h, e_h^i, e_h^j, \eta_h\big) = \big(\mathrm{Curl}_{\mathcal{T}_h} b_h^{ij} - \mathrm{Curl}\,\widehat{b}_h^{ij}; \eta_h \overline{\vartheta}_h^j\big) \le C\|\eta_h\|_{L^\infty(\mathbb{T}^2)}.$$

Using that

$$\mathrm{curl}\,\big(\eta_h \overline{\vartheta}_h^j\big) = \sum_{\alpha=1}^n \Big\{\mathrm{Curl}\,\eta_h \cdot \big[e_h^{j,\alpha} \nabla u_h^\alpha\big] + \eta_h \big[\mathrm{Curl}\, e_h^{j,\alpha}\big] \cdot \nabla u_h^\alpha\Big\}$$

we infer with a $\mathcal{T}_h$-elementwise integration by parts that

$$\begin{aligned}\Theta_2\big(u_h, e_h^i, e_h^j, \eta_h\big) &= \big(\mathrm{Curl}_{\mathcal{T}_h} b_h^{ij} - \mathrm{Curl}\,\widehat{b}_h^{ij}; \eta_h \overline{\vartheta}_h^j\big)\\ &= -\sum_{\alpha=1}^n \Big\{\big(b_h^{ij} - \widehat{b}_h^{ij}; \mathrm{Curl}\,\eta_h \cdot \big[e_h^{j,\alpha}\nabla u_h^\alpha\big]\big) + \big(b_h^{ij} - \widehat{b}_h^{ij}; \eta_h \big[\mathrm{Curl}\, e_h^{j,\alpha}\big] \cdot \nabla u_h^\alpha\big)\Big\}\\ &\quad + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \big(b_h^{ij} - \widehat{b}_h^{ij}\big)\big(\eta_h \overline{\vartheta}_h^j\big) \cdot \tau_K \,\mathrm{d}t,\end{aligned} \tag{2.11}$$

where $\tau_K$ is a unit tangent to $\partial K$ for each $K \in \mathcal{T}_h$. For the first term on the right-hand side of (2.11) we have by Lemma 1.5.8 and with $\|\mathrm{Curl}\,\eta_h\|_{L^\infty(\mathbb{T}^2)} = \|\nabla \eta_h\|_{L^\infty(\mathbb{T}^2)}$ that

$$\begin{aligned}\sum_{\alpha=1}^n \big(b_h^{ij} - \widehat{b}_h^{ij}; \mathrm{Curl}\,\eta_h \cdot \big[e_h^{j,\alpha}\nabla u_h^\alpha\big]\big) &\le Ch\|h_{\mathcal{T}_h}^{-1}\big(b_h^{ij} - \widehat{b}_h^{ij}\big)\| \,\|\nabla\eta_h\|_{L^\infty(\mathbb{T}^2)}\|e_h^j\|_{L^\infty(\mathbb{T}^2)}\|\nabla u_h\|\\ &\le Ch\|\nabla\eta_h\|_{L^\infty(\mathbb{T}^2)}.\end{aligned}$$

The second term on the right-hand side of (2.11) is estimated by

$$\begin{aligned}\sum_{\alpha=1}^n \big(b_h^{ij} - \widehat{b}_h^{ij}; \eta_h\big[\mathrm{Curl}\, e_h^{j,\alpha}\big] \cdot \nabla u_h^\alpha\big) &= \sum_{z\in\mathcal{N}_h}\sum_{\alpha=1}^n \eta_h(z)\big(b_h^{ij} - \widehat{b}_h^{ij}; \varphi_z\big[\mathrm{Curl}\, e_h^{j,\alpha}\big] \cdot \nabla u_h^\alpha\big)\\ &\le C\sum_{z\in\mathcal{N}_h} |\eta_h(z)|\,\|b_h^{ij} - \widehat{b}_h^{ij}\|_{L^2(\omega_z)}\|\nabla e_h^j\|_{L^\infty(\omega_z)}\|\nabla u_h\|_{L^2(\omega_z)}\\ &\le C\sum_{z\in\mathcal{N}_h} |\eta_h(z)|\, h_z\|\mathrm{Curl}_{\mathcal{T}_h} b_h^{ij}\|_{L^2(\widehat{\omega}_z)}\|\nabla e_h^j\|_{L^\infty(\omega_z)}\|\nabla u_h\|_{L^2(\omega_z)}.\end{aligned}$$

Notice that the vector field $\eta_h\overline{\vartheta}_h^j = \eta_h \sum_{\alpha=1}^n e_h^{j,\alpha}\nabla u_h^\alpha$ has continuous tangential components across interelement boundaries and $\widehat{b}_h^{ij}$ is continuous so that the boundary contributions to the right-hand side of (2.11) can be recast as

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K}\big(b_h^{ij} - \widehat{b}_h^{ij}\big)\big(\eta_h\overline{\vartheta}_h^j\big)\cdot\tau_K\,\mathrm{d}t = \sum_{E\in\mathcal{E}_h}\int_E \big[b_h^{ij}\big]\big(\eta_h\overline{\vartheta}_h^j\big)\cdot\tau_E\,\mathrm{d}t.$$

Since $b_h^{ij} \in \mathcal{S}_\#^{1,NC}(\mathcal{T}_h)$, the jump $\big[b_h^{ij}\big]|_E$ across $E$ is affine and vanishes at the midpoint of $E$ where $b_h^{ij}$ is continuous so that $\int_E \big[b_h^{ij}\big]\,\mathrm{d}t = 0$. This enables us to subtract an arbitrary constant $c_E \in \mathbb{R}^2$

from the second factor and to employ a Poincaré inequality on $E$ and discrete trace inequalities to estimate a typical contribution to the right-hand side as

$$\begin{aligned}\int_E \big[b_h^{ij}\big]\big(\eta_h\overline{\vartheta}_h^j\big)\cdot\tau_E\,\mathrm{d}t &= \int_E \big[b_h^{ij}\big]\big(\eta_h\overline{\vartheta}_h^j - c_E\big)\cdot\tau_E\,\mathrm{d}t \\ &\le C h_E \big\|\partial\big[b_h^{ij}\big]/\partial t\big\|_{L^2(E)} h_E^{-1/2}\big\|\eta_h\overline{\vartheta}_h^j - c_E\big\|_{L^2(K_E)} \\ &\le C h_E^{1/2}\big\|\mathrm{Curl}_{\mathcal{T}_h} b_h^{ij}\big\|_{L^2(K_E)} h_E^{1/2}\big\|\nabla\big(\eta_h\overline{\vartheta}_h^j\big)\big\|_{L^2(K_E)}\end{aligned} \tag{2.12}$$

with $K_E\in\mathcal{T}_h$ such that $E\subset\partial K_E$. Noting that $\big|\nabla\big(\eta_h\overline{\vartheta}_h^j\big)\big|\le|\nabla\eta_h|\,\big|\overline{\vartheta}_h^j\big|+|\eta_h|\big|\nabla e_h^j\big|\,\big|\nabla u_h\big|$ we verify with the above estimates that

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}\int_{\partial K}\big(b_h^{ij}-\widehat{b}_h^{ij}\big)\big(\eta_h\overline{\vartheta}_h^j\big)\cdot\tau_K\,\mathrm{d}t \\ &\qquad\le Ch\big\|\nabla\eta_h\big\|_{L^\infty(\mathbb{T}^2)} + C\sum_{z\in\mathcal{N}_h} h_z\big\|\nabla e_h^j\big\|_{L^\infty(\omega_z)}\big\|\mathrm{Curl}_{\mathcal{T}_h} b_h^{ij}\big\|_{L^2(\widehat{\omega}_z)}\big\|\nabla e_h^i\big\|_{L^2(\omega_z)}|\eta_h(z)|.\end{aligned}$$

A combination of the estimates derived in (i)-(v) proves the statement. ☐

### 2.2.4 Convergence as $h\to 0$

With the preparations of the previous lemmas, we can investigate convergence behaviour of the different quantities as the maximal mesh-size decays to zero.

**Lemma 2.2.6.** *Let $\big(\mathcal{T}_h\big)_{h>0}$ be a sequence of logarithmically right-angled triangulations. Let $\Phi\in L^2(\mathbb{T}^2;\mathbb{R}^2)$ and let $\psi_h := \mathcal{G}_h\Phi\in\mathcal{S}^1_\#(\mathcal{T}_h)$ satisfy $\int_{\mathbb{T}^2}\psi_h\,\mathrm{d}x = 0$ and*

$$\big(\nabla\psi_h;\nabla v_h\big) = \big(\Phi;\nabla v_h\big)$$

*for all $v_h\in\mathcal{S}^1_\#(\mathcal{T}_h)$. Then,*

$$\big|\Lambda_4\big(e_h^i,e_h^j,\psi_h\big)\big| + \big|\Lambda_5\big(e_h^i,e_h^j,\psi_h\big)\big|\to 0$$

*as $h\to 0$.*

*Proof.* Owing to the definition of $\Lambda_4$ and Lemma 1.4.7 we have

$$\begin{aligned}\Lambda_4\big(e_h^i,e_h^j,\psi_h\big) &\le C\big\|e_h^i\big\|\,\big\|e_h^j\big\|\,\big\|\mathbf{A}(\psi_h)-\mathbf{A}^{\mathrm{T}}(\psi_h)\big\|_{L^\infty(\mathbb{T}^2)} \\ &\le C\big\|e_h^i\big\|\,\big\|e_h^j\big\|\sup_{K\in\mathcal{T}_h}\min_{\gamma=1,2,3}|\cos\alpha_{K,\gamma}|\,\|\psi_h\|_{L^\infty(\mathbb{T}^2)}.\end{aligned}$$

The inverse estimate of Lemma 1.4.9 guarantees that

$$\|\psi_h\|_{L^\infty(\mathbb{T}^2)}\le C\log h_{min}^{-1}\big\|\nabla\psi_h\big\|$$

and the definition of $\psi_h$ provides the estimate

$$\big\|\nabla\psi_h\big\|\le\|\Phi\|.$$

The combination of the estimates and the definition of logarithmically right-angled triangulations proves the asserted limit for $\Lambda_4$. The same arguments lead to the assertion for $\Lambda_5$ (where the uniform bounds $\big\|e_h^i\big\|_{L^\infty(\mathbb{T}^2)}\le 1$ may be employed for a more direct proof). ☐

**Lemma 2.2.7.** *Let $1 \le i \le k$ and $f_h^i \in \mathcal{S}^1_\#(\mathcal{T}_h)$ be as in Lemma 2.2.5. There exist $\big(t_\iota^i\big)_{\iota\in\mathbb{N}} \subset \mathbb{R}$ and $\big(y_\iota^i\big)_{\iota\in\mathbb{N}} \subset \mathbb{T}^2$ such that $\sum_{\iota\in\mathbb{N}} |t_\iota^i|^{2/3} \le C$ and for an appropriate subsequence (which is not relabeled) and every $\eta \in C^\infty(\mathbb{T}^2)$ we have*

$$\big(f_h^i; \mathcal{I}_h\eta\big)_h \to \sum_{\iota\in\mathbb{N}} t_\iota^i \eta(y_\iota^i).$$

*Proof.* We define $F_h \in C(\mathbb{T}^2)^*$ as

$$F_h(\eta) := \big(f_h^i; \mathcal{I}_h\eta\big)_h.$$

Since $F_h$ is uniformly bounded in $C(\mathbb{T}^2)^*$ there exists $F \in C(\mathbb{T}^2)^*$ such that (for a subsequence) we have $F_h \rightharpoonup^* F$ in $C(\mathbb{T}^2)^*$. We fix $\delta > 0$ and define

$$\sum\nolimits_{\delta,h} := \big\{ z \in \mathcal{N}_h \colon h_z \max_{j=1,2,\ldots,k} \big\|\nabla e_h^j\big\|_{L^\infty(\omega_z)} > \delta \big\}.$$

Then, using that $h_z\big\|\nabla e_h^j\big\|_{L^\infty(\omega_z)} \le C\big\|\nabla e_h^j\big\|_{L^2(\omega_z)}$ for each $z \in \mathcal{N}_h$ we infer that

$$\begin{aligned}
\operatorname{card} \sum\nolimits_{\delta,h} &\le \delta^{-2} \sum_{z\in\mathcal{N}_h} h_z^2 \max_{j=1,2,\ldots,k} \big\|\nabla e_h^j\big\|^2_{L^\infty(\omega_z)} \\
&\le C\delta^{-2} \sum_{z\in\mathcal{N}_h} \sum_{j=1}^{k} \big\|\nabla e_h^j\big\|^2_{L^2(\omega_z)} \le C\delta^{-2} \sum_{j=1}^{k} \big\|\nabla e_h^j\big\|^2 \le C\delta^{-2},
\end{aligned}$$

i.e., the cardinality of the set $\sum_{\delta,h}$ is uniformly bounded with respect to $h$ and therefore, for an appropriate subsequence which is not relabeled we have

$$\sum\nolimits_{\delta,h} \to \sum\nolimits_{\delta} = \big\{x_1^\delta, x_2^\delta, \ldots., x_{L^\delta}^\delta\big\}$$

as $h \to 0$. With $F_h^z := F_h\big(\varphi_z\big) \in \mathbb{R}$ for each $z \in \mathcal{N}_h$ we have

$$F_h(\eta) = \sum_{z\in\mathcal{N}_h} F_h^z \eta(z) = \sum_{z\in\Sigma_{\delta,h}} F_h^z\eta(z) + \sum_{z\in\mathcal{N}_h\setminus\Sigma_{\delta,h}} F_h^z\eta(z) =: F_{\delta,h}^{bad}(\eta) + F_{\delta,h}^{good}(\eta).$$

With the estimates of Lemma 2.2.5 we infer

$$\begin{aligned}
\big|F_{\delta,h}^{good}(\eta)\big| &\le Ch\big\|\nabla\eta\big\|_{L^\infty(\mathbb{T}^2)} + C \sum_{z\in\mathcal{N}_h\setminus\Sigma_{\delta,h}} h_z\big\|\nabla e_h^i\big\|_{L^\infty(\omega_z)} \gamma_{h,z}^2 |\eta(z)| \\
&\le Ch\big\|\nabla\eta\big\|_{L^\infty(\mathbb{T}^2)} + C\delta\|\eta\|_{L^\infty(\mathbb{T}^2)},
\end{aligned}$$

in particular we have (after passage to a subsequence) that $F_{\delta,h}^{good} \rightharpoonup^* F_\delta^{good}$ in $C(\mathbb{T}^2)^*$ as $h \to 0$ with $F_\delta^{good} \in C(\mathbb{T}^2)^*$ such that $\big\|F_\delta^{good}\big\|_{C(\mathbb{T}^2)^*} \le C\delta$. For $F_{\delta,h}^{bad}$ we have that

$$\big|F_{\delta,h}^{bad}(\eta)\big| \le Ch\big\|\nabla\eta\big\|_{L^\infty(\mathbb{T}^2)} + C \sum_{z\in\Sigma_{\delta,h}} \gamma_{h,z}^3 |\eta(z)|.$$

An application of Lemma 1.9.3 shows that for a subsequence we have $F_{\delta,h}^{bad} \rightharpoonup^* F_\delta^{bad} = \sum_{\iota=1}^{L^\delta} \rho_\iota^\delta \delta_{x_\iota^\delta}$ as $h \to 0$ for $\rho_\iota^\delta \in \mathbb{R}$ such that $\sum_{\iota=1}^{L^\delta} |\rho_\iota^\delta|^{2/3} \le C$ independently of $\delta$. We thus have

$$\big\|F - F_\delta^{bad}\big\|_{C(\mathbb{T}^2)^*} \le C\delta.$$

Employing Lemma 1.9.1 we verify the assertion. □

**Lemma 2.2.8.** *Suppose that $(u_h)_{h>0}$ is a bounded sequence in $W^{1,2}(\mathbb{T}^2;\mathbb{R}^n)$ such that $u_h \in \mathcal{S}^1_\#(\mathcal{T}_h)^n$ for all $h>0$ and $u_h(z)\in N$ for all $z\in\mathcal{N}_h$. For each $h>0$ let $(e^i_h)_{i=1,2,\dots,k}$ be an orthonormal frame for $u_h^{-1}TN$ which is optimal in the sense of Lemma 2.2.3 so that $\max_{i=1,2,\dots,k}\|\nabla e^i_h\|\le C$. Then, for every accumulation point $u\in W^{1,2}(\mathbb{T}^2;\mathbb{R}^n)$ of the sequence $(u_h)_{h>0}$ and an appropriate subsequence, which is not relabeled in the following, we have:*
*(i) $u(x)\in N$ for almost every $x\in\mathbb{T}^2$ and*

$$u_h \rightharpoonup u \quad \text{in } W^{1,2}(\mathbb{T}^2;\mathbb{R}^n);$$

*(ii) there exist $(e^i)_{i=1,2,..,k}\subset W^{1,2}(\mathbb{T}^2;\mathbb{R}^n)$ such that*

$$e^i_h \rightharpoonup e^i \quad \text{in } W^{1,2}(\mathbb{T}^2;\mathbb{R}^n)$$

*and $(e^i)_{i=1,2,\dots,k}$ is an orthonormal frame for $u^{-1}TN$, i.e., for almost every $x\in\mathbb{T}^2$ the vectors $e^1(x),e^2(x),\dots,e^k(x)$ form an orthonormal basis for $T_{u(x)}N$;*
*(iii) for $\omega^{ij}:=e^{j,\mathrm{T}}\nabla e^i\in L^2(\mathbb{T}^2;\mathbb{R}^2)$ we have*

$$\omega^{ij}_h,\,\overline{\omega}^{ij}_h \rightharpoonup \omega^{ij} \quad \text{in } L^2(\mathbb{T}^2;\mathbb{R}^n);$$

*(iv) for $\vartheta^i:=e^{i,\mathrm{T}}\nabla u\in L^2(\mathbb{T}^2;\mathbb{R}^2)$ we have*

$$\vartheta^i_h,\,\overline{\vartheta}^i_h \rightharpoonup \vartheta^i \quad \text{in } L^2(\mathbb{T}^2;\mathbb{R}^2);$$

*(v) there exist $b^{ij}\in W^{1,2}(\mathbb{T}^2)$ and $H^{ij}\in L^2(\mathbb{T}^2;\mathbb{R}^2)$ such that for $a^{ij}_h$, $\widehat{b}^{ij}_h$, and $H^{ij}_h$ as in Lemma 2.2.4 we have*

$$a^{ij}_h \rightharpoonup 0 \quad \text{in } W^{1,2}(\mathbb{T}^2), \qquad \widehat{b}^{ij}_h \rightharpoonup b^{ij} \quad \text{in } W^{1,2}(\mathbb{T}^2), \qquad H^{ij}_h \to H^{ij} \quad \text{in } L^2(\mathbb{T}^2;\mathbb{R}^2)$$

*and $\omega^{ij}=\operatorname{Curl} b^{ij}+H^{ij}$.*

*Proof.* (i) For every accumulation point $u\in W^{1,2}(\mathbb{T}^2;\mathbb{R}^n)$ of the bounded sequence $(u_h)_{h>0}$ there exists a subsequence which we do not relabel such that $u_h\rightharpoonup u$ in $W^{1,2}(\mathbb{T}^2;\mathbb{R}^n)$. Lemma 1.8.1 then implies that $u(x)\in N$ for almost every $x\in\mathbb{T}^2$.
(ii) Since each sequence $(e^i_h)_{h>0}$ is bounded in $W^{1,2}(\mathbb{T}^2;\mathbb{R}^n)$ each admits a weak limit $e^i\in W^{1,2}(\mathbb{T}^2;\mathbb{R}^n)$ of an appropriate subsequence. By successive extraction of subsequences we may assume that the same subsequence converges weakly for each $1\le i\le k$. For $1\le i,j\le k$ with $i\ne j$ we have $e^i_h(z)\cdot e^j_h(z)=0$ for all $z\in\mathcal{N}_h$ and a nodal interpolation estimate proves

$$\|e^i_h\cdot e^j_h\|\le Ch\|\nabla(e^i_h\cdot e^j_h)\|\le Ch\|e^{j,\mathrm{T}}_h\nabla e^i_h+e^{i,\mathrm{T}}_h\nabla e^j_h\|\le Ch,$$

where we used that $|e^i_h|,|e^j_h|\le 1$ almost everywhere in $\mathbb{T}^2$ and $\|\nabla e^i_h\|,\|\nabla e^j_h\|\le C$, independently of $h$. Hence, $e^i_h\cdot e^j_h\to 0$ in $L^2(\mathbb{T}^2)$ and in particular, $e^i_h(x)\cdot e^j_h(x)\to 0$ for almost every $x\in\mathbb{T}^2$. Since also $e^i_h(x)\to e^i(x)$ and $e^j_h(x)\to e^j(x)$ for almost every $x\in\mathbb{T}^2$ we deduce that $e^i\cdot e^j=0$ almost everywhere in $\mathbb{T}^2$. Similarly, using that $|e^i_h(z)|=1$ for all $z\in\mathcal{N}_h$ we estimate

$$\big\||e^i_h|^2-1\big\|\le Ch\|\nabla|e^i_h|^2\|\le Ch\|e^{i,\mathrm{T}}_h\nabla e^i_h\|\le Ch,$$

which implies $|e^i_h|^2\to 1$ in $L^2(\mathbb{T}^2)$ and hence $|e^i|=1$ almost everywhere in $\mathbb{T}^2$. It remains to show that for $\ell=k+1,\dots,n$ we have $e^i\cdot(\nu^\ell\circ u)=0$ almost everywhere in $\mathbb{T}^2$. Since $\nu^\ell$ is locally $C^1$ and

$e_h^i(x) \to e^i(x)$, $u_h(x) \to u(x)$ as $h \to 0$ for almost every $x \in \mathbb{T}^2$ it suffices to show that for every $\delta > 0$ and almost every $x \in M$ there exists $h_0 = h_0(x)$ such that for all $h < h_0$ we have

$$\big|e_h^i(x) \cdot \nu^\ell\big(u_h(x)\big)\big| \leq \delta.$$

Fix $\delta > 0$ and define for $h > 0$

$$\textstyle\sum_{\delta,h} := \big\{z \in \mathcal{N}_h \colon \|\nabla u_h\|_{L^2(\omega_z)} + \|\nabla e_h^i\|_{L^2(\omega_z)} > \delta\big\}.$$

Then, card $\sum_{\delta,h} \leq C\delta^{-2}$ for all $h > 0$ and hence $\sum_{\delta,h} \to \sum_\delta = \big\{x_1^\delta, x_2^\delta, ..., x_{L_\delta}^\delta\big\}$ for $x_1^\delta, x_2^\delta, ..., x_{L_\delta}^\delta \in \mathbb{T}^2$ as $h \to 0$. For each $x \in \mathbb{T}^2 \setminus \sum_\delta$ there exists $h_0$ such that for every $h < h_0$ there exists $z_h^x \in \mathcal{N}_h \setminus \sum_{\delta,h}$ such that $x \in \omega_{z_h^x}$. Then we have, using that $e_h^i(z) \cdot \nu^\ell\big(u_h(z)\big) = 0$ for all $z \in \mathcal{N}_h$, that

$$\begin{aligned}
\big|e_h^i(x) \cdot \nu^\ell\big(u_h(x)\big)\big| &= \Big|e_h^i(z_h^x) \cdot \big[\nu^\ell\big(u_h(x)\big) - \nu^\ell\big(u_h(z_h^x)\big)\big] + \big[e_h^i(z_h^x) - e_h^i(x)\big] \cdot \nu^\ell\big(u_h(x)\big)\Big| \\
&\leq \|D\overline{\nu}^\ell\|_{L^\infty(B_{h_0}(u(x)))}\big|u_h(x) - u_h(z_h^x)\big| + \big|e_h^i(z_h^x) - e_h^i(x)\big| \\
&\leq Ch\big(\|\nabla u_h\|_{L^\infty(\omega_{z_h^x})} + \|\nabla e_h^i\|_{L^\infty(\omega_{z_h^x})}\big) \\
&\leq C\big(\|\nabla u_h\|_{L^2(\omega_{z_h^x})} + \|\nabla e_h^i\|_{L^2(\omega_{z_h^x})}\big) \\
&\leq C\delta,
\end{aligned}$$

which proves the statement.
(iii) For all $\eta \in C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ we have, using $e_h^i \to e^i$ in $L^2(\mathbb{T}^2;\mathbb{R}^n)$ and $\nabla e_h^{j,\alpha} \rightharpoonup \nabla e^{j,\alpha}$ in $L^2(\mathbb{T}^2;\mathbb{R}^2)$, $\alpha = 1, 2, ..., n$, that

$$\begin{aligned}
\big(\overline{\omega}_h^{ij} - \omega^{ij}; \eta\big) &= \sum_{\alpha=1}^n \big(e_h^{j,\alpha}\nabla e_h^{i,\alpha} - e^{j,\alpha}\nabla e^{i,\alpha}; \eta\big) \\
&= \sum_{\alpha=1}^n \Big\{\big(\big[e_h^{j,\alpha} - e^{j,\alpha}\big]\nabla e_h^{i,\alpha}; \eta\big) + \big(\big[\nabla e_h^{j,\alpha} - \nabla e^{j,\alpha}\big] \cdot e^{i,\alpha}; \eta\big)\Big\} \to 0
\end{aligned}$$

as $h \to 0$. Therefore $\overline{\omega}_h^{ij} \rightharpoonup \omega^{ij}$ in $L^2(\mathbb{T}^2;\mathbb{R}^2)$ as $h \to 0$. Using

$$\big\|\mathbf{A}\big(e_h^{j,\alpha}\big) - e_h^{j,\alpha}\mathbf{I}\big\| \leq Ch\big\|\nabla e_h^{j,\alpha}\big\|$$

we find that $\mathbf{A}\big(e_h^{j,\alpha}\big) \to e^{j,\alpha}\mathbf{I}$ in $L^2(\mathbb{T}^2;\mathbb{R}^{2\times 2})$ for $\alpha = 1, 2, .., n$ and thus also $\omega_h^{ij} \rightharpoonup \omega^{ij}$ in $L^2(\mathbb{T}^2;\mathbb{R}^2)$.
(iv) This follows exactly as the assertion in (iii).
(v) For $\phi \in C^\infty(\mathbb{T}^2)$ and $\phi_h := \mathcal{I}_h\phi \in \mathcal{S}^1_\#(\mathcal{T}_h)$ we have

$$\big(\nabla a_h^{ij}; \nabla\phi\big) = \big(\nabla a_h^{ij}; \nabla\phi_h\big) + \big(\nabla a_h^{ij}; \nabla\big[\phi - \phi_h\big]\big)$$

and the second term on the right-hand side vanishes as $h \to 0$ owing to uniform boundedness of $a_h^{ij}$ in $W^{1,2}(\mathbb{T}^2)$ and nodal interpolation results. By definition of $a_h^{ij}$ and Lemma 2.2.3 we have

$$\big(\nabla a_h^{ij}; \nabla\phi_h\big) = \big(\omega_h^{ij}; \nabla\phi_h\big) = \Lambda_4\big(e_h^i, e_h^j, \phi_h\big) + \Lambda_5\big(e_h^i, e_h^j, \phi_h\big).$$

The definition of $\Lambda_4$ and $\Lambda_5$ and the estimates of Lemma 1.4.6 we derive

$$\big|\Lambda_4(e_h^i, e_h^j, \phi_h) + \Lambda_5(e_h^i, e_h^j, \phi_h)\big| \leq Ch\|\nabla e_h^i\|\,\|\nabla e_h^j\|\,\|\nabla \phi_h\|_{L^\infty(M)}$$

and thus $a_h^{ij} \rightharpoonup 0$ as $h \to 0$. Since $H_h^{ij}$ is a bounded sequence in a finite-dimensional space, cf. Lemma 1.5.6, there exists $H^{ij}$ such that, for an appropriate subsequence, we have $H_h^{ij} \to H^{ij}$ in $L^2(\mathbb{T}^2;\mathbb{R}^2)$. Since $\|\mathrm{Curl}_{\mathcal{T}_h} b_h^{ij}\|$ is bounded uniformly we find, using Lemma 1.5.8 that $b_h^{ij} - \widehat{b}_h^{ij} \to 0$ in $L^2(\mathbb{T}^2)$ and $\widehat{b}_h^{ij} \rightharpoonup b^{ij}$ for some $b^{ij} \in W^{1,2}(\mathbb{T}^2)$ (and an appropriate subsequence). For every $\psi \in C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ and $\psi_h := \mathcal{I}_h\psi$ we find with an elementwise integration by parts as in the proof of Lemma 2.9 that

$$\begin{aligned}(\omega_h^{ij};\psi_h) &= \big(\mathrm{Curl}_{\mathcal{T}_h} b_h^{ij};\psi_h\big) + \big(H_h^{ij};\psi_h\big)\\ &= -\big(b_h^{ij};\mathrm{curl}\,\psi_h\big) + \big(H_h^{ij};\psi_h\big) + \sum_{E\in\mathcal{E}_h}\int_E \big[b_h^{ij}\big](\psi_h - c_E)\tau_E\,\mathrm{d}t,\end{aligned}$$

where $c_E \in \mathbb{R}^2$ is an arbitrary constant vector for each $E \in \mathcal{E}_h$. Arguing as in (2.12), passing to the limit for $h \to 0$, and integrating by parts we verify that $\omega^{ij} = \mathrm{Curl}\, b^{ij} + H^{ij}$. This finishes the proof of the lemma. ☐

**Remark 2.2.9.** *We remark that if $N$ is orientable than the last part of item (ii) in the proof of Lemma 2.2.8 can be significantly simplified: With $\overline{\nu}^\ell : \mathbb{R}^n \to \mathbb{R}^n$ as in Lemma 1.6.8 we have $e_h^i(z)\cdot\overline{\nu}^\ell\big(u_h(z)\big) = 0$ for all $z \in \mathcal{N}_h$ and $\ell = k+1,\dots,n$ and therefore*

$$\|e_h^i\cdot(\overline{\nu}^\ell\circ u_h)\| \leq Ch\big\|\nabla\big[e_h^i\cdot(\overline{\nu}^\ell\circ u_h)\big]\big\| \leq Ch\big(\big\|(\overline{\nu}^\ell\circ u_h)^{\mathrm{T}}\nabla e_h^i\big\| + \big\|e_h^{i,\mathrm{T}}\nabla(\overline{\nu}^\ell\circ u_h)\big\|\big) \leq Ch.$$

*Pointwise convergence almost everywhere implies $e^i\cdot\big(\overline{\nu}^\ell\circ u\big) = 0$.*

The following result is based on P.L. Lions' concentrated compactness principle [Lio85]. For a discussion of the assertion in the periodic setting we refer to [FMS98].

**Lemma 2.2.10.** *[FMS98, Equation (2.4)] Suppose that $(b_h)_{h>0}$, $(e_h)_{h>0}$ and $(f_h)_{h>0}$ are bounded sequences in $W^{1,2}(\mathbb{T}^2)$ with weak limits $b, e, f \in W^{1,2}(\mathbb{T}^2)$, respectively, and assume that $e_h$ is bounded in $L^\infty(\mathbb{T}^2)$. Then, there exist $(s_\iota)_{\iota\in\mathbb{N}} \subset \mathbb{R}$ satisfying $\sum_{\iota\in\mathbb{N}}|s_\iota| \leq C$ and $(x_\iota)_{\iota\in\mathbb{N}} \subset \mathbb{T}^2$ such that for (a subsequence and) all $\eta \in C^\infty(\mathbb{T}^2)$ we have*

$$\big(\mathrm{Curl}\, b_h; e_h\eta\nabla f_h\big) \to \big(\mathrm{Curl}\, b; e\eta\nabla f\big) + \sum_{\iota\in\mathbb{N}} s_\iota\eta(x_\iota)$$

*as $h \to 0$.*

### 2.2.5 Statement of the main result

We are now in position to prove the asserted weak convergence result for a sequence of periodic, almost discrete harmonic maps.

**Theorem 2.2.11.** *Let $(\mathcal{T}_h)_{h>0}$ be a sequence of logarithmically right-angled triangulations of $\mathbb{T}^2$ and let $(u_h)_{h>0}$ be such that for all $h > 0$ we have $u_h \in \mathcal{S}^1_\#(\mathcal{T}_h)^n$, $u_h(z) \in N$ for all $z \in \mathcal{N}_h$, and*

$$\|\nabla u_h\| \leq C.$$

*Suppose that there exists a sequence of linear functionals* $\mathcal{R}es_h \colon \mathcal{S}^1_{\#}(\mathcal{T}_h)^n \to \mathbb{R}$ *such that*

$$\sup_{v_h \in \mathcal{S}^1_{\#}(\mathcal{T}_h)^n \setminus \{0\}} \frac{\big|\mathcal{R}es_h(v_h)\big|}{\|v_h\|_{W^{1,2}(\mathbb{T}^2)}} \to 0$$

*as* $h \to 0$ *and assume that for all* $v_h \in \mathcal{S}^1_{\#}(\mathcal{T}_h)^n$ *satisfying* $v_h(z) \in T_{u_h(z)}N$ *for all* $z \in \mathcal{N}_h$ *we have*

$$\big(\nabla u_h; \nabla v_h\big) = \mathcal{R}es_h(v_h).$$

*Then every weak accumulation point of* $\big(u_h\big)_{h>0} \subset W^{1,2}(\mathbb{T}^2;\mathbb{R}^n)$ *is a harmonic map into* $N$.

*Proof.* We let $u \in W^{1,2}(\mathbb{T}^2;\mathbb{R}^n)$ be a weak accumulation point of $\big(u_h\big)_{h>0}$ and we do not relabel the corresponding subsequence so that $u_h \rightharpoonup u$ as $h \to 0$. Let $\eta \in C^\infty(\mathbb{T}^2)$ and for $h > 0$ set $\eta_h := \mathcal{I}_h \eta_h$. For each $h > 0$ let $\big(e^i_h\big)_{i=1,2,\dots,k}$ be an orthonormal frame for $u_h^{-1}TN$ which is optimal in the sense of Lemma 2.2.3. Then, Lemma 2.2.4 and Lemma 2.2.5 imply that

$$\begin{aligned}\mathcal{R}es_h\big(\mathcal{I}_h\big[\eta_h e^i_h\big]\big) &= \big(\nabla u_h; \nabla \mathcal{I}_h\big[\eta_h e^i_h\big]\big)\\ &= \sum_{j=1}^k \Big\{\big(\operatorname{Curl}\widehat{b}^{ij}_h \cdot \overline{\vartheta}^j_h; \eta_h\big) + \big(H^{ij}_h \cdot \overline{\vartheta}^j_h; \eta_h\big)\Big\} + \big(\vartheta^i_h; \nabla\eta_h\big) + \big(f^i_h; \eta_h\big)_h.\end{aligned}$$

With the limits $b^{ij}$, $H^{ij}$, $\vartheta^j$ and $\omega^{ij}$ of appropriate subsequences identified in Lemmas 2.2.7, 2.2.8, and 2.2.10, and with the assumptions on $\mathcal{R}es_h$ we find that

$$0 = \sum_{j=1}^k \Big\{\big(\operatorname{Curl} b^{ij} \cdot \vartheta^j; \eta\big) + \big(H^{ij} \cdot \vartheta^j; \eta\big)\Big\} + \big(\vartheta^i; \nabla\eta\big) + \sum_{\iota\in\mathbb{N}} s_\iota \eta(x_\iota) + \sum_{\iota\in\mathbb{N}} t_\iota \eta(y_\iota)$$

and $\omega^{ij} = \operatorname{Curl} b^{ij} + H^{ij}$ so that

$$\sum_{j=1}^k \big(\omega^{ij} \cdot \vartheta^j_h; \eta\big) + \big(\vartheta^i; \nabla\eta\big) = \sum_{\iota\in\mathbb{N}} s_\iota \eta(x_\iota) + \sum_{\iota\in\mathbb{N}} t_\iota \eta(y_\iota).$$

The left-hand side of this identity belongs to $L^1(M) + H^{-1}(M)$, which does not contain Dirac measures, see [FMS98] for details. Therefore, $s_\iota = t_\iota = 0$ for all $\iota \in \mathbb{N}$ and Proposition 1.7.2 implies that the weak limit $u$ is a harmonic map into $N$. ☐

## 2.3 Reduction of the general case to a periodic setting

In this section we discuss the generalization of Theorem 2.2.11 to general flat domains. The main assertion is the following.

**Theorem 2.3.1.** *Suppose that* $M = M_h \subset \mathbb{R}^2 \times \{0\}$ *is a bounded Lipschitz domain in* $\mathbb{R}^2$ *with polyhedral boundary and assume that* $\big(\mathcal{T}_h\big)_{h>0}$ *is a sequence of asymptotically right-angled, regular triangulations of* $M$. *Let* $\big(u_h\big)_{h>0}$ *be such that for each* $h > 0$ *we have* $u_h \in \mathcal{S}^1(\mathcal{T}_h)^n$, $u_h(z) \in N$ *for all* $z \in \mathcal{N}_h$, $u_h|_{\Gamma_{\mathrm{D},h}} = u_{\mathrm{D},h}$, *and*

$$\big\|\nabla_{M_h} u_h\big\| \le C.$$

*For each $h > 0$ let $\mathcal{R}es_h \colon \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^n \to \mathbb{R}$ satisfy*

$$\sup_{v_h \in \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^n \setminus \{0\}} \frac{\big|\mathcal{R}es_h(v_h)\big|}{\|v_h\|_{W^{1,2}(M;\mathbb{R}^n)}} \to 0$$

*as $h \to 0$ and assume that for all $v_h \in \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^n$ satisfying $v_h(z) \in T_{u_h(z)}N$ for all $z \in \mathcal{N}_h$ we have*

$$\big(\nabla_{M_h} u_h; \nabla_{M_h} v_h\big) = \mathcal{R}es_h(v_h).$$

*If in addition $u_{\mathrm{D},h} \to u_{\mathrm{D}}$ in $L^2(\Gamma_{\mathrm{D}};\mathbb{R}^3)$ as $h \to 0$ then every weak accumulation point of the sequence $\big(u_h\big)_{h>0} \subset W^{1,2}(M;\mathbb{R}^3)$ is a harmonic map into $N$ with $u|_{\Gamma_{\mathrm{D}}} = u_{\mathrm{D}}$.*

*Proof.* Let $u \in W^{1,2}(M;\mathbb{R}^n)$ be a weak accumulation point of the sequence $\big(u_h\big)_{h>0}$. Then, Lemma 1.8.1 and Lemma 1.8.4 imply that $u(x) \in N$ for almost every $x \in M$ and $u|_{\Gamma_{\mathrm{D}}} = u_{\mathrm{D}}$, respectively. It remains to show that $u$ is a harmonic map, i.e., that

$$\big(\nabla u; \nabla v\big) = 0$$

for all $v \in W^{1,2}_0(M;R^n)$ satisfying $v(x) \in T_{u(x)}N$ for almost every $x \in M$. Given some fixed $\delta > 0$ it suffices to prove this identity for all $v \in W^{1,2}_0(Q;R^n)$ for all cubes $Q \subset M$ with sides of length at most $\delta$ and parallel to the coordinate axis. We fix such a cube $Q$ and may assume without loss of generality that $Q = Q_{1/4}(a)$ is centered at $a = (1/4, 1/4)$ and has sides of length $1/4$. Also, we may assume that $Q_{1/2}(a) \subset M$. For $h$ sufficiently small we consider the cube $Q_{1/2-2h}(a)$ and introduce the subset $\widetilde{\mathcal{T}}_h$ of the triangulation $\mathcal{T}_h$ by setting

$$\widetilde{\mathcal{T}}_h := \big\{K \in \mathcal{T}_h : K \cap Q_{1/2-2h}(a) \neq \emptyset\big\}.$$

Then, $\widetilde{\mathcal{T}}_h$ covers $Q_{1/2-2h}(a)$ and is contained in $Q_{1/2}(a)$, cf. Figure 2.1.

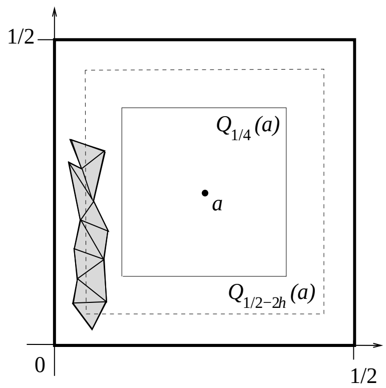


Figure 2.1: Cubes $Q_{1/4}(a)$ and $Q_{1/2-2h}(a)$ for $a = (1/4, 1/4)$ and typical triangles of the subtriangulation $\widetilde{\mathcal{T}}_h$.

In order to reduce to the periodic setting discussed in the previous section, we (1) extend $\widetilde{\mathcal{T}}_h$ to a regular triangulation $\widehat{\mathcal{T}}_h$ of $(0,1/2)^2$, (2) extend $u_h|_{\cup\widetilde{\mathcal{T}}_h}$ to a function $\widehat{u}_h \in \mathcal{S}^1(\widehat{\mathcal{T}}_h)^n$ such that $\widehat{u}_h(z) \in N$ for all $z \in \widehat{\mathcal{N}}_h$, the nodes of the triangulation $\widehat{\mathcal{T}}_h$, and such that $\big\|\nabla \widehat{u}_h\big\|_{L^2((0,1/2)^2)} \leq C$,

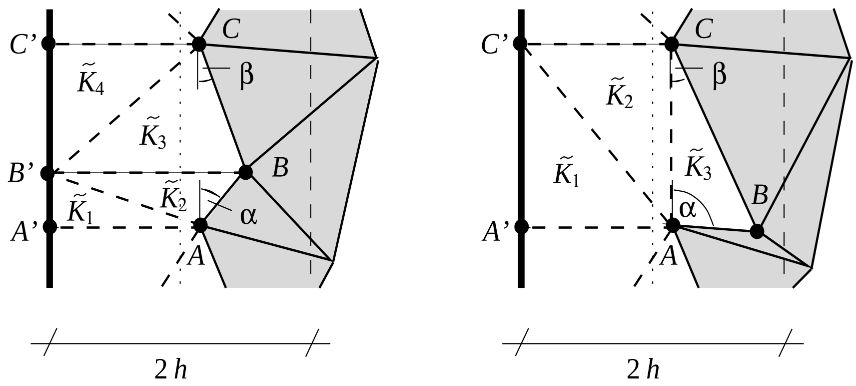


Figure 2.2: Typical scenarios in the extension of the triangulation $\widetilde{\mathcal{T}}_h$ (shaded) to a triangulation of $(0,1/2)^2$. The angle $\alpha$ is not critical in the left plot and critical in the right one.

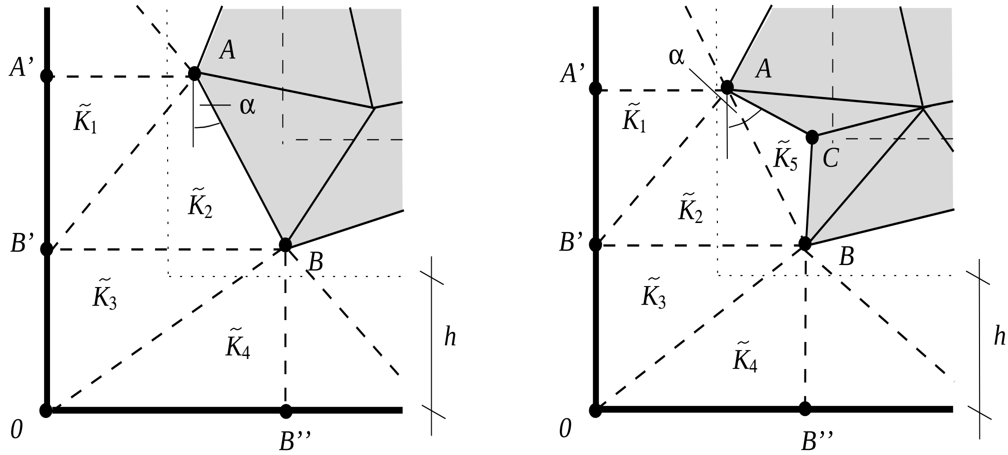


Figure 2.3: Typical scenarios in the extension of the triangulation $\widetilde{\mathcal{T}}_h$ (shaded) to a triangulation of $(0,1/2)^2$ at a corner. The angle $\alpha$ is not critical in the left plot and critical in the right one.

and (3) finally reflect $\widehat{u}_h$ across the lines $\{1/2\} \times \mathbb{R}$ and $\mathbb{R} \times \{1/2\}$ in order to obtain a periodic function on $[0,1]^2$ to which we can apply the theory of the previous section.
*Step 1.* The task to extend $\widetilde{\mathcal{T}}_h$ to a regular triangulation of $(0,1/2)^2$ is simple if $\mathcal{T}_h$ is a uniform triangulation consisting of halved squares with sides parallel to the coordinate axis. The general case is slightly more involved and in order to guarantee shape-regularity of the extended triangulation we discuss a few typical scenarios. Consider first the situation depicted in the left plot of Figure 2.2 and assume that the angles $\alpha, \beta$ are not critical in the sense that they satisfy $-\pi/2+c_0 \le \alpha, \beta \le \pi/2-c_0$ with a uniform (small) constant $c_0 > 0$. We then introduce the new triangles $\widetilde{K}_1, \widetilde{K}_2, \widetilde{K}_3, \widetilde{K}_4$ as shown. A typical critical angle $\alpha$ is depicted in the right plot of Figure 2.2. In this case, we connect the points $A$ and $C$ to obtain a new triangle $\widetilde{K}_3$. We can then proceed as in the previous case. Next, we examine a typical situation at a corner of the cube $Q_{1/2-2h}(a)$. Again, the construction of the extension depends on the angle $\alpha$ defined in Figure 2.3. In the left plot, $\alpha$ satisfies $-\pi/2 + c_0 \le \alpha \le \pi/2 - c_0$ and we introduce the new triangles $\widetilde{K}_1, \widetilde{K}_2, \widetilde{K}_3, \widetilde{K}_4$. The case

of a critical angle is depicted in the right plot of Figure 2.3. We connect the vertices $A$ and $B$ to introduce the new triangle $\widetilde{K}_5$. We are then in the situation described above. We remark that in case that there are triangles with $|A-B| \ll h$ then the new triangles $\widetilde{K}_i$ can be refined further in order to maintain shape regularity.

*Step 2.* In the situations discussed above, we extend $u_h$ by setting $\widehat{u}_h(A') := u_h(A)$, $\widehat{u}_h(B') := u_h(B)$, $\widehat{u}_h(C') := u_h(C)$ and $\widehat{u}_h(A') := u_h(A)$, $\widehat{u}_h(C') := u_h(C)$ in the situations depicted in the left and right plot Figure 2.2, respectively. We set $\widehat{u}_h(A') := u_h(A)$, $\widehat{u}_h(B') = \widehat{u}_h(B'') = \widehat{u}_h(0) := u_h(B)$, $\widehat{u}_h(C') := u_h(C)$ in the scenarios shown in Figure 2.3. In order to show that we do not increase the gradient of $u_h$ we notice that, e.g., in the situation of the right plot of Figure 2.2 we have

$$\begin{aligned}\big|\nabla \widehat{u}_h|_{\widetilde{K}_2}\big| \le h^{-1}\big|\widehat{u}_h(C') - \widehat{u}_h(A)\big| &\le h^{-1}\big(\big|u_h(A) - u_h(B)\big| + \big|u_h(B) - u_h(C)\big|\big) \\ &\le \big|\nabla u_h|_{K_A}\big| + \big|\nabla u_h|_{K_B}\big|,\end{aligned}$$

where $K_A, K_B \in \widetilde{\mathcal{T}}_h$ are such that $A \in K_A$ and $B \in K_B$.

*Step 3.* We reflect the triangulation $\widehat{\mathcal{T}}_h$ and the function $\widehat{u}_h$ across the lines $\{1/2\} \times \mathbb{R}$ and $\mathbb{R} \times \{1/2\}$ to obtain a triangulation $\mathcal{T}_h^{\#}$ of $\mathbb{T}^2$ and a function $u_h^{\#} \in \mathcal{S}^1_{\#}(\mathcal{T}_h^{\#})$ periodic function on $[0,1]^2$, cf. Figure 2.4.

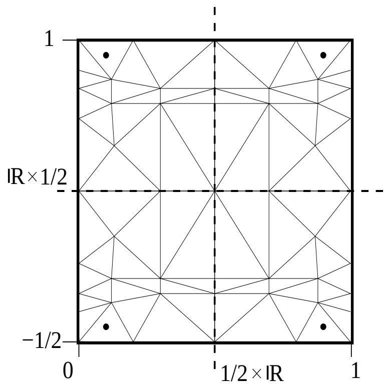


Figure 2.4: Reflection of the triangulation $\widetilde{\mathcal{T}}_h$ of $(0,1/2)^2$ to obtain a ("periodic") triangulation of $\mathbb{T}^2$ with fundamental domain $(-1/2,1/2)^2$. (The dot is included for better visualization.)

We may now apply Theorem 2.2.11 to the sequence $\big(u_h^{\#}\big)$. The only thing we have to modify is that test functions are supported in $Q_{1/4}(a)$. This shows that $u$ is harmonic in $Q_{1/4}(a)$ and finishes the proof of the theorem. □

## 2.4 Various extensions

The condition that a sequence of triangulations is logarithmically right-angled severely restricts the applicability of the convergence result of the previous section. In the first part of this section we show that the condition is not required if the sequence of discrete harmonic maps is uniformly bounded in $W^{1,2+\sigma}(M;\mathbb{R}^n)$ for some $\sigma > 0$. This suggests that the angle condition is mostly needed to cope with technical difficulties. Moreover, this observation justifies approximation schemes on non-flat submanifolds $M \subset \mathbb{R}^3$ for which right-angled triangulations may not be available at all.

Related aspects for the approximation of harmonic maps on curved surfaces are discussed in the second part of the section. The last part of this section discusses in brief the transfer of the above convergence result to the numerical approximation of the harmonic map heat flow into general submanifolds. Motivation for this is that some of the algorithms for the approximation of harmonic maps introduced in Chapter 3 are based on gradient flow approaches.

### 2.4.1 Convergence on regular triangulations

The logarithmical right-angled condition implies that the discrete connection forms $\omega_h^{ij}$ in Coulomb gauge are discrete divergence free, at least asymptotically, see Lemma 2.2.3. The following proposition shows that this is also the case if the sequence $\big(u_h\big)_{h>0}$ admits higher integrability properties and can be used to replace Lemma 2.2.6 in the proof of Theorem 2.2.11.

**Proposition 2.4.1.** *Suppose that $\big(\mathcal{T}_h\big)_{h>0}$ is a sequence of regular triangulations of $\mathbb{T}^2$ with maximal mesh-size $h \to 0$ and for each $h > 0$ let $u_h \in \mathcal{S}^1_\#(\mathcal{T}_h)^n$ be such that $u_h(z) \in N$ for all $z \in \mathcal{N}_h$. Suppose that there exists $\sigma > 0$ and $C > 0$ such that*

$$\big\|\nabla u_h\big\|_{L^{2+\sigma}(\mathbb{T}^2)} \le C$$

*for all $h > 0$. Given $\eta \in C^\infty(\mathbb{T}^2)$ set $\eta_h := \mathcal{I}_h[\eta] \in \mathcal{S}^1_\#(\mathcal{T}_h)$ and let $\psi_h^j := \mathcal{G}_h\big[\eta_h\overline{\vartheta}_h^j\big] \in \mathcal{S}^1_\#(\mathcal{T}_h)$ be defined through $\int_{\mathbb{T}^2}\psi_h^j \,\mathrm{d}x = 0$ and*

$$\big(\nabla\psi_h^j;\nabla v_h\big) = \big(\eta_h\overline{\vartheta}_h^j;\nabla v_h\big)$$

*for all $v_h \in \mathcal{S}^1_\#(\mathcal{T}_h)$ and with $\overline{\vartheta}_h^j$ as in Definition 2.2.1. We then have*

$$\big|\Lambda_4\big(e_h^i, e_h^j, \psi_h^j\big)\big| + \big|\Lambda_5\big(e_h^i, e_h^j, \psi_h^j\big)\big| \to 0$$

*as $h \to 0$, where $\Lambda_4$ and $\Lambda_5$ are as in Lemma 2.2.3.*

*Proof.* Given $a, b \in L^\infty(\mathbb{T}^2)$ and $c \in L^q(\mathbb{T}^2)$ for $q \in (1,\infty)$ we have

$$\big(ab;c\big) \le \|a\|^{1/q}_{L^\infty(\mathbb{T}^2)}\|b\|^{1/q}_{L^\infty(\mathbb{T}^2)}\|a\|^{1/q'}\|b\|^{1/q'}\|c\|_{L^q(\mathbb{T}^2)}.$$

By definition of $\Lambda_4\big(e_h^i, e_h^j, \psi_h^j\big)$ we verify with $q := 2+\sigma$ that

$$\begin{aligned}\big|\Lambda_4\big(e_h^i, e_h^j, \psi_h\big)\big| &\le \big\|\nabla e_h^i\big\|^{1/q}_{L^\infty(\mathbb{T}^2)}\big\|\nabla e_h^j\big\|^{1/q}_{L^\infty(\mathbb{T}^2)}\big\|\nabla e_h^i\big\|^{1/q'}\big\|\nabla e_h^j\big\|^{1/q'}\big\|\psi_h^j - \mathbf{A}\big(\psi_h^j\big)\big\|_{L^q(\mathbb{T}^2)}\\ &\le Ch\big\|\nabla e_h^i\big\|^{1/q}_{L^\infty(\mathbb{T}^2)}\big\|\nabla e_h^j\big\|^{1/q}_{L^\infty(\mathbb{T}^2)}\big\|\nabla e_h^i\big\|^{1/q'}\big\|\nabla e_h^j\big\|^{1/q'}\big\|\nabla\psi_h^j\big\|_{L^q(\mathbb{T}^2)},\end{aligned}$$

where we incorporated the estimate of Lemma 1.4.6. An inverse estimate and $\big\|e_h^\ell\big\|_{L^\infty(\mathbb{T}^2)} \le 1$ and $\big\|\nabla e_h^\ell\big\| \le C$ for $\ell = i, j$ imply

$$\big|\Lambda_4\big(e_h^i, e_h^j, \psi_h\big)\big| \le Ch^{1-2/q}\big\|\nabla\psi_h^j\big\|_{L^q(\mathbb{T}^2)}. \tag{4.13}$$

We define $\mathcal{G}\colon L^2(\mathbb{T}^2;\mathbb{R}^2) \to \overset{\circ}{W}{}^{1,2}(\mathbb{T}^2)$ for $\Phi \in L^2(\mathbb{T}^2;\mathbb{R}^2)$ by

$$\big(\nabla\mathcal{G}[\Phi];\nabla v\big) = \big(\Phi;\nabla v\big)$$

for all $v \in \overset{\circ}{W}{}^{1,2}(\mathbb{T}^2) := \{w \in W^{1,2}(\mathbb{T}) : \int_{\mathbb{T}^2} w \,\mathrm{d}x = 0\}$. Moreover, we let $\mathcal{R}G_h \colon W^{1,2}(\mathbb{T}^2) \to \overset{\circ}{\mathcal{S}}{}^1_\#(\mathcal{T}_h)$ denote the Ritz-Galerkin projection onto $\overset{\circ}{\mathcal{S}}{}^1_\#(\mathcal{T}_h) := \mathcal{S}^1_\#(\mathcal{T}_h) \cap \overset{\circ}{W}{}^{1,2}(\mathbb{T}^2)$ defined for $a \in W^{1,2}(\mathbb{T}^2)$ through

$$\big(\nabla \mathcal{R}G_h[a]; \nabla v_h\big) = \big(\nabla a; \nabla v_h\big)$$

for all $v_h \in \overset{\circ}{\mathcal{S}}{}^1_\#(\mathcal{T}_h)$. We then have $\mathcal{G}_h = \mathcal{R}G_h \circ \mathcal{G}$ and in particular that

$$\psi_h = \big(\mathcal{R}G_h \circ \mathcal{G}\big)\big[\eta_h \overline{\vartheta}^j_h\big].$$

By definition, the Ritz-Galerkin projection satisfies $\|\nabla \mathcal{R}G_h[a]\| \le \|\nabla a\|$ for all $a \in W^{1,2}(\mathbb{T}^2)$. Results in [BTW03] imply that it also satisfies $\|\nabla \mathcal{R}G_h[a]\|_{L^\infty(\mathbb{T}^2)} \le C\|\nabla a\|_{L^\infty(\mathbb{T}^2)}$ for all $a \in W^{1,\infty}(\mathbb{T}^2)$. Interpolation of operators, see [BL76], thus implies that

$$\|\nabla \mathcal{R}G_h[a]\|_{L^q(\mathbb{T}^2)} \le C\|\nabla a\|_{L^q(\mathbb{T}^2)}$$

for all $a \in W^{1,q}(\mathbb{T}^2)$. Since the operator $\mathcal{G}$ satisfies, see [Iwa83],

$$\|\nabla \mathcal{G}[\Phi]\|_{L^q(\mathbb{T}^2)} \le C\|\Phi\|_{L^q(\mathbb{T}^2)}$$

we have that

$$\begin{aligned}
\|\nabla \psi_h\|_{L^q(\mathbb{T}^2)} &= \|\nabla \mathcal{G}_h[\eta_h \overline{\vartheta}^j_h]\|_{L^q(\mathbb{T}^2)} \\
&= \|\nabla \big(\mathcal{R}G_h \circ \mathcal{G}\big)[\eta_h \overline{\vartheta}^j_h]\|_{L^q(\mathbb{T}^2)} \\
&\le C\|\nabla \mathcal{G}[\eta_h \overline{\vartheta}^j_h]\|_{L^q(\mathbb{T}^2)} \\
&\le C\|\eta_h \overline{\vartheta}^j_h\|_{L^q(\mathbb{T}^2)} \\
&\le C\|\nabla u_h\|_{L^q(\mathbb{T}^2)}\|\eta_h\|_{L^\infty(\mathbb{T}^2)},
\end{aligned}$$

where we incorporated the definition of $\overline{\vartheta}^j_h$ and the bound $\|e^j_h\|_{L^\infty(\mathbb{T}^2)} \le 1$ in the last estimate. Using this bound and the assumed estimate $\|\nabla u_h\|_{L^q(\mathbb{T}^2)} \le C$ in (4.13) we deduce

$$\big|\Lambda_4\big(e^i_h, e^j_h, \psi_h\big)\big| \le Ch^{1-2/q}\|\eta_h\|_{L^\infty(\mathbb{T}^2)},$$

which proves the asserted limit for $\Lambda_4$ since $q > 2$. The same argumentation implies the asserted limit for $\Lambda_5$. □

### 2.4.2 Convergence of discrete harmonic maps on curved surfaces

The analysis carried out in Section 2.2 transfers to a large extent to discrete harmonic maps on curved surfaces. Here, we briefly discuss necessary modifications of the argumentation.

Suppose that $M \subset \mathbb{R}^3$ is a two-dimensional, smooth, compact, and orientable submanifold without boundary and for each $h > 0$ we are given an approximation $M_h$ of $M$ defined through a set of triangles $\mathcal{T}_h$ as in Section 1.3. For each $h > 0$ let $u_h \in \mathcal{S}^1(\mathcal{T}_h)^n$ be such that $u_h(z) \in N$ for all $z \in \mathcal{N}_h$. Since the discrete product rule and the discrete Helmholtz decomposition also hold on $M_h$, Definition 2.2.1 as well as Lemmas 2.2.2 and 2.2.3 hold verbatim with $\mathcal{S}^1_\#(\mathcal{T}_h)$ replaced by

$\mathcal{S}^1(\mathcal{T}_h)$, $\nabla$ substituted by $\nabla_{M_h}$, and noting that $\omega_h^{ij}, \overline{\omega}_h^{ij}, \vartheta_h^i, \overline{\vartheta}_h^i \in L^2(M_h;\mathbb{R}^3)$. Also, Lemmas 2.2.4 and 2.2.5 only require notational changes. The assertions of Lemmas 2.2.6 and 2.2.7 are still valid on curved surfaces, however, a logarithmical right-angled condition appears very restrictive for curved surfaces since right-angled triangulations may not be available for certain curved surfaces at all. Nevertheless, the right-angled condition may be replaced by a regularity assumption as in Proposition 2.4.1, we refer to the end of this section for details. By employing the lifting operator of Section 1.3 one may also pass to limits of the various functions as in Lemma 2.2.8. Since the proof of Lemma 2.2.10 is based on Wente's inequality which also holds on Riemannian surfaces, see [Hél02], the conclusion of Lemma 2.2.10 may also be drawn for liftings of functions defined on the sequence of approximate surfaces $\big(M_h\big)_{h>0}$ although not all technical details have been checked. To state the result of Theorem 2.2.11 for curved surfaces it remains to show that integrals over $M_h$ and those of the corresponding lifted functions on $M$ have the same limits as $h \to 0$: For functions $b_h, e_h, f_h \in \mathcal{S}^1(\mathcal{T}_h)$ and $\eta \in C^\infty(M)$ we let $\widetilde{b}_h, \widetilde{e}_h, \widetilde{f}_h \in L^\infty(M)$ denote their liftings onto $M$ and choose $\check{\eta}$ such that $\widetilde{\check{\eta}} = \eta$. Recalling the identity $\nabla_{M_h} b_h = \big(\mathbf{GD}_h\big)^{\mathrm{T}} \nabla_M \widetilde{b}_h$ from the proof of Lemma 1.3.5 as well as $\mathrm{Curl}_{M_h} = \mu_h \times \nabla_{M_h}$ and $\mathrm{Curl}_M = \mu \times \nabla_M$ we infer that

$$\begin{aligned}
&\int_{M_h} e_h \check{\eta} \mathrm{Curl}_{M_h} b_h \cdot \nabla_{M_h} f_h \,\mathrm{d}s_h - \int_M \widetilde{e}_h \eta \,\mathrm{Curl}_M \widetilde{b}_h \cdot \nabla_M \widetilde{f}_h \,\mathrm{d}s \\
&\quad = \int_M \widetilde{e}_h \eta \Big(\widetilde{\mu}_h \times \big[\big(\mathbf{GD}_h\big)^{\mathrm{T}} \nabla_M \widetilde{b}_h\big]\Big) \cdot \big[\big(\mathbf{GD}_h\big)^{\mathrm{T}} \nabla_M \widetilde{f}_h\big] \big\{Q_h/Q - 1\big\} \,\mathrm{d}s \\
&\qquad + \int_M \widetilde{e}_h \eta \Big\{ \Big(\widetilde{\mu}_h \times \big[\big(\mathbf{GD}_h\big)^{\mathrm{T}} \nabla_M \widetilde{b}_h\big]\Big) \cdot \big[\big(\mathbf{GD}_h\big)^{\mathrm{T}} \nabla_M \widetilde{f}_h\big] - \big(\mu \times \nabla_M \widetilde{b}_h\big) \cdot \nabla_M \widetilde{f}_h \Big\} \,\mathrm{d}s \\
&\quad = \int_M \widetilde{e}_h \eta \Big(\widetilde{\mu}_h \times \big[\big(\mathbf{GD}_h\big)^{\mathrm{T}} \nabla_M \widetilde{b}_h\big]\Big) \cdot \big[\big(\mathbf{GD}_h\big)^{\mathrm{T}} \nabla_M \widetilde{f}_h\big] \big\{Q_h/Q - 1\big\} \,\mathrm{d}s \\
&\qquad + \int_M \widetilde{e}_h \eta \Big\{ \Big([\widetilde{\mu}_h - \mu] \times \big[\big(\mathbf{GD}_h\big)^{\mathrm{T}} \nabla_M \widetilde{b}_h\big]\Big) \cdot \big[\big(\mathbf{GD}_h\big)^{\mathrm{T}} \nabla_M \widetilde{f}_h\big] \\
&\qquad\qquad + \Big(\mu \times \big[\big(\mathbf{GD}_h - \mathbf{I}_{3\times 3}\big)^{\mathrm{T}} \nabla_M \widetilde{b}_h\big]\Big) \cdot \big[\big(\mathbf{GD}_h\big)^{\mathrm{T}} \nabla_M \widetilde{f}_h\big] \\
&\qquad\qquad + \big(\mu \times \nabla_M \widetilde{b}_h\big) \cdot \big[\big(\mathbf{GD}_h - \mathbf{I}_{3\times 3}\big)^{\mathrm{T}} \nabla_M \widetilde{f}_h\big] \Big\} \,\mathrm{d}s.
\end{aligned}$$

One may now argue as in the proof of Lemma 1.3.5 to deduce that the right-hand side tends to zero as $h \to 0$ and therefore

$$\int_{M_h} e_h \check{\eta} \mathrm{Curl}_{M_h} b_h \cdot \nabla_{M_h} f_h \,\mathrm{d}s_h \to \int_M e \eta \,\mathrm{Curl}_M b \cdot \nabla_M f \,\mathrm{d}s + \sum_{\iota \in \mathbb{N}} s_\iota \eta(x_\iota)$$

as $h \to 0$. Similarly, we verify that

$$\int_{M_h} H_h^{ij} \cdot \overline{\vartheta}_h^j \eta_h \,\mathrm{d}s_h - \int_M \widetilde{H}_h^{ij} \cdot \widetilde{\overline{\vartheta}}_h^j \widetilde{\eta}_h \,\mathrm{d}s \to 0$$

and

$$\int_{M_h} \vartheta_h^i \cdot \nabla_{M_h} \eta_h \,\mathrm{d}s_h - \int_M \overline{\vartheta}_h^i \widetilde{\eta}_h \,\mathrm{d}s \to 0$$

as $h \to 0$. This finishes the sketch of the convergence proof for discrete harmonic maps on logarithmically right-angled triangulations of curved surfaces.

For sequences of regular triangulations we assume higher integrability of the lifted sequence $\big(\widetilde{u}_h\big)_{h>0}$ and aim at bounding the projection onto discrete gradients of $\eta_h\vartheta_h^j$ in $L^q(M_h)$ for some $q$ greater than 2 as in Proposition 2.4.1. Since stability of projections onto gradient fields is unclear for non-smooth submanifolds such as $M_h$ but is known for the smooth submanifold $M$ owing to [ISS99], we have to adjust the argumentation of Proposition 2.4.1. For $\phi \in L^2(M_h)$ we define

$$\overline{\mathcal{G}}_h\big[\phi\big] := \mathcal{R}G_h\check{\psi},$$

where $\psi = \mathcal{G}_M\big[\widetilde{\phi}\big] \in \overset{\circ}{W}{}^{1,2}(M)$ satisfies $\big(\nabla\psi;\nabla_M\eta\big) = \big(\phi;\nabla_M\eta\big)$ for all $\eta \in \overset{\circ}{W}{}^{1,2}(M)$. We have by choice of $a_h^{ij}$ and $\Lambda_{4,5}$ that

$$\begin{aligned}
\big(\nabla_{M_h}a_h^{ij};\eta_h\vartheta_h^i\big) &= \big(\nabla_{M_h}a_h^{ij};\nabla\mathcal{G}_h\big[\eta_h\vartheta_h^j\big]\big)\\
&= \big(\omega_h^{ij};\nabla_{M_h}\mathcal{G}_h\big[\eta_h\vartheta_h^j\big]\big)\\
&= \big(\omega_h^{ij};\nabla_{M_h}\overline{\mathcal{G}}_h\big[\eta_h\vartheta_h^j\big]\big) + \big(\omega_h^{ij};\nabla_{M_h}\big\{\mathcal{G}_h-\overline{\mathcal{G}}_h\big\}\big[\eta_h\vartheta_h^j\big]\big)\\
&= \Lambda_{4,5}\big(e_h^i,e_h^j,\overline{\mathcal{G}}_h\big[\eta_h\vartheta_h^j\big]\big) + \big(\omega_h^{ij};\nabla_{M_h}\big\{\mathcal{G}_h-\overline{\mathcal{G}}_h\big\}\big[\eta_h\vartheta_h^j\big]\big)
\end{aligned}$$

and notice that the first term on the right-hand side can be treated as in Proposition 2.4.1 so that it remains to show that the second term on the right-hand side tends to zero as $h \to 0$. By definition of $\overline{\mathcal{G}}_h$ and properties of the transfer operators we have

$$\begin{aligned}
\big(\nabla_{M_h}\overline{\mathcal{G}}_h\big[\phi\big];\nabla_{M_h}v_h\big) &= \big(\nabla_{M_h}\check{\psi};\nabla_{M_h}v_h\big)\\
&= \big(\nabla_M\mathcal{G}_M\big[\widetilde{\phi}\big];\nabla_M\widetilde{v}_h\big) + o(1)\\
&= \big(\widetilde{\phi};\nabla_M\widetilde{v}_h\big) + o(1)\\
&= \big(\phi;\nabla_{M_h}v_h\big) + o(1)\\
&= \big(\nabla_{M_h}\mathcal{G}_h\big[\phi\big];\nabla_{M_h}v_h\big) + o(1)
\end{aligned}$$

for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^n$ and this finishes the discussion of the modification of Proposition 2.4.1 for curved surfaces.

### 2.4.3 Approximation of the $L^2$ flow of harmonic maps

The $L^2$ flow of harmonic maps into a submanifold $N$ describes a function $u\colon [0,T]\times M \to N$ that satisfies $u(0) = u_0$ in the sense of traces for some given $u_0 \in W^{1,2}(M;\mathbb{R}^n)$ such that $u_0(x) \in N$ for almost every $x \in M$ and

$$\big(\partial_t u(t,\cdot);v\big) + \big(\nabla_M u(t,\cdot);\nabla_M v\big) = 0 \tag{4.14}$$

for almost every $t \in (0,T)$ and all $v \in W^{1,2}(M;\mathbb{R}^n)$ such that $v(x) \in T_{u(t,x)}N$ for almost every $x \in M$.

Suppose that for a uniform partition of $(0,T)$ defined through a time-step size $\tau > 0$ an approximation scheme provides piecewise affine and constant functions $u_h^a, u_h^c \colon [0,T] \to \mathcal{S}^1(\mathcal{T}_h)^n$ such that $u_h^c(t,z) \in N$ for almost every $t \in (0,T)$ and all $z \in \mathcal{N}_h$, $u_h^a(j\tau) = u_h^c(j\tau)$ for $j = 0,1,2,...,J_T$, and

$$\big(\partial_t u_h^a(t,\cdot); v_h\big) + \big(\nabla_{M_h} u_h^c(t,\cdot); \nabla_{M_h} v_h\big) = 0$$

for almost all $t \in (0,T)$ and all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^n$ such that $v_h(z) \in T_{u_h^c(t,z)}N$ for all $z \in \mathcal{N}_h$. Suppose that we also have for all $T' \in (0,T)$ that

$$\int_0^{T'} \big\|\partial_t u_h^a\big\|^2 \,\mathrm{d}t + \frac{1}{2}\big\|\nabla_{M_h} u_h^c(T')\big\|^2 \le \frac{1}{2}\big\|\nabla_{M_h} \mathcal{I}_h u_0\big\|^2. \tag{4.15}$$

Then, there exists $u \in H^1\big[0,T; L^2(M;\mathbb{R}^n)\big] \cap L^\infty\big[0,T; W^{1,2}(M;\mathbb{R}^n)\big]$ such that for an appropriate subsequence we have

$$\partial_t u_h^a \rightharpoonup \partial_t u \quad \text{in } L^2\big[0,T; L^2(M;\mathbb{R}^n)\big]$$

and

$$u_h^c \rightharpoonup^* u \quad \text{in } L^\infty\big[0,T; W^{1,2}(M;\mathbb{R}^n)\big]$$

as $(h,\tau) \to 0$. A natural question that arises is whether $u$ is a solution of the continuous problem (4.14). The previous sections provided conditions which guarantee

$$\big(\nabla_{M_h} u_h^c; \nabla_{M_h} \mathcal{I}_h\big[\eta e_h^i\big]\big) \to \big(\nabla_M u; \nabla_M\big[\eta e^i\big]\big).$$

If the time-dependent orthonormal frame converges strongly in $L^2\big[0,T; L^2(M;\mathbb{R}^n)\big]$, i.e., $e_h^i \to e^i$ strongly in $L^2((0,T)\times M;\mathbb{R}^n)$ for $i = 1,2,...,k$ (for an appropriate subsequence), one can modify the argumentation of the previous sections to show that for the time-dependent problem we get

$$\int_0^T \Big\{\big(\partial_t u_h^a; \mathcal{I}_h\big[\eta e_h^i\big]\big) + \big(\nabla_{M_h} u_h^c; \nabla_{M_h} \mathcal{I}_h\big[\eta e_h^i\big]\big)\Big\}\,\mathrm{d}t \\ \to \int_0^T \Big\{\big(\partial_t u; \eta e^i\big) + \big(\nabla_M u; \nabla_M \eta e^i\big)\Big\}\,\mathrm{d}t$$

as $(h,\tau) \to 0$. We notice however that this requires a different gauge of the frame. Since the energy estimate (4.15) also holds in the limit, $u(0) = u_0$ provided that $\mathcal{I}_h u_0 \to u_0$ in $W^{1,2}(M;\mathbb{R}^n)$, and $u(t,x) \in N$ for almost every $(t,x) \in (0,T)\times M$ one may then show that the sequence $u_h^a$ approximates weak solutions of the harmonic map heat flow into $N$ in the sense of [Str85].

## 2.5 Weak convergence of discrete harmonic maps into spheres

If $N = S^{n-1}$ then convergence of discrete harmonic maps follows in arbitrary dimensions. The key towards proving this is the following lemma which shows that the wedge product and the surface gradient on curved surfaces behave as in the "flat" case. The asserted identity is essential for the convergence analysis presented below since it allows to avoid products of gradients of weakly convergent sequences in $W^{1,2}(M)$.

**Lemma 2.5.1.** *Let $\underline{D}_\gamma$, $\gamma = 1, 2, ..., m$, denote the components of $\nabla_M$. Then, for all $v \in W^{1,2}(M;\mathbb{R}^n) \cap L^\infty(M;\mathbb{R}^n)$ and $\phi \in W^{1,2}(M;\Lambda^2(\mathbb{R}^n)) \cap L^\infty(M;\Lambda^2(\mathbb{R}^n))$ we have*

$$\big(\nabla_M v; \nabla_M *^{-1}[*\phi \wedge v]\big) = \sum_{\gamma=1}^m \big(\underline{D}_\gamma v; *^{-1}[(\underline{D}_\gamma *\phi) \wedge v]\big),$$

*where we identified $\mathbb{R}^n$ and $\Lambda(\mathbb{R}^n)$.*

*Proof.* For $\gamma = 1, 2, ..., m$ we have according to Lemma 1.2.3 and bilinearity of the wedge product that

$$\underline{D}_\gamma[*\phi \wedge v] = \big(\underline{D}_\gamma *\phi\big) \wedge v + *\phi \wedge \big(\underline{D}_\gamma v\big).$$

Hence, using the property of the wedge product that $a \wedge a = 0$ and $a \cdot b = a \wedge *b$ as well as $a \wedge (b \wedge c) = (c \wedge a) \wedge b$ for $a, b \in \mathbb{R}^n$ and $c \in \Lambda^\ell(\mathbb{R}^n)$ we deduce that

$$\begin{aligned}\big(\nabla_M v; \nabla_M *^{-1}[*\phi \wedge v]\big) &= \sum_{\gamma=1}^m \big(\underline{D}_\gamma v; *^{-1}\underline{D}_\gamma[*\phi \wedge v]\big)\\ &= \sum_{\gamma=1}^m \big(\underline{D}_\gamma v; *^{-1}[(\underline{D}_\gamma *\phi) \wedge v]\big) + \sum_{\gamma=1}^m \big(\underline{D}_\gamma v; *^{-1}[*\phi \wedge \underline{D}_\gamma v]\big)\\ &= \sum_{\gamma=1}^m \big(\underline{D}_\gamma v; *^{-1}[(\underline{D}_\gamma *\phi) \wedge v]\big),\end{aligned}$$

which proves the lemma. ☐

**Theorem 2.5.2.** *Let $(u_h)_{h>0}$ be such that $u_h \in \mathcal{S}^1(\mathcal{T}_h)^n$, $u_h(z) \in S^{n-1}$ for all $z \in \mathcal{N}_h$, $u_h|_{\Gamma_{\mathrm{D},h}} = u_{\mathrm{D},h}$, and*

$$\big\|\nabla_{M_h} u_h\big\| \le C$$

*for all $h > 0$. Suppose that for each $h > 0$ the linear functional $\mathcal{R}es_h \colon \mathcal{S}^1_\mathrm{D}(\mathcal{T}_h)^n \to \mathbb{R}$ satisfies*

$$\sup_{v_h \in \mathcal{S}^1_\mathrm{D}(\mathcal{T}_h)^n \setminus \{0\}} \frac{\big|\mathcal{R}es_h(v_h)\big|}{\|\widetilde{v}_h\|_{W^{1,2}(M;\mathbb{R}^n)}} \to 0$$

*as $h \to 0$ and assume that for all $v_h \in \mathcal{S}^1_\mathrm{D}(\mathcal{T}_h)^3$ satisfying $v_h(z) \cdot u_h(z) = 0$ for all $z \in \mathcal{N}_h$ we have*

$$\big(\nabla_{M_h} u_h; \nabla_{M_h} v_h\big) = \mathcal{R}es_h(v_h).$$

*If $u_{\mathrm{D},h} \to u_\mathrm{D}$ in $L^2(\Gamma_\mathrm{D};\mathbb{R}^n)$ then every weak accumulation point of the lifted sequence $(\widetilde{u}_h)_{h>0} \subset W^{1,2}(M;\mathbb{R}^n)$ is a harmonic map into $S^{n-1}$ with $u|_{\Gamma_\mathrm{D}} = u_\mathrm{D}$.*

*Proof.* For a weak accumulation point $u \in W^{1,2}(M;\mathbb{R}^n)$ and a subsequence (which is not relabeled in the following) of $(\widetilde{u}_h)_{h>0}$ such that $\widetilde{u}_h \rightharpoonup u$ in $W^{1,2}(M;\mathbb{R}^n)$ we have $\widetilde{u}_h \to u$ in $L^2(M;\mathbb{R}^n)$. Lemma 1.8.1 and Lemma 1.8.4 show that $u(x) \in N = S^{n-1}$ for almost every $x \in M$ and that $u|_{\Gamma_\mathrm{D}} = u_\mathrm{D}$. To show that $u$ is a harmonic map we fix $\phi \in C_c^\infty(M;\Lambda^2(\mathbb{R}^n))$ and let $\check{\phi} := \phi \circ \mathcal{P}_h$ so that the lifting of $\check{\phi}$ coincides with $\phi$, i.e., $\widetilde{\check{\phi}} = \phi$. By stability of nodal interpolation in $W^{1,2}(K)$, $K \in \mathcal{T}_h$, and Lemmas 1.2.3 and 1.3.7 we have that

$$\begin{aligned}
\big\|\nabla_{M_h}\mathcal{I}_h\big[*\check{\phi}\wedge u_h\big]\big\| &\le C\big\|\nabla_{M_h}\big[*\check{\phi}\wedge u_h\big]\big\| \\
&\le C\big(\|u_h\|_{L^\infty(M_h)}\big\|\nabla_{M_h}\check{\phi}\big\| + \big\|\check{\phi}\big\|_{L^\infty(M_h)}\big\|\nabla_{M_h}u_h\big\|\big) \\
&\le C\big(\|\phi\|_{L^\infty(M)} + \big\|\nabla_M\phi\big\|\big),
\end{aligned} \tag{5.16}$$

where we also used that $\|u_h\|_{L^\infty(M_h)} \le 1$ and $\big\|\nabla_{M_h}u_h\big\| \le C$. With $\mathbf{F}_h$ as in Lemma 1.3.5 we have

$$\begin{aligned}
\mathcal{R}es_h\big(\mathcal{I}_h*^{-1}\big[*\check{\phi}\wedge u_h\big]\big) &= \big(\nabla_{M_h}u_h;\nabla_{M_h}\mathcal{I}_h *^{-1}\big[*\check{\phi}\wedge u_h\big]\big) \\
&= \big(\nabla_{M_h}u_h;\nabla_{M_h}*^{-1}\big[*\check{\phi}\wedge u_h\big]\big) + \big(\nabla_{M_h}u_h;\nabla_{M_h}*^{-1}\big(\mathcal{I}_h\big[*\check{\phi}\wedge u_h\big] - *\check{\phi}\wedge u_h\big)\big) \\
&= \big(\nabla_M\widetilde{u}_h;\nabla_M*^{-1}\big[*\phi\wedge\widetilde{u}_h\big]\big) + \big(\mathbf{F}_h\nabla_M\widetilde{u}_h;\nabla_M*^{-1}\big[\phi\wedge\widetilde{u}_h\big]\big) \\
&\quad + \big(\nabla_{M_h}u_h;\nabla_{M_h}*^{-1}\big\{\mathcal{I}_h\big[*\check{\phi}\wedge u_h\big] - *\check{\phi}\wedge u_h\big\}\big) =: I + II + III.
\end{aligned} \tag{5.17}$$

By assumptions on $\mathcal{R}es_h$ and (5.16) we have

$$\mathcal{R}es_h\big(\mathcal{I}_h *^{-1}\big[*\check{\phi}\wedge u_h\big]\big) \to 0$$

as $h \to 0$. Since $\big\|F_h\big\|_{L^\infty(M)} \to 0$ as $h \to 0$ and $\big\|\nabla_M*^{-1}\big[*\phi\wedge\widetilde{u}_h\big]\big\| \le C$ we also verify that

$$II = \big(\mathbf{F}_h\nabla_M\widetilde{u}_h;\nabla_M*^{-1}\big[*\phi\wedge\widetilde{u}_h\big]\big) \to 0$$

as $h \to 0$. The estimate (4.4) in Chapter 1 and Lemma 1.3.7 imply that for each $K \in \mathcal{T}_h$ we have

$$\begin{aligned}
\big\|\nabla_{M_h} *^{-1}\big(\mathcal{I}_h\big[*\check{\phi}\wedge u_h\big]-*\check{\phi}\wedge u_h\big)\big\|_{L^2(K)} &\le Ch\big\|D^2_{M_h}*^{-1}\big[*\check{\phi}\wedge u_h\big]\big\|_{L^2(K)} \\
&\le Ch\big(\big\|\nabla_{M_h}u_h\big\|\big\|\nabla_{M_h}\check{\phi}\big\|_{L^\infty(K)} + \|u_h\|_{L^\infty(M_h)}\big\|D^2_{M_h}\check{\phi}\big\|_{L^2(K)}\big) \\
&\le Ch\big(\big\|\nabla_M\phi\big\|_{L^\infty(\widetilde{K})} + \big\|D^2_M\phi\big\|_{L^2(\widetilde{K})}\big)
\end{aligned}$$

so that

$$III = \big(\nabla_{M_h}u_h;\nabla_{M_h}*^{-1}\big(\mathcal{I}_h\big[*\check{\phi}\wedge u_h\big] - *\check{\phi}\wedge u_h\big)\big) \to 0$$

as $h \to 0$. By Lemma 2.5.1 and the convergence properties of $\widetilde{u}_h$, i.e., $\widetilde{u}_h \to u$ in $L^2(M;\mathbb{R}^n)$ and

$\widetilde{u}_h \rightharpoonup u$ in $W^{1,2}(M;\mathbb{R}^n)$ we infer that

$$
\begin{aligned}
I = \big(\nabla_M \widetilde{u}_h; \nabla_M *^{-1}\big[*\phi \wedge \widetilde{u}_h\big]\big) &= \sum_{\gamma=1}^{m} \big(\underline{D}_\gamma \widetilde{u}_h; *^{-1}\big[(\underline{D}_\gamma *\phi) \wedge \widetilde{u}_h\big]\big) \\
&\to \sum_{\gamma=1}^{m} \big(\underline{D}_\gamma u; *^{-1}\big[(\underline{D}_\gamma *\phi) \wedge u\big) = \big(\nabla_M u; \nabla_M *^{-1}\big[*\phi \wedge u\big]\big),
\end{aligned}
\tag{5.18}
$$

as $h \to 0$. The combination of the identified limits shows that

$$\big(\nabla_M u; \nabla_M *^{-1}\big[*\phi \wedge u\big]\big) = 0$$

and a density argument proves that this identity holds for all $\phi \in W_0^{1,2}(M;\Lambda^2(\mathbb{R}^n)) \cap L^\infty(M;\Lambda^2(\mathbb{R}^n))$. Given any $\psi \in W_0^{1,2}(M;\mathbb{R}^n) \cap L^\infty(M;\mathbb{R}^n)$ such that $u \cdot \psi = 0$ almost everywhere in $M$ there exists $\phi \in W_0^{1,2}(M;\Lambda^2(R^n)) \cap L^\infty(M;\Lambda^2(R^n))$ such that $\psi = *^{-1}\big[*\phi \wedge u\big]$ and thus we deduce that the identity

$$\big(\nabla_M u; \nabla_M \psi\big) = 0$$

is satisfied for all $\psi \in W_0^{1,2}(M;\mathbb{R}^n) \cap L^\infty(M;\mathbb{R}^n)$ such that $\psi \cdot u = 0$ almost everywhere. Proposition 1.7.2 implies that $u$ is a harmonic map. $\square$

# Chapter 3

# Iterative algorithms for the computation of discrete harmonic maps

We analyze and introduce various iterative schemes for the computation of discrete harmonic maps into a large class of submanifolds $N$ in this chapter. The schemes are motivated by $H^1$ and $L^2$ gradient flows of the harmonic map problem or realize a Newton scheme for an equivalent saddle-point formulation. Besides well posedness of the algorithms we discuss stability bounds and convergence on the discrete level and we investigate optimality of constraints on ratios of damping parameters and mesh-sizes or angle conditions of triangulations. Convergence to continuous harmonic maps into submanifolds as the maximal mesh-size and termination criterion tend to zero can then be deduced with the results of the previous chapter.

## 3.1 Discrete harmonic maps

As in the first chapter we assume that the smooth, compact, orientable, $d$-dimensional submanifold $M \subset \mathbb{R}^{d+1}$ either has no boundary or is a Lipschitz domain in $\mathbb{R}^d \times \{0\}$ with polyhedral boundary. In the latter case we let $\Gamma_{\mathrm{D}} \subseteq \partial M$ be such that $\Gamma_{\mathrm{D}}$ is either empty or of positive $(d-1)$-dimensional surface measure. Whenever we are given a triangulation $\mathcal{T}_h$ of a submanifold $M$ with non-empty boundary we suppose that $\Gamma_{\mathrm{D}}$ is matched exactly by the union of edges on $\Gamma_{\mathrm{D}}$.

**Definition 3.1.1.** *Set*

$$\mathring{\mathcal{S}}^1(\mathcal{T}_h) := \begin{cases} \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h) = \{v_h \in \mathcal{S}^1(\mathcal{T}_h) : v_h|_{\Gamma_{\mathrm{D}}} = 0\} & \text{if } \Gamma_{\mathrm{D}} \neq \emptyset, \\ \{v_h \in \mathcal{S}^1(\mathcal{T}_h) : \int_{M_h} v_h \,\mathrm{d}x = 0\} & \text{if } \partial M = \emptyset. \end{cases}$$

*Given* $u_{\mathrm{D}} = \overline{u}_{\mathrm{D}}|_{\Gamma_{\mathrm{D}}}$ *for some* $\overline{u}_{\mathrm{D}} \in W^{1,2}(M;\mathbb{R}^n)$ *such that* $u_{\mathrm{D}} \in C(\Gamma_{\mathrm{D}};\mathbb{R}^n)$ *and* $u_{\mathrm{D}}(x) \in N$ *for all* $x \in \Gamma_{\mathrm{D}}$*, we define* $\overline{u}_{\mathrm{D},h} \in \mathcal{S}^1(\mathcal{T}_h)^n$ *and* $u_{\mathrm{D},h} := \overline{u}_{\mathrm{D},h}|_{\Gamma_{\mathrm{D}}}$ *by setting*

$$\overline{u}_{\mathrm{D},h}(z) := \begin{cases} u_{\mathrm{D}}(z) & \text{for } z \in \mathcal{N}_h \cap \Gamma_{\mathrm{D}}, \\ 0 & \text{for } z \in \mathcal{N}_h \setminus \Gamma_{\mathrm{D}}. \end{cases}$$

**Definition 3.1.2.** *A vector field* $u_h \in \mathcal{S}^1(\mathcal{T}_h)^n$ *is called* a discrete harmonic map into $N$ subject to the boundary data $u_{\mathrm{D},h}$ *if and only if* $u_h|_{\Gamma_\mathrm{D}} = u_{\mathrm{D},h}$, $u_h(z) \in N$ *for all* $z \in \mathcal{N}_h$, *and* $u_h$ *is stationary for*

$$v_h \mapsto \frac{1}{2}\int_{M_h} \left|\nabla_{M_h} v_h\right|^2 \mathrm{d}x$$

*among all* $v_h \in \mathcal{S}^1(\mathcal{T}_h)^n$ *such that* $v_h|_{\Gamma_\mathrm{D}} = u_{\mathrm{D},h}$ *and* $v_h(z) \in N$ *for all* $z \in \mathcal{N}_h$.

**Proposition 3.1.3.** *Given* $u_{\mathrm{D},h}$ *as in Definition 3.1.2 there exists a discrete harmonic map into* $N$.

*Proof.* This follows immediately from the coercivity of the energy functional, the observation that admissible discrete vector fields exist, and the fact that $\mathcal{S}^1(\mathcal{T}_h)^n$ is finite-dimensional. ☐

As in the previous chapter we let $N \subset \mathbb{R}^n$ denote a compact, $k$-dimensional $C^\ell$ submanifold, $\ell \geq 2$, without boundary. Occasionally we will assume that $N$ is orientable and given by smooth level set functions.

**Assumption (O).** $N$ is orientable and there exist functions $f^{k+1}, ..., f^n \in C^\ell(\mathbb{R}^n)$ such that

$$N = \left\{p \in \mathbb{R}^n : f^\ell(p) = 0 \text{ for } \ell = k+1, ..., n\right\}$$

and for all $p \in N$ the vectors

$$\nabla f^{k+1}(p), ..., \nabla f^n(p)$$

are linearly independent.

We remark that if Assumption (O) is satisfied then $\nabla f^\ell(p) \perp T_pN$ for all $p \in N$ and $\ell = k+1, ..., n$ so that we may define

$$\overline{\nu}^\ell(q) := \frac{\nabla f^\ell(q)}{\left|\nabla f^\ell(q)\right|}$$

for $q$ in an appropriate neighborhood of $N$ and $\ell = k+1, ..., n$. According to Lemma 1.6.8 we will assume that $\overline{\nu}^\ell$ is defined in the entire $\mathbb{R}^n$ though not necessarily given by the above expression.

The following assertions characterize discrete harmonic maps and are essential for the definition of the iterative schemes discussed in the following sections.

**Lemma 3.1.4.** *A function* $u_h \in \mathcal{S}^1(\mathcal{T}_h)^n$ *satisfying* $u_h|_{\Gamma_\mathrm{D}} = u_{\mathrm{D},h}$ *is a discrete harmonic map into* $N$ *subject to the boundary data* $u_{\mathrm{D},h}$ *if and only if*
*(a) there holds* $u_h(z) \in N$ *for all* $z \in \mathcal{N}_h$ *and*

$$\left(\nabla_{M_h} u_h; \nabla_{M_h} v_h\right) = 0$$

*for all* $v_h \in \mathcal{S}^1_\mathrm{D}(\mathcal{T}_h)^n$ *such that* $v_h(z) \in T_{u_h(z)}N$ *for all* $z \in \mathcal{N}_h$.
*In case that Assumption (O) is satisfied then (a) holds if and only if*
*(b) there exist* $\lambda_h^\ell \in \mathcal{S}^1(\mathcal{T}_h)$, $\ell = k+1, ..., n$, *such that*

$$\begin{aligned} \left(\nabla_{M_h} u_h; \nabla_{M_h} v_h\right) + \sum_{\ell=k+1}^n \left(\lambda_h^\ell; (\overline{\nu}^\ell \circ u_h) \cdot v_h\right)_h &= 0, \\ \sum_{\ell=k+1}^n \left(\varrho_h^\ell; f^\ell \circ u_h\right)_h &= 0 \end{aligned}$$

*for all* $\left(v_h, (\varrho_h^{k+1}, ..., \varrho_h^n)\right) \in \mathcal{S}^1_\mathrm{D}(\mathcal{T}_h)^n \times \mathcal{S}^1(\mathcal{T}_h)^{n-k}$.

*Proof.* We first assume that Assumption (O) is satisfied and prove equivalence of (a) and (b). Suppose that (b) is satisfied. Then, choosing $\varrho_h^\ell = \varphi_z$ in the second equation and incorporating properties of reduced integration yields that $f^\ell\big(u_h(z)\big) = 0$ for $\ell = k+1, ..., n$ and hence $u_h(z) \in N$ for all $z \in \mathcal{N}_h$. For $v_h \in \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^n$ such that $v_h(z) \in T_{u_h(z)}N$ for all $z \in \mathcal{N}_h$ we have $(\overline{\nu}^\ell \circ u_h)(z) \cdot v_h(z) = 0$ for all $z \in \mathcal{N}_h$ and hence the first equation in (b) reduces to

$$\big(\nabla_{M_h} u_h; \nabla_{M_h} v_h\big) = 0$$

which is the equation in (a).
Conversely, if (a) is satisfied then we define $\lambda_h^\ell \in \mathcal{S}^1(\mathcal{T}_h)$ through

$$\lambda_h^\ell(z) := -\beta_z^{-1}\big(\nabla_{M_h} u_h; \nabla_{M_h}\big[(\overline{\nu}^\ell \circ u_h)(z)\varphi_z\big]\big)$$

for all $z \in \mathcal{N}_h$ and $\ell = k+1, ..., n$. Given any $v_h \in \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^n$ we let $v_h^{nor} \in \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^n$ denote the function that satisfies

$$v_h^{nor}(z) = \sum_{\ell=k+1}^{n} \big\{(\overline{\nu}_\ell \circ u_h)(z) \cdot v_h(z)\big\}(\overline{\nu}_\ell \circ u_h)(z)$$

for all $z \in \mathcal{N}_h$. Then, $v_h^{tan} := v_h - v_h^{nor}$ satisfies $v_h^{tan}(z) \in T_{u_h(z)}N$ for all $z \in \mathcal{N}_h$. Since $\big(\nabla_{M_h} u_h; \nabla_{M_h} v_h^{tan}\big) = 0$ and $u_h(z) \in N$ for all $z \in \mathcal{N}_h$, we deduce with the definition of $\lambda_h^\ell$ that

$$\begin{aligned}
\big(\nabla_{M_h} u_h; \nabla_{M_h} v_h\big) &= \big(\nabla_{M_h} u_h; \nabla_{M_h} v_h^{nor}\big) \\
&= \sum_{z\in\mathcal{N}_h} \sum_{\ell=k+1}^{n} \Big(\nabla_{M_h} u_h; \nabla_{M_h}\big[\big\{(\overline{\nu}_\ell \circ u_h)(z) \cdot v_h(z)\big\}(\overline{\nu}_\ell \circ u_h)(z)\varphi_z\big]\Big) \\
&= \sum_{z\in\mathcal{N}_h} \sum_{\ell=k+1}^{n} \big\{(\overline{\nu}_\ell \circ u_h)(z) \cdot v_h(z)\big\}\Big(\nabla_{M_h} u_h; \nabla_{M_h}\big[(\overline{\nu}_\ell \circ u_h)(z)\varphi_z\big]\Big) \\
&= -\sum_{z\in\mathcal{N}_h} \sum_{\ell=k+1}^{n} \big\{(\overline{\nu}_\ell \circ u_h)(z) \cdot v_h(z)\big\}\beta_z \lambda_h^\ell(z) \\
&= -\sum_{\ell=k+1}^{n} \big(\lambda_h^\ell; (\overline{\nu}_\ell \circ u_h) \cdot v_h\big)_h,
\end{aligned} \tag{1.1}$$

which is the first identity in (b). The second identity follows immediately since $u_h(z) \in N$ for all $z \in \mathcal{N}_h$ implies that $f^\ell\big(u_h(z)\big) = 0$ for $\ell = k+1, ..., n$ and all $z \in \mathcal{N}_h$. The properties of the discrete inner product then show that

$$\big(\varrho_h; f^\ell \circ u_h\big)_h = 0$$

for all $\varrho_h \in \mathcal{S}^1(\mathcal{T}_h)$. Therefore, (b) is satisfied.
It remains to show that (a) is equivalent to Definition 3.1.2. If $u_h$ is a discrete harmonic map into $N$ subject to the boundary data $u_{\mathrm{D},h}$ then for all $v_h \in \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^n$ we have (for sufficiently small $t$ so that $u_h(z) + tv_h(z) \in U_{\delta_N}(N)$ for all $z \in \mathcal{N}_h$, cf. Section 1.6)

$$\lim_{t\to 0} t^{-1}\Big(\frac{1}{2}\int_{M_h} \big|\nabla_{M_h}\mathcal{I}_h\pi_N(u_h + tv_h)\big|^2 \,\mathrm{d}x - \frac{1}{2}\int_{M_h} \big|\nabla_{M_h} u_h\big|^2 \,\mathrm{d}x\Big) = 0, \tag{1.2}$$

where $\mathcal{I}_h \pi_N(u_h + tv_h)$ denotes that function in $\mathcal{S}^1(\mathcal{T}_h)^n$ whose nodal values coincide with $\pi_N\big(u_h(z) + tv_h(z)\big)$ for all $z \in \mathcal{N}_h$. Noting that $\pi_N$ is $C^1$ in a neighborhood of $N$ we find that for all $z \in \mathcal{N}_h$ the function $w_h^t := \mathcal{I}_h\big[\pi_N(u_h + tv_h)\big]$ satisfies

$$w_h^t(z) = \pi_N(u_h + tv_h)(z) = \pi_N\big(u_h(z)\big) + tD\pi_N\big(u_h(z)\big)v_h(z) + o(t)$$

for all $z \in \mathcal{N}_h$, provided that $t$ is small enough. In particular, if $v_h(z) \in T_{u_h(z)}N$ for all $z \in \mathcal{N}_h$ then we verify, using $\pi_N\big(u_h(z)\big) = u_h(z)$ for all $z \in \mathcal{N}_h$ and $D\pi_N(p)|_{T_pN} = \mathrm{id}_{T_pN}$ for all $p \in N$, that

$$w_h^t = u_h + tv_h + o(t).$$

This implies that

$$\frac{1}{2}\int_{M_h} \big|\nabla_{M_h} w_h^t\big|^2 \,\mathrm{d}x - \frac{1}{2}\int_{M_h} \big|\nabla_{M_h} u_h\big|^2 \,\mathrm{d}x = t\int_{Mh} \nabla_{M_h} u_h \cdot \nabla_{M_h} v_h \,\mathrm{d}x + o(t)$$

and hence, (1.2) reduces to

$$\big(\nabla_{M_h} u_h; \nabla_{M_h} v_h\big) = 0.$$

This proves that if $u_h$ is a discrete harmonic map then (a) is satisfied.
Conversely, suppose that (a) holds. For fixed $\varepsilon > 0$ and all $t \in (-\varepsilon, \varepsilon)$ let $w_h^t \in \mathcal{S}^1(\mathcal{T}_h)^n$ be such that $w_h^t(z) \in N$ for all $z \in \mathcal{N}_h$ and $w_h^t|_{\Gamma_\mathrm{D}} = u_{\mathrm{D},h}$. Moreover, suppose that $w_h^0 = u_h$ and the mapping $t \mapsto w_h^t$ is continuously differentiable. We want to show that the function

$$g\colon t \mapsto \frac{1}{2}\int_{M_h} \big|\nabla_{M_h} w_h^t\big|^2 \,\mathrm{d}x$$

satisfies $g'(0) = 0$. A Taylor expansion about $t = 0$ yields

$$w_h^t = u_h + tv_h + o(t),$$

where $v_h \in \mathcal{S}^1_\mathrm{D}(\mathcal{T}_h)^n$ is defined by $v_h(z) := \frac{d}{dt}\big|_{t=0} w_h^t(z)$ for all $z \in \mathcal{N}_h$. Then, $v_h(z) \in T_{u_h(z)}N$ for all $z \in \mathcal{N}_h$ and owing to (a) we have

$$\frac{1}{2}\int_{M_h} \big|\nabla_{M_h} \pi_N(u_h + tv_h)\big|^2 \,\mathrm{d}x - \frac{1}{2}\int_{M_h} \big|\nabla_{M_h} u_h\big|^2 \,\mathrm{d}x = t\big(\nabla_{M_h} u_h; \nabla_{M_h} v_h\big) + o(t) = o(t)$$

which proves that $g'(0) = 0$. This finishes the proof of the lemma. □

**Remark 3.1.5.** *The equations in (b) of the previous lemma characterize a stationary point of the functional*

$$\widehat{E}\big(u_h, (\lambda_h^{k+1}, ..., \lambda_h^n)\big) := \frac{1}{2}\big\|\nabla_{M_h} u_h\big\|^2 + \sum_{\ell=k+1}^{n} \big(\lambda_h^\ell; f^\ell(u_h)\big)_h.$$

*The use of reduced integration is essential to guarantee that $u_h(z) \in N$ for all $z \in \mathcal{N}_h$. Considering the $(n-1)$-dimensional unit sphere $N = S^{n-1}$ and $f^n(p) := |p|^2 - 1$, the condition that*

$$\big(\varrho_h; |u_h|^2 - 1\big) = 0$$

*holds for all $\varrho_h \in \mathcal{S}^1(\mathcal{T}_h)$ only implies that the constraint is satisfied in an averaged sense, namely that $P_h\big[|u_h|^2\big] = 1$, where $P_h$ is the $L^2$ orthogonal projection onto $\mathcal{S}^1(\mathcal{T}_h)$. This does in general not imply that $|u_h(z)| = 1$ holds for all nodes $z \in \mathcal{N}_h$.*

## 3.2 Stability and convergence of an $H^1$ gradient flow approach

The first approximation scheme discussed in this chapter is motivated by an approach proposed in [Alo97] on a continuous level for harmonic maps into spheres. The scheme can best be motivated by considering a higher order gradient flow for the minimization of the constrained Dirichlet energy. The $H^1$ gradient flow for harmonic maps into $N$ describes a function $u\colon (0,\infty)\times M \to N$ such that $u(0,\cdot) = u_0$, $u(t,\cdot)|_{\Gamma_D} = u_D$, and

$$\big(\nabla_M \partial_t u; \nabla_M v\big) + \big(\nabla_M u; \nabla_M v\big) = 0$$

for almost every $t \in (0,\infty)$ and all $v \in W^{1,2}_D(M;\mathbb{R}^n)$ such that $v(x) \in T_{u(t,x)}N$ for almost every $x \in M$. Noting that $\partial_t u(t,x) \in T_{u(t,x)}N$ for almost every $x \in M$ yields a symmetric reformulation of the problem in which $w = \partial_t u$ is the unknown rather than $u$ itself. An explicit discretization in time with a time-step size (or damping parameter) $\kappa > 0$ based on these observations reads as follows (where we assume for ease of presentation that $\Gamma_D \neq \emptyset$).

**Algorithm I** (Explicit $H^1$ flow semi-discretization)**.** *Input:* damping parameter $\kappa > 0$.

1. Choose $u^0 \in W^{1,2}(M;\mathbb{R}^n)$ such that $u^0|_{\Gamma_D} = u_D$ and $u^0(x) \in N$ for almost every $x \in M$. Set $i := 0$.
2. Compute $w^i \in W^{1,2}_D(M;\mathbb{R}^n)$ such that $w^i(x) \in T_{u^i(x)}N$ for almost every $x \in M$ and
$$\big(\nabla_M w^i; \nabla_M v\big) + \big(\nabla_M u^i; \nabla_M v\big) = 0$$
for all $v \in W^{1,2}_D(M;\mathbb{R}^n)$ such that $v(x) \in T_{u(x)}N$ for almost every $x \in M$.
3. Set
$$u^{i+1}(x) := \pi_N\big(u^i(x) + \kappa w^i(x)\big)$$
for almost every $x \in M$.
4. Set $i := i + 1$ and go to 2.

The same algorithm can be derived by considering a linearization of the constraint $u(x) \in N$ for almost every $x \in M$: suppose $u^i$ satisfies $u^i(x) \in N$ for almost every $x \in M$ and assume that it serves as an approximation of a harmonic map into $N$. If we look for a correction $w$ so that $u^i + \kappa w$ is a harmonic map, then a linearization about $u^i(x)$ of the condition $(u^i + \kappa w)(x) \in N$ reads $w(x) \in T_{u^i(x)}N$ for almost every $x \in M$. An iterative scheme that alternatingly computes a correction $w^i$ subject to the linearized constraint and then projects the temporary update $u^{i+1} := u^i + \kappa w^i$ onto $N$ leads to the algorithm above in which $\kappa$ can be thought of as a damping parameter. Yet another interpretation of the algorithm is through a simplification of a Newton iteration, see Remarks 3.5.1 (iii) for details.

Below we are not primarily interested in the question whether the output of our algorithms approximate the time-dependent problem described above, but aim at understanding under what conditions on the time-step size (respectively damping parameter) the iterations are well-defined and whether convergence of the iterates $u^i$ to a discrete harmonic map can be established. We refer to [AJ06, BKP07] for a related algorithm for the approximation of the Landau-Lifshitz-Gilbert equation of micromagnetics (with the target manifold being the unit sphere), and to [BBFP07] for a scheme that approximates the $p$-harmonic heat flow into spheres.

### 3.2.1 Full discretization

A fully discrete version of Algorithm I replaces the pointwise operations and conditions such as the projection onto $N$ by corresponding conditions at the nodes of a triangulation and reads as follows.

**Algorithm A.** *Input:* triangulation $\mathcal{T}_h$, damping parameter $\kappa > 0$, stopping criterion $\varepsilon > 0$.

1. Choose $u_h^0 \in \mathcal{S}^1(\mathcal{T}_h)^n$ such that $u_h^0|_{\Gamma_{\mathrm{D}}} = u_{\mathrm{D},h}$ and $u_h^0(z) \in N$ for all $z \in \mathcal{N}_h \setminus \Gamma_{\mathrm{D}}$. Set $i := 0$.
2. Compute $w_h^i \in \mathring{\mathcal{S}}^1(\mathcal{T}_h)^n$ such that $w_h^i(z) \in T_{u_h^i(z)}N$ for all $z \in \mathcal{N}_h$ and
$$\big(\nabla_{M_h} w_h^i; \nabla_{M_h} v_h\big) = -\big(\nabla_{M_h} u_h^i; \nabla_{M_h} v_h\big)$$
for all $v_h \in \mathring{\mathcal{S}}^1(\mathcal{T}_h)^n$ such that $v_h(z) \in T_{u_h^i(z)}N$ for all $z \in \mathcal{N}_h$.
3. Stop if $\|\nabla_{M_h} w_h^i\| \le \varepsilon$.
4. Define $u_h^{i+1} \in \mathcal{S}^1(\mathcal{T}_h)^n$ by setting
$$u_h^{i+1}(z) := \pi_N\big(u_h^i(z) + \kappa w_h^i(z)\big)$$
for all $z \in \mathcal{N}_h$.
5. Set $i := i + 1$ and go to 2.

*Output:* $u_h^* := u_h^i$.

**Remark 3.2.1.** *The solution $w_h^i$ in Step 2 can be computed from a saddle-point problem which seeks $\big(w_h, (\lambda_h^{k+1}, ..., \lambda_h^n)\big) \in \mathring{\mathcal{S}}^1(\mathcal{T}_h)^n \times \mathcal{S}^1(\mathcal{T}_h)^{n-k}$ such that*

$$\begin{aligned}
\big(\nabla_{M_h} w_h; \nabla_{M_h} v_h\big) \quad + \sum_{\ell=k+1}^{n} \big(\lambda_h^\ell; (\overline{\nu}^\ell \circ u_h^i) \cdot v_h\big)_h &= \big(\nabla_{M_h} u_h^i; \nabla_{M_h} v_h\big), \\
\sum_{\ell=k+1}^{n} \big(\varrho_h^\ell; (\overline{\nu}^\ell \circ u_h^i) \cdot v_h\big)_h &= 0,
\end{aligned}$$

*for all $\big(v_h, (\varrho_h^{k+1}, ..., \varrho_h^n)\big) \in \mathring{\mathcal{S}}^1(\mathcal{T}_h)^n \times \mathcal{S}^1(\mathcal{T}_h)^{n-k}$. Notice that the normal vectors $\overline{\nu}^\ell$ are only required at the nodes and need not be globally continuous.*

In the following subsections we provide sufficient conditions that guarantee that Algorithm A is well-defined, stable, and convergent to a discrete harmonic map into $N$ as $\varepsilon \to 0$.

### 3.2.2 Well-posedness

In case that the target manifold $N$ is not the boundary of a convex set then mild but dimension-dependent conditions on the damping parameter $\kappa$ ensure that the projection onto $N$ is well-defined in each step of the iteration. For ease of readability we fix $i$ and drop the superscripts in the following lemma.

**Lemma 3.2.2.** *(i) Given any $u_h \in \mathcal{S}^1(\mathcal{T}_h)^n$ satisfying $u_h(z) \in N$ for all $z \in \mathcal{N}_h \setminus \Gamma_D$ there exists a unique $w_h \in \mathring{\mathcal{S}}^1(\mathcal{T}_h)^n$ such that $w_h(z) \in T_{u_h^i(z)}N$ for all $z \in \mathcal{N}_h$ and*

$$\big(\nabla_{M_h} w_h; \nabla_{M_h} v_h\big) = -\big(\nabla_{M_h} u_h; \nabla_{M_h} v_h\big)$$

*for all $v_h \in \mathring{\mathcal{S}}^1(\mathcal{T}_h)^n$ such that $v_h(z) \in T_{u_h^i(z)}N$ for all $z \in \mathcal{N}_h$.*
*(ii) There exists a constant $C_{N,\mathcal{T}_h} > 0$ such that the function $w_h$ satisfies*

$$\|w_h\|_{L^\infty(M_h)} \le C_{N,\mathcal{T}_h} h_{min}^{1-d/2} \log h_{min}^{-1} \big\|\nabla_{M_h} u_h\big\|.$$

*In particular, if $\kappa \le \big(C_0 C_{N,\mathcal{T}_h}\big)^{-1} h_{min}^{d/2-1} \log h_{min}^{-1} \omega_N$ for $C_0 := \big\|\nabla_{M_h} u_h\big\|$ then*

$$\operatorname{dist}\big(u_h(z) + \kappa w_h(z), N\big) \le \delta_N$$

*so that $\pi_N\big(u_h(z) + \kappa w_h(z)\big)$ is well-defined for all $z \in \mathcal{N}_h$.*

**Remark 3.2.3.** *Recall from Section 1.6 that $\omega_N = \infty$ if $N = \partial\mathcal{C}$ for a convex set $\mathcal{C} \subseteq \mathbb{R}^n$ so that in this case the projection in Step 4 is always well-defined.*

*Proof.* The set

$$L_h := \Big\{ v_h \in \mathring{\mathcal{S}}^1(\mathcal{T}_h)^n : v_h(z) \in T_{u_h(z)}N \text{ for all } z \in \mathcal{N}_h \Big\}$$

is a subspace of $\mathring{\mathcal{S}}^1(\mathcal{T}_h)^n$. The Lax-Milgram lemma guarantees the existence of a unique $w_h \in L_h$ such that

$$\big(\nabla_{M_h} w_h; \nabla_{M_h} v_h\big) = -\big(\nabla_{M_h} u_h; \nabla_{M_h} v_h\big)$$

for all $v_h \in L_h$, in particular,

$$\big\|\nabla_{M_h} w_h\big\| \le \big\|\nabla_{M_h} u_h\big\|.$$

The Poincaré inequality of Lemma 1.2.2 and the inverse estimate of Lemma 1.4.9 show

$$\|w_h\|_{L^\infty(M_h)} \le C h_{min}^{1-d/2} \log h_{min}^{-1} \big\|\nabla_{M_h} w_h\big\|.$$

If $\kappa \le \big(C_0 C\big)^{-1} h_{min}^{d/2-1} \log h_{min}^{-1} \omega_N$ then we have by definition of $\omega_N$ in Lemma 1.6.4 for all $z \in \mathcal{N}_h$ that

$$\operatorname{dist}\big(u_h(z) + \kappa w_h(z), N\big) \le \delta_N$$

which ensures that $\pi_N\big(u_h(z) + \kappa w_h(z)\big)$ is well-defined for all $z \in \mathcal{N}_h$. □

### 3.2.3 Stability

Slightly more restrictive conditions on the damping parameter $\kappa$ and regularity of $N$ are required to ensure uniform boundedness of iterates of Algorithm A in $W^{1,2}(M;\mathbb{R}^n)$.

**Lemma 3.2.4.** *In addition to the above assumptions suppose that $N$ is a $C^3$ submanifold and $d \le 4$. Then, there exist constants $C', C'' > 0$ such that if*

$$\kappa \le C' \min\big\{h_{min}, \omega_N h_{min}^{d/2-1} |\log h_{min}|^{-1}\big\}$$

*then we have for $J \in \mathbb{N}$ and iterates $u_h^0, u_h^1, ..., u_h^{J+1}$ and $w_h^1, w_h^2, ..., w_h^{J+1}$ of Algorithm A that*

$$\big(1 - C''\kappa h_{min}^{-1}\big)\kappa \sum_{i=0}^{J} \big\|\nabla_{M_h} w_h^i\big\|^2 + \frac{1}{2}\big\|\nabla_{M_h} u_h^{J+1}\big\|^2 \le \frac{1}{2}\big\|\nabla_{M_h} u_h^0\big\|^2.$$

*The constants $C', C'' > 0$ only depend on $C_0 := \big\|\nabla_{M_h} u_h^0\big\|$, $N$, and the geometry of $\mathcal{T}_h$.*

*Proof.* Owing to Lemma 3.2.2 and the assumptions on $\kappa$ we have that all steps of Algorithm A are well-defined. Using that $\pi_N$ is twice continuously differentiable in a neighborhood of $N$, cf. Theorem 1.6.1, recalling that $D\pi_N(p)|_{T_pN} = \mathrm{id}|_{T_pN}$ for all $p \in N$, and employing the fact that $N$ is compact so that $|u_h^i(z)| \le C$ for all $z \in \mathcal{N}_h$, we verify that the identity

$$\begin{aligned} u_h^{i+1}(z) &= \pi_N\big(u_h^i(z) + \kappa w_h^i(z)\big) \\ &= \pi_N\big(u_h^i(z)\big) + \kappa D\pi_N\big(u_h^i(z)\big) w_h^i(z) + \mathcal{O}\big(\big|\kappa w_h^i(z)\big|^2\big) \\ &= u_h^i(z) + \kappa w_h^i(z) + \mathcal{O}\big(\big|\kappa w_h^i(z)\big|^2\big) \end{aligned}$$

is satisfied for all $z \in \mathcal{N}_h$. We define $r_h^{i+1} := \big(u_h^{i+1} - u_h^i\big) - \kappa w_h^i$ and deduce from the last estimate that

$$\big|r_h^{i+1}(z)\big| \le C\kappa^2 \big|w_h^i(z)\big|^2,$$

with a constant that only depends on $N$. From this estimate we derive the bound

$$\big\|r_h^{i+1}\big\|^2 \le C\kappa^4 \big\|w_h^i\big\|_{L^4(M_h)}^4, \tag{2.3}$$

for which we employed (4.6) of Chapter 1. Owing to the first equation of Algorithm A we have, upon choosing $v_h = w_h^i = \kappa^{-1}(u_h^{i+1} - u_h^i) - \kappa^{-1} r_h^{i+1}$ and employing the binomial identity $b(b-a) = (b-a)^2/2 + (b^2 - a^2)/2$,

$$\begin{aligned} \big\|\nabla_{M_h} w_h^i\big\|^2 &= -\big(\nabla_{M_h} u_h^i; \nabla_{M_h} w_h^i\big) \\ &= -\kappa^{-1}\big(\nabla_{M_h} u_h^i; \nabla_{M_h}(u_h^{i+1} - u_h^i)\big) + \kappa^{-1}\big(\nabla_{M_h} u_h^i; \nabla_{M_h} r_h^{i+1}\big) \\ &= \kappa^{-1}\big(\nabla_{M_h}(u_h^{i+1} - u_h^i); \nabla_{M_h}(u_h^{i+1} - u_h^i)\big) - \kappa^{-1}\big(\nabla_{M_h} u_h^{i+1}; \nabla_{M_h}(u_h^{i+1} - u_h^i)\big) \\ &\quad + \kappa^{-1}\big(\nabla_{M_h} u_h^i; \nabla_{M_h} r_h^{i+1}\big) \\ &= \kappa^{-1}\big\|\nabla_{M_h}(u_h^{i+1} - u_h^i)\big\|^2 - \frac{1}{2\kappa}\big\|\nabla_{M_h}(u_h^{i+1} - u_h^i)\big\|^2 \\ &\quad - \frac{1}{2\kappa}\big(\big\|\nabla_{M_h} u_h^{i+1}\big\|^2 - \big\|\nabla_{M_h} u_h^i\big\|^2\big) + \kappa^{-1}\big(\nabla_{M_h} u_h^i; \nabla_{M_h} r_h^{i+1}\big), \end{aligned}$$

or equivalently,

$$\begin{aligned}\big\|\nabla_{M_h} w_h^i\big\|^2 + \frac{1}{2\kappa}\big(\big\|\nabla_{M_h} u_h^{i+1}\big\|^2 - \big\|\nabla_{M_h} u_h^i\big\|^2\big)& \\ = \frac{1}{2\kappa}\big\|\nabla_{M_h}(u_h^{i+1}-u_h^i)\big\|^2 + \kappa^{-1}\big(\nabla_{M_h} u_h^i; \nabla_{M_h} r_h^{i+1}\big).&\end{aligned} \tag{2.4}$$

To bound the first term on the right-hand side we note that, according to the definition of $r_h^{i+1}$,

$$\begin{aligned}\frac{1}{2\kappa}\big\|\nabla_{M_h}(u_h^{i+1}-u_h^i)\big\|^2 &= \frac{\kappa}{2}\big\|\nabla_{M_h}(u_h^{i+1}-u_h^i)/\kappa\big\|^2 \\ &\le \kappa\big\|\nabla_{M_h} w_h^i\big\|^2 + \kappa\big\|\kappa^{-1}\nabla_{M_h} r_h^{i+1}\big\|^2.\end{aligned}$$

An inverse estimate, the bound (2.3), and the Sobolev estimate $\|w_h^i\|_{L^4(M_h)} \le C\|\nabla_{M_h} w_h^i\|$ for $d \le 4$, cf. Theorem 1.2.1, show that

$$\begin{aligned}\kappa^{-1}\big\|\nabla_{M_h} r_h^{i+1}\big\|^2 &\le C\kappa^{-1}h_{min}^{-2}\big\|r_h^{i+1}\big\|^2 \\ &\le C\kappa^{-1}h_{min}^{-2}\kappa^4\big\|w_h^i\big\|_{L^4(M_h)}^4 \\ &\le C\kappa^3 h_{min}^{-2}\big\|\nabla_{M_h} w_h^i\big\|^4.\end{aligned}$$

Suppose that $\|\nabla_{M_h} u_h^i\| \le C_0$ (which is, by definition of $C_0$, satisfied for $i = 0$). Then we clearly have $\|\nabla_{M_h} w_h^i\| \le C_0$ and hence

$$\kappa^{-1}\big\|\nabla_{M_h} r_h^{i+1}\big\|^2 \le CC_0^2\kappa^3 h_{min}^{-2}\big\|\nabla_{M_h} w_h^i\big\|^2.$$

The previous estimates show

$$\frac{1}{2\kappa}\big\|\nabla_{M_h}(u_h^{i+1}-u_h^i)\big\|^2 \le \big(1 + CC_0^2\kappa^2 h_{min}^{-2}\big)\kappa\big\|\nabla_{M_h} w_h^i\big\|^2. \tag{2.5}$$

The second term on the right-hand side of (2.4) is bounded using $\|\nabla_{M_h} u_h^i\| \le C_0$, an inverse estimate, and (2.3) by

$$\begin{aligned}\kappa^{-1}\big(\nabla_{M_h} u_h^i; \nabla_{M_h} r_h^{i+1}\big) &\le \kappa^{-1}\big\|\nabla_{M_h} u_h^i\big\|\,\big\|\nabla_{M_h} r_h^{i+1}\big\| \\ &\le \kappa^{-1}C_0\big\|\nabla_{M_h} r_h^{i+1}\big\| \\ &\le \kappa^{-1}C_0 h_{min}^{-1}\big\|r_h^{i+1}\big\| \\ &\le \kappa^{-1}C_0 h_{min}^{-1}\kappa^2\big\|w_h^i\big\|_{L^4(M_h)}^2 \\ &\le \kappa^{-1}CC_0 h_{min}^{-1}\kappa^2\big\|\nabla_{M_h} w_h^i\big\|^2.\end{aligned} \tag{2.6}$$

The combination of (2.4) with (2.5) and (2.6) implies, upon using $\kappa h_{min}^{-1} \le C'$, that

$$\big(1 - C\kappa h_{min}^{-1}\big)\big\|\nabla_{M_h} w_h^i\big\|^2 + \frac{1}{2\kappa}\big(\big\|\nabla_{M_h} u_h^{i+1}\big\|^2 - \big\|\nabla_{M_h} u_h^i\big\|^2\big) \le 0.$$

This estimate shows that $\|\nabla_{M_h} u_h^{i+1}\| \le C_0$ if $\|\nabla_{M_h} u_h^i\| \le C_0$ and hence justifies the above assumption that $\|\nabla_{M_h} u_h^i\| \le C_0$. This finishes the proof of the theorem. □

### 3.2.4 A sharp refinement for convex targets

As noted in Remark 3.2.3, if $N = \partial\mathcal{C}$ is the boundary of a convex set then Algorithm A is well-defined for all choices of $\kappa$. Provided that the underlying triangulation $\mathcal{T}_h$ is weakly acute and $\kappa < 2$, Algorithm A is also unconditionally stable, owing to the fact that the projection $\pi_N$ is non-expanding. The regularity assumptions on $N$ of Lemma 3.2.4 are not required in the following assertion.

**Lemma 3.2.5.** *Suppose that $\mathcal{T}_h$ is weakly acute, $\kappa < 2$, and $N = \partial\mathcal{C}$ for a bounded, open, convex set $\mathcal{C} \subset \mathbb{R}^n$. Then, for $J \in \mathbb{N}$ and iterates $u_h^0, u_h^1, ..., u_h^{J+1}$ and $w_h^1, w_h^2, ..., w_h^{J+1}$ of Algorithm A we have*

$$\kappa\big(1-\kappa/2\big)\sum_{i=0}^{J}\big\|\nabla_{M_h} w_h^i\big\|^2 + \frac{1}{2}\big\|\nabla_{M_h} u_h^{J+1}\big\|^2 \le \frac{1}{2}\big\|\nabla_{M_h} u_h^0\big\|^2.$$

*Proof.* Since $\big\|\nabla_{M_h} w_h^i\big\|^2 = -\big(\nabla_{M_h} u_h^i; \nabla_{M_h} w_h^i\big)$ we have

$$\begin{aligned}
\frac{1}{2}\big\|\nabla_{M_h}\big(u_h^i + \kappa w_h^i\big)\big\|^2 &= \frac{1}{2}\big\|\nabla_{M_h} u_h^i\big\|^2 + \kappa\big(\nabla_{M_h} u_h^i; \nabla_{M_h} w_h^i\big) + \frac{\kappa^2}{2}\big\|\nabla_{M_h} w_h^i\big\|^2 \\
&= \frac{1}{2}\big\|\nabla_{M_h} u_h^i\big\|^2 - \kappa\big(1-\kappa/2\big)\big\|\nabla_{M_h} w_h^i\big\|^2.
\end{aligned}$$

Employing the fact that for weakly acute triangulations we have that $\mathbf{K}_{z,z'} \le 0$ for distinct $z, z' \in \mathcal{N}_h$, cf. Definition 1.4.2, noting that $u_h^i(z) + \kappa w_h^i(z) \notin \mathcal{C}$ for all $z \in \mathcal{N}_h$, and recalling that the projection $\pi_N\colon \mathbb{R}^n \setminus \mathcal{C} \to N$ is Lipschitz continuous with Lipschitz constant less than or equal to 1 we infer with Lemma 1.4.4 that

$$\begin{aligned}
\big\|\nabla_{M_h} u_h^{i+1}\big\|^2 &= -\frac{1}{2}\sum_{z,z'\in\mathcal{N}_h}\mathbf{K}_{z,z'}\big|u_h^{i+1}(z) - u_h^{i+1}(z')\big|^2 \\
&= -\frac{1}{2}\sum_{z,z'\in\mathcal{N}_h}\mathbf{K}_{z,z'}\big|\pi_N\big(u_h^i(z) + \kappa w_h^i(z)\big) - \pi_N\big(u_h^i(z') + \kappa w_h^i(z')\big)\big|^2 \\
&\le -\frac{1}{2}\sum_{z,z'\in\mathcal{N}_h}\mathbf{K}_{z,z'}\big|\big(u_h^i(z) + \kappa w_h^i(z)\big) - \big(u_h^i(z') + \kappa w_h^i(z')\big)\big|^2 \\
&= \big\|\nabla_{M_h}\big(u_h^i + \kappa w_h^i\big)\big\|^2.
\end{aligned}$$

Here we also used that contributions to the sum are trivial for $z = z'$. A combination of the estimates implies

$$\kappa\big(1-\kappa/2\big)\big\|\nabla_{M_h} w_h^i\big\|^2 + \frac{1}{2}\big\|\nabla_{M_h} u_h^{i+1}\big\|^2 - \frac{1}{2}\big\|\nabla_{M_h} u_h^i\big\|^2 \le 0,$$

which is the asserted bound after summation over $i = 0, 1, 2, .., J$. □

Weak acuteness of a triangulation is not just a technical detail to ensure stability of Step 4 in Algorithm A for $N = \partial\mathcal{C}$ and $0 < \kappa < 2$. The angle condition is sharp in the sense of the following example which is a refinement of an example from [Bar05b].

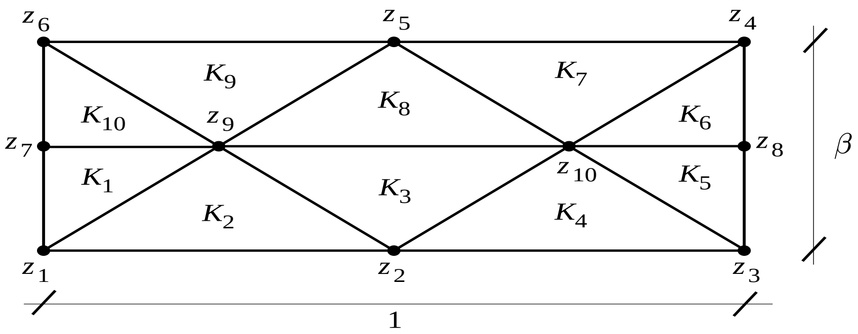


Figure 3.1: Triangulation in Example 3.2.6 which is weakly acute if and only if $\beta \geq 1/2$.

**Example 3.2.6.** *Let $0 < \beta \leq 1$, $\kappa > 0$, $M := (0,1) \times (0,\beta)$, and let $\mathcal{T}_h$ be the triangulation of $M$ defined through the nodes*

$$\begin{aligned} &z_1 := (0,0), \quad && z_2 := (1/2,0), \quad && z_3 := (1,0), \quad && z_4 := (1,\beta), \quad && z_5 := (1/2,\beta), \\ &z_6 := (0,\beta), && z_7 := (0,\beta/2), && z_8 := (1,\beta/2), && z_9 := (1/4,\beta/2), && z_{10} := (3/4,\beta/2) \end{aligned}$$

*and triangles*

$$\begin{aligned} &K_1 := \operatorname{conv}\{z_1,z_7,z_9\}, \quad K_2 := \operatorname{conv}\{z_1,z_2,z_9\}, \quad K_3 := \operatorname{conv}\{z_2,z_9,z_{10}\}, \quad K_4 := \operatorname{conv}\{z_2,z_3,z_{10}\}, \\ &K_5 := \operatorname{conv}\{z_3,z_8,z_{10}\}, \quad K_6 := \operatorname{conv}\{z_4,z_8,z_{10}\}, \quad K_7 := \operatorname{conv}\{z_4,z_5,z_{10}\}, \quad K_8 := \operatorname{conv}\{z_5,z_9,z_{10}\}, \\ &K_9 := \operatorname{conv}\{z_5,z_6,z_9\}, \quad K_{10} := \operatorname{conv}\{z_6,z_7,z_9\}, \end{aligned}$$

*cf. Figure 3.1. Set $s := 1/2 - \beta$ and let $u_h^i, w_h^i \in \mathcal{S}^1(\mathcal{T}_h)^n$, $n \geq 2$, be the functions satisfying*

$$u_h^i(z_j) = (1,0,...,0) \quad \textit{for } j = 1,2,...,8, \quad u_h^i(z_9) = (-1,0,...,0), \quad u_h^i(z_{10}) = (1,0,...,0),$$

*and*

$$w_h^i(z_j) = 0 \quad \textit{for } j = 1,2,...,9, \quad w_h^i(z_{10}) = (0,-s/\kappa,0,...,0).$$

*Then, $u_h^i(z) \in S^{n-1}$ and $w_h^i(z) \in T_{u_h^i(z)}S^{n-1}$ for all $z \in \mathcal{N}_h$. For $u_h^{i+1} \in \mathcal{S}^1(\mathcal{T}_h)^n$ defined through $u_h^{i+1}(z) := \pi_{S^{n-1}}\big(u_h^i(z) + \kappa w_h^i(z)\big)$ for all $z \in \mathcal{N}_h$ we have*

$$\big\|\nabla_{M_h} u_h^{i+1}\big\| \leq \big\|\nabla_{M_h}\big(u_h^i + \kappa w_h^i\big)\big\|$$

*if and only if $\mathcal{T}_h$ is weakly acute, i.e., if and only if $\beta \geq 1/2$.*

*Proof.* The proof of Lemma 3.2.5 shows that the estimate holds if $\mathcal{T}_h$ is weakly acute and this is the case if and only if $\beta \geq 1/2$. Suppose that $\beta < 1/2$ and abbreviate $v_h := u_h^i + \kappa w_h^i$ and $u_h := u_h^{i+1}$.

Owing to symmetry of $\mathbf{K}$, Lemma 1.4.4, and $v_h(z_j) = u_h(z_j)$ for $j = 1, 2, ..., 9$ we have

$$\begin{aligned}
\delta &:= \big\|\nabla_{M_h} v_h\big\|^2 - \big\|\nabla_{M_h} u_h\big\|^2 \\
&= -\frac{1}{2} \sum_{z,z' \in \mathcal{N}_h} \mathbf{K}_{z,z'} \big(|v_h(z) - v_h(z')|^2 - |u_h(z) - u_h(z')|^2\big) \\
&= -\sum_{j=1}^{9} \mathbf{K}_{z_j,z_{10}} \big(|v_h(z_j) - v_h(z_{10})|^2 - |u_h(z_j) - u_h(z_{10})|^2\big) \\
&= -\sum_{j=1}^{8} \mathbf{K}_{z_j,z_{10}} \big(|(1,0,...,0) - v_h(z_{10})|^2 - |(1,0,...,0) - u_h(z_{10})|^2\big) \\
&\qquad - \mathbf{K}_{z_9,z_{10}} \big(|(-1,0,...,0) - v_h(z_{10})|^2 - |(-1,0,...,0) - u_h(z_{10})|^2\big).
\end{aligned}$$

We have $|(1,0,...,0) - v_h(z_{10})|^2 = s^2$, $|(-1,0,...,0) - v_h(z_{10})|^2 = 4 + s^2$, and

$$\begin{aligned}
t_1^2 &:= |(1,0,...,0) - u_h(z_{10})|^2 = 2 - 2/\sqrt{1+s^2}, \\
t_2^2 &:= |(-1,0,...,0) - u_h(z_{10})|^2 = 2 + 2/\sqrt{1+s^2}.
\end{aligned}$$

Since $\sum_{j=1}^{10} \mathbf{K}_{z_j,z_{10}} = 0$ we have $\sum_{j=1}^{8} \mathbf{K}_{z_j,z_{10}} = -\mathbf{K}_{z_9,z_{10}} - \mathbf{K}_{z_{10},z_{10}}$ and hence

$$\begin{aligned}
\delta &= (s^2 - t_1^2)\big(\mathbf{K}_{z_9,z_{10}} + \mathbf{K}_{z_{10},z_{10}}\big) - \mathbf{K}_{z_9,z_{10}}(4 + s^2 - t_2^2) \\
&= \mathbf{K}_{z_{10},z_{10}}(s^2 - t_1^2) - \mathbf{K}_{z_9,z_{10}}(4 + t_1^2 - t_2^2).
\end{aligned}$$

Direct calculations yield to

$$\mathbf{K}_{z_{10},z_{10}} = (12\beta^2 + 5)/(4\beta), \qquad \mathbf{K}_{z_9,z_{10}} = (1 - 4\beta^2)/(4\beta).$$

Setting $\phi(s^2) := \sqrt{1+s^2} - 1 - s^2/2$ we have

$$4\beta\sqrt{1+s^2}\,\delta = \big(12\beta^2 + 5\big)\big(s^4/2 + s^2\phi(s^2) - 2\phi(s^2)\big) - \big(1 - 4\beta^2\big)\big(2s^2 + 4\phi(s^2)\big).$$

Since $\beta^2 = 1/4 - s + s^2$ we verify that

$$\begin{aligned}
4\beta\sqrt{1+s^2}\,\delta &= \big(8 - 12s + 12s^2\big)\big(s^4/2 + s^2\phi(s^2) - 2\phi(s^2)\big) - 16(s - s^2)\big(s^2/2 + \phi(s^2)\big) \\
&= -8s^3 + 12s^4 - 6s^5 + 6s^6 + \phi(s^2)\big(-16_9 s - 12s^3 + 12s^4\big) \\
&= -6s^3(1 - 2s) - 6s^5(1 - s) + 4s\phi(s^2)\big(2 - 3s^2 + 3s^3\big) - 2\big(s^3 + 8\phi(s^2)\big).
\end{aligned}$$

Since $0 < s < 1/2$ and $\phi(s^2) < 0$, the first three terms on the right-hand side are negative. A Taylor expansion reveals $-s^4/8 \le \phi(s^2)$ and implies that the last term on the right-hand side is non-positive. This shows $\delta < 0$ if $\beta < 1/2$ and implies the assertion of the example. □

### 3.2.5 Termination and convergence

In case that Algorithm A is well-defined and stable, the iterates converge to a discrete harmonic map into $N$. Notice that in the following theorem the discretization is fixed.

**Theorem 3.2.7.** *Suppose that the conditions of Lemma 3.2.4 or Lemma 3.2.5 are satisfied. Then, Algorithm A terminates within a finite number of iterations and the output* $u_h^*$ *satisfies* $u_h^*(z) \in N$ *for all* $z \in \mathcal{N}_h$, $u_h^*|_{\Gamma_D} = u_{D,h}$, *and*

$$\big(\nabla_{M_h} u_h^*; \nabla_{M_h} v_h\big) = \mathcal{R}es_h\big(v_h\big),$$

*for all* $v_h \in \overset{\circ}{\mathcal{S}}{}^1(\mathcal{T}_h)^n$ *such that* $v_h(z) \in T_{u_h^*(z)}N$ *for all* $z \in \mathcal{N}_h$, *where the linear functional* $\mathcal{R}es_h : \mathcal{S}^1_D(\mathcal{T}_h)^n \to \mathbb{R}$ *satisfies*

$$\big|\mathcal{R}es_h\big(v_h\big)\big| \le \varepsilon \,\|\nabla_{M_h} v_h\|$$

*for all* $v_h \in \mathcal{S}^1_D(\mathcal{T}_h)^n$. *Moreover, for a sequence* $\big(\varepsilon_J\big)_{J\in\mathbb{N}}$ *of positive numbers such that* $\varepsilon_J \to 0$ *as* $J \to \infty$, *every accumulation point of the corresponding bounded sequence of outputs* $\big(u_h^{*,J}\big)_{J\in\mathbb{N}}$ *of Algorithm A is a discrete harmonic map into* $N$ *subject to the boundary data* $u_{D,h}$.

*Proof.* By Lemma 3.2.4 or Lemma 3.2.5 we have that for all $J \in \mathbb{N}$ the bound

$$\sum_{i=0}^{J} \big\|\nabla_{M_h} w_h^i\big\|^2 \le C\big(h, \kappa, u_h^0\big)$$

is satisfied. Therefore, $w_h^i \to 0$ as $i \to \infty$ and hence Algorithm A terminates within a finite number of iterations. If $u_h^* = u_h^{i^*}$ for some $i^* \in \mathbb{N}$ is the output of Algorithm A, we verify the first part of the theorem upon defining $\mathcal{R}es_h : \mathcal{S}^1_D(\mathcal{T}_h)^n \to \mathbb{R}$ through

$$\mathcal{R}es_h\big(v_h\big) := \big(\nabla_{M_h} w_h^{i^*}; \nabla_{M_h} v_h\big)$$

for $v_h \in \mathcal{S}^1_D(\mathcal{T}_h)^n$ and recalling that by the termination criterion we have $\big\|\nabla_{M_h} w_h^{i^*}\big\| \le \varepsilon$. For a sequence $\varepsilon_J \to 0$, $J \in \mathbb{N}$, the corresponding sequence of outputs $\big(u_h^{*,J}\big)_{J\in\mathbb{N}}$ obtained with the same $u_h^0$ is uniformly bounded. Let $u_h \in \mathcal{S}^1(\mathcal{T}_h)^n$ be the limit of a subsequence of $\big(u_h^{*,J}\big)_{J\in\mathbb{N}}$ which is not relabeled in the following. Then, $u_h(z) = \lim_{J\to\infty} u_h^{*,J}(z)$ for every $z \in \mathcal{N}_h$ so that $u_h(z) \in N$ for all $z \in \mathcal{N}_h$ by continuity of $N$. Moreover, we trivially have $u_h|_{\Gamma_D} = u_{D,h}$. For each $J \in \mathbb{N}$ we define $\lambda_h^{J,\ell} \in \mathcal{S}^1(\mathcal{T}_h)$ by setting

$$\lambda_h^{J,\ell}(z) := -\beta_z^{-1}\Big(\nabla_{M_h} u_h^{*,J}; \nabla_{M_h}\big[(\nu^\ell \circ u_h^{*,J})(z)\varphi_z\big]\Big)$$

for all $z \in \mathcal{N}_h$ and $\ell = k+1, ..., n$. Given any $v_h \in \mathcal{S}^1_D(\mathcal{T}_h)^n$ we let $v_h^{J,nor} \in \mathcal{S}^1_D(\mathcal{T}_h)^n$ be defined through

$$v_h^{J,nor}(z) := \sum_{\ell=k+1}^{n} \big[(\nu_\ell \circ u_h^{*,J})(z) \cdot v_h(z)\big](\nu_\ell \circ u_h^{*,J})(z)$$

for all $z \in \mathcal{N}_h$. Then, $v_h^{J,tan} := v_h - v_h^{J,nor} \in \mathcal{S}^1_D(\mathcal{T}_h)^n$ satisfies $v_h^{J,tan}(z) \in T_{u_h^{*,J}(z)}N$ for all $z \in \mathcal{N}_h$. Arguing as in the derivation of (1.1) in the proof of Lemma 3.1.4 we verify that

$$\big(\nabla_{M_h} u_h^{*,J}; \nabla_{M_h} v_h^{J,nor}\big) = -\sum_{\ell=k+1}^{n} \big(\lambda_h^{J,\ell}; (\overline{\nu}_\ell \circ u_h^{*,J}) \cdot v_h\big)_h.$$

With the correction $w_h^{*,J} \in \overset{\circ}{\mathcal{S}}^1(\mathcal{T}_h)^n$ provided by Algorithm A which satisfies $\|\nabla_{M_h} w_h^{*,J}\| \le \varepsilon_J$ and upon noting that we may subtract a constant from $v_h^{J,tan}$ to deduce the equation

$$\big(\nabla_{M_h} u_h^{*,J}; \nabla_{M_h} v_h^{J,tan}\big) = \big(\nabla_{M_h} w_h^{*,J+1}; \nabla_{M_h} v_h^{J,tan}\big)$$

we verify that

$$\big(\nabla_{M_h} u_h^{*,J}; \nabla_{M_h} v_h\big) + \sum_{\ell=k+1}^{n} \big(\lambda_h^{J,\ell}; (\nu^\ell \circ u_h^{*,J}) \cdot v_h\big)_h = \big(\nabla_{M_h} w_h^{*,J+1}; \nabla_{M_h} v_h^{J,tan}\big)$$

for all $v_h \in \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^n$. Since $w_h^J \to 0$ and $\lambda_h^{J,\ell} \to \lambda_h^\ell$ as $J \to \infty$ for functions $\lambda_h^\ell \in \mathcal{S}^1(\mathcal{T}_h)$, $\ell = k+1, ..., n$, we deduce with Lemma 3.1.4 (a) that $u_h$ is a discrete harmonic map into $N$ subject to the boundary data $u_{\mathrm{D},h}$. □

## 3.3 $L^2$ gradient flow approach for harmonic maps into 2-spheres

The above considerations for the discretization of the $H^1$ gradient flow of harmonic maps can also be carried out for the $L^2$ gradient flow which reads

$$\big(\partial_t u; v\big) + \big(\nabla_M u; \nabla_M v\big) = 0$$

for almost every $t \in (0, \infty)$ and for all $v \in W^{1,2}(M; \mathbb{R}^n)$ such that $v(x) \in T_{u(t,x)}N$ for almost every $x \in M$. We aim at developing an implicit discretization and assume that $N = S^2$. In this case the strong formulation of the equation reads

$$\partial_t u - \Delta_M u = \big|\nabla_M u\big|^2 u$$

and taking the cross product of this identity with $u$ twice and using $u \times u = 0$ we have

$$u \times \big(u \times \partial_t u\big) - u \times \big(u \times \Delta_M u\big) = 0.$$

The Graßmann identity

$$a \times \big(b \times c\big) = b\big(a \cdot c\big) - c\big(a \cdot b\big)$$

valid for $a, b, c \in \mathbb{R}^3$ together with the properties of $u$ that $u \cdot u = 1$ and $u \cdot \partial_t u = 0$ then yields that for strong solutions we have

$$\partial_t u + u \times \big(u \times \Delta_M u\big) = 0. \tag{3.7}$$

To derive a discretization of this equation we consider a sequence $\big(u^i\big)_{i=0,1,...,J+1}$ of approximations at time-steps $i\kappa$ and set

$$u_h^{i+1/2} := \big(u^i + u^{i+1}\big)/2 \qquad \text{and} \qquad d_t u^{i+1} := \big(u^{i+1} - u^i\big)/\kappa$$

for $i = 0, 1, 2, ..., J$. Our Crank-Nicolson type discretization of (3.7) follows [BP07] and reads as follows.

**Algorithm II** (Implicit $L^2$ flow semi-discretization)**.** *Input:* time-step size $\kappa > 0$.

1. Choose $u^0 \in W^{1,2}(M;\mathbb{R}^n)$ such that $u^0(x) \in S^2$ for almost every $x \in M$. Set $i := 0$.
2. Compute $u^{i+1} \in W^{1,2}(M;\mathbb{R}^n)$ such that

$$\big(d_t u^{i+1}; v\big) + \big(u^{i+1/2} \times (u^{i+1/2} \times \Delta_M u^{i+1/2}); v\big) = 0$$

   for all $v \in W^{1,2}(M;\mathbb{R}^n)$.
3. Set $i := i+1$ and go to 2.

Notice that the equation in Step 2 of Algorithm II defines an unconstrained problem which, however, requires the solution of a nonlinear system of equations. The striking property of the algorithm is that for $v = u^{i+1/2}$ the second term in the right-hand side of the equation in Step 2 disappears so that

$$d_t \big\| u^{i+1} \big\|^2 = \big(d_t u^{i+1}; u^{i+1/2}\big) = 0.$$

This will imply unconditional well posedness and conservation of the unit length constraint, provided that the spatial discretization is done appropriately. Even though the scheme follows from a discretization of a strong formulation, we will show that approximations converge to (weakly) harmonic maps without making any regularity assumptions on an exact solution. For a convergence proof of iterates to weak solutions of the $L^2$ gradient flow of ($p$-) harmonic maps into spheres in a Euclidean setting we refer the reader to [BP06, BP07].

Throughout this section we restrict to

$$N = S^2 \qquad \text{and} \qquad \Gamma_{\mathrm{D}} = \emptyset.$$

While the second restriction is made to avoid technical difficulties, the first one is essential. Not only do we need that the target manifold is a sphere, it also has to be two-dimensional.

### 3.3.1 Implicit discretization of the $L^2$ gradient flow

To fully discretize (3.7) we define a discrete Laplace operator $\widetilde{\Delta}_{M_h} : \mathcal{S}^1(\mathcal{T}_h) \to \mathcal{S}^1(\mathcal{T}_h)$ by requiring that for $v_h \in \mathcal{S}^1(\mathcal{T}_h)$ the identity

$$-\big(\widetilde{\Delta}_{M_h} v_h; \chi_h\big)_h = \big(\nabla_{M_h} v_h; \nabla_{M_h} \chi_h\big)$$

is satisfied for all $\chi_h \in \mathcal{S}^1(\mathcal{T}_h)$. For vector valued functions, $\widetilde{\Delta}_{M_h}$ is obtained by applying it to each of the components of the vector field.

**Algorithm B.** *Input:* triangulation $\mathcal{T}_h$, time-step size $\kappa > 0$, stopping criterion $\varepsilon > 0$, approximation tolerance $\delta \geq 0$.

1. Choose $u_h^0 \in \mathcal{S}^1(\mathcal{T}_h)^3$ such that $|u_h^0(z)| = 1$ for all $z \in \mathcal{N}_h$. Set $i := 0$.
2. Compute $u_h^{i+1} \in \mathcal{S}^1(\mathcal{T}_h)^3$ and $r_h^{i+1} \in \mathcal{S}^1(\mathcal{T}_h)^3$ such that $\|r_h^{i+1}\|_h \leq \delta$ and
$$\kappa^{-1}\big(u_h^{i+1} - u_h^i; v_h\big)_h + \big(u_h^{i+1/2} \times (u_h^{i+1/2} \times \widetilde{\Delta}_{M_h} u_h^{i+1/2}); v_h\big)_h = \big(u_h^{i+1/2} \times r_h^{i+1}; v_h\big)_h$$
for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$.
3. Stop if $\kappa^{-1}\|u_h^{i+1} - u_h^i\|_h \leq \varepsilon$.
4. Set $i := i + 1$ and go to 2.

*Output:* $u_h^{*,1/2} := u_h^{i+1/2}$ and $u_h^* := u_h^{i+1}$.

We remark that the Crank-Nicolson type discretization together with the use of reduced integration is essential here to guarantee conservation of the constraint at the nodes and unconditional stability. The additional right-hand side in Step 2, defined through a small but a priori unspecified function $r_h^{i+1} \in \mathcal{S}^1(\mathcal{T}_h)^3$, models an inexact solution of the original nonlinear equation. The special structure of this residual is important to guarantee that iterates satisfy the pointwise constraint $|u_h^{i+1}(z)| = 1$ exactly for all $z \in \mathcal{N}_h$. For $\delta = 0$, hence $r_h^{i+1} = 0$, existence of iterates follows from a fixed-point argument which will also motivates an iterative solver for the equation in Step 2.

**Proposition 3.3.1.** *Given any $u_h^i \in \mathcal{S}^1(\mathcal{T}_h)^3$ there exists $u_h^{i+1} \in \mathcal{S}^1(\mathcal{T}_h)^3$ such that*
$$\kappa^{-1}\big(u_h^{i+1} - u_h^i; v_h\big)_h + \big(u_h^{i+1/2} \times (u_h^{i+1/2} \times \widetilde{\Delta}_{M_h} u_h^{i+1/2}); v_h\big)_h = 0$$
*for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$, i.e., the identity in Step 2 of Algorithm B holds with $r_h^{i+1} = 0$.*

*Proof.* For $w_h \in \mathcal{S}^1(\mathcal{T}_h)^3$ define
$$\Phi(w_h) := \frac{2}{\kappa} w_h - \frac{2}{\kappa} u_h^i - w_h \times \big(w_h \times \widetilde{\Delta}_{M_h} w_h\big).$$
By Young's inequality and properties of the cross product we have for all $w_h \in \mathcal{S}^1(\mathcal{T}_h)^3$ that
$$\big(\Phi(w_h); w_h\big)_h = \frac{2}{\kappa}\|w_h\|_h^2 - \frac{2}{\kappa}(u_h^i; w_h)_h \geq \frac{1}{\kappa}\|w_h\|_h^2 - \frac{1}{\kappa}\|u_h^i\|_h^2.$$
Hence we deduce that $\big(\Phi(w_h); w_h\big)_h \geq 0$ for all $w_h \in \mathcal{S}^1(\mathcal{T}_h)^3$ satisfying $\|w_h\|_h \geq \|u_h^i\|_h$. This implies (cf. [GR86, Corollary 1.1, p. 279]) that there exists $w_h^* \in \mathcal{S}^1(\mathcal{T}_h)^3$ with $\big(\Phi(w_h^*); v_h\big)_h = 0$ for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$. Setting $u_h^{i+1} := 2w_h^* - u_h^i$ so that $u_h^{i+1/2} = w_h^*$ and $u_h^{i+1/2} - u_h^i = (u_h^{i+1} - u_h^i)/2$, the definition of $\Phi$ yields that
$$0 = \big(\Phi(u_h^{i+1/2}); v_h\big)_h = \frac{1}{\kappa}\big(u_h^{i+1} - u_h^i; v_h\big)_h - \big(u_h^{i+1/2} \times (u_h^{i+1/2} \times \widetilde{\Delta}_{M_h} u_h^{i+1/2}); v_h\big)_h$$
for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$. Thus, the identity in Step 2 of Algorithm B is satisfied with $r_h^{i+1} = 0$. ∎

### 3.3.2 Constraint-conservation, stability, and termination

As a consequence of the symmetric discretization of (3.7) in time we obtain conservation of the unit-length constraint as well as unconditional stability of the iteration.

**Lemma 3.3.2.** *Suppose that* $0 \le \delta \le 1$. *For* $J \in \mathbb{N}$ *and iterates* $u_h^0, u_h^1, ..., u_h^{J+1}$ *of Algorithm B we have*

$$|u_h^i(z)| = 1$$

*for* $0 \le i \le J+1$ *and all* $z \in \mathcal{N}_h$ *as well as*

$$\frac{1}{2}\big\|\nabla_{M_h} u_h^{J+1}\big\|^2 + \big(1-\delta\big)\kappa \sum_{i=0}^{J} \big\|u_h^{i+1/2} \times \widetilde{\Delta}_{M_h} u_h^{i+1/2}\big\|_h^2 \le \frac{1}{2}\big\|\nabla_{M_h} u_h^0\big\|^2 + \kappa(J+1)\delta/4$$

*and*

$$\frac{1}{2}\big\|\nabla_{M_h} u_h^{J+1}\big\|^2 + (1-\delta)^2\kappa \sum_{i=0}^{J} \big\|\kappa^{-1}\big[u_h^{i+1} - u_h^i\big]\big\|_h^2 \le \frac{1}{2}\big\|\nabla_{M_h} u_h^0\big\|^2 + (5/4)\kappa(J+1)\delta.$$

*Proof.* The choice $v_h = u_h^{i+1/2}(z)\varphi_z$ for $z \in \mathcal{N}_h$ in Step 2 of Algorithm B yields

$$\begin{aligned}\frac{1}{2\kappa}\big(|u_h^{i+1}(z)|^2 - |u_h^i(z)|^2\big) &= \kappa^{-1}\big[u_h^{i+1}(z) - u_h^i(z)\big] \cdot u_h^{i+1/2}(z) \\ &= \beta_z^{-1}\big(\kappa^{-1}\big[u_h^{i+1} - u_h^i\big]; u_h^{i+1/2}(z)\varphi_z\big)_h = 0\end{aligned}$$

and implies that $|u_h^{i+1}(z)| = 1$ provided that $|u_h^i(z)| = 1$. With $v_h = \widetilde{\Delta}_{M_h} u_h^{i+1/2}$ in Step 2 of Algorithm B we deduce, using $\big\|u_h^{i+1/2}\big\|_{L^\infty(M_h)} \le 1$ and $\big\|r_h^{i+1}\big\|_h \le \delta$,

$$\begin{aligned}&\frac{1}{2\kappa}\big(\big\|\nabla_{M_h} u_h^{i+1}\big\|^2 - \big\|\nabla_{M_h} u_h^i\big\|^2\big) + \big\|u_h^{i+1/2} \times \widetilde{\Delta}_{M_h} u_h^{i+1/2}\big\|_h^2 \\ &\quad = -\big(\kappa^{-1}\big[u_h^{i+1} - u_h^i\big]; \widetilde{\Delta}_{M_h} u_h^{i+1/2}\big)_h - \big(u_h^{i+1/2} \times (u_h^{i+1/2} \times \widetilde{\Delta}_{M_h} u_h^{i+1/2}); \widetilde{\Delta}_{M_h} u_h^{i+1/2}\big)_h \\ &\quad = -\big(u_h^{i+1/2} \times r_h^{i+1}; \widetilde{\Delta}_{M_h} u_h^{i+1/2}\big)_h \\ &\quad \le \delta\,\big\|u_h^{i+1/2} \times \widetilde{\Delta}_{M_h} u_h^{i+1/2}\big\|_h \\ &\quad \le \delta/4 + \delta\big\|u_h^{i+1/2} \times \widetilde{\Delta}_{M_h} u_h^{i+1/2}\big\|_h^2.\end{aligned}$$

Hence

$$\frac{1}{2\kappa}\big(\big\|\nabla_{M_h} u_h^{i+1}\big\|^2 - \big\|\nabla_{M_h} u_h^i\big\|^2\big) + \big(1-\delta\big)\big\|u_h^{i+1/2} \times \widetilde{\Delta}_{M_h} u_h^{i+1/2}\big\|_h^2 \le \delta/4. \tag{3.8}$$

Multiplication by $\kappa$ and summation over $i = 0, 1, 2, ..., J$ provides the first estimate. We choose $v_h = \kappa^{-1}\big[u_h^{i+1} - u_h^i\big]$ in Step 2 of Algorithm B to verify with $\big\|u_h^{i+1/2}\big\|_{L^\infty(M_h)} \le 1$, $\big\|r_h^{i+1}\big\|_h \le \delta$, and Young's inequality, that

$$\begin{aligned}&\big\|\kappa^{-1}\big[u_h^{i+1} - u_h^i\big]\big\|_h^2 \\ &\quad = \big(u_h^{i+1/2} \times \widetilde{\Delta}_{M_h} u_h^{i+1/2}, u_h^{i+1/2} \times \kappa^{-1}\big[u_h^{i+1} - u_h^i\big]\big)_h + \big(u_h^{i+1/2} \times r_h^{i+1}, \kappa^{-1}\big[u_h^{i+1} - u_h^i\big]\big)_h \\ &\quad \le \big\|u_h^{i+1/2} \times \widetilde{\Delta}_{M_h} u_h^{i+1/2}\big\|_h \big\|\kappa^{-1}\big[u_h^{i+1} - u_h^i\big]\big\|_h + \delta\,\big\|\kappa^{-1}\big[u_h^{i+1} - u_h^i\big]\big\|_h \\ &\quad \le \frac{1}{2}\big\|u_h^{i+1/2} \times \widetilde{\Delta}_{M_h} u_h^{i+1/2}\big\|_h^2 + \frac{1}{2}\big\|\kappa^{-1}\big[u_h^{i+1} - u_h^i\big]\big\|_h^2 + \frac{\delta}{2} + \frac{\delta}{2}\big\|\kappa^{-1}\big[u_h^{i+1} - u_h^i\big]\big\|_h^2,\end{aligned}$$

i.e., after subtraction of the second term on the right-hand side and multiplication by $2(1-\delta)$,

$$(1-\delta)^2\big\|\kappa^{-1}\big[u_h^{i+1}-u_h^i\big]\big\|_h^2 \le (1-\delta)\big\|u_h^{i+1/2}\times \widetilde{\Delta}_{M_h}u_h^{i+1/2}\big\|_h^2 + \delta(1-\delta). \tag{3.9}$$

Insertion of (3.9) into (3.8) and a subsequent summation over $i = 0, 1, ..., J$ together with the estimate $\delta(1-\delta) \le \delta$ finish the proof of the lemma. □

Algorithm B terminates provided that $\delta$ is sufficiently small.

**Lemma 3.3.3.** *Suppose that $\delta \le \min\{1/2, \varepsilon^2/10\}$. Then, Algorithm B terminates within at most $J_{max} \le 8C_0/\big(\kappa\varepsilon^2\big) - 1$ iterations where $C_0 := \big\|\nabla_{M_h}u_h^0\big\|^2$.*

*Proof.* If for $i = 0, 1, 2, ..., J$ we have

$$\big\|\kappa^{-1}\big[u_h^{i+1}-u_h^i\big]\big\|_h \ge \varepsilon$$

then Lemma 3.3.2 implies

$$(1/4)\kappa(J+1)\varepsilon^2 \le (1-\delta)^2\kappa\sum_{i=0}^{J}\big\|\kappa^{-1}\big[u_h^{i+1}-u_h^i\big]\big\|_h^2 \le \frac{1}{2}\big\|\nabla_{M_h}u_h^0\big\|^2 + (5/4)\kappa(J+1)\delta$$

or, equivalently,

$$\kappa(J+1)\big(\varepsilon^2-5\delta\big) \le 4C_0.$$

Since $\varepsilon^2 - 5\delta \ge \varepsilon^2/2$ this implies that $J \le 8C_0(\kappa\varepsilon^2)^{-1} - 1$. □

**Remark 3.3.4.** *If we are interested in an approximation of the $L^2$ gradient flow then $(J+1)\kappa \approx T$ for some time horizon $T > 0$ is fixed and the choice $\delta = o(1)$ as $h \to 0$ is sufficient to guarantee stability of the iteration and validity of a correct energy estimate as $(h,\kappa) \to 0$.*

### 3.3.3 Convergence to a continuous harmonic map

As opposed to the analysis for Algorithm A, it turns out that it is preferable to verify directly that a sequence of outputs $\big(u_h^{*,J+1/2}\big)_{J\in\mathbb{N}}$ converges unconditionally to a harmonic map into $S^2$ as $h \to 0$, $\varepsilon_J \to 0$, and $\delta_J \to 0$ for $J \to \infty$. Notice that we do not assume quasi-uniformity of triangulations in the following theorem and that $\kappa$ need not tend to zero.

**Theorem 3.3.5.** *(i) For sequences $\big(\varepsilon_J\big)_{J\in\mathbb{N}}$ and $\big(\delta_J\big)_{J\in\mathbb{N}}$ such that $\varepsilon_J \to 0$ as $J \to \infty$ and $\delta_J \le \min\{1/2, \varepsilon_J^2/10\}$ for all $J \in \mathbb{N}$ every accumulation point of the sequence of outputs $\big(u_h^{*,J+1/2}\big)$ of Algorithm B is a discrete harmonic map into $S^2$.*
*(ii) If in addition to the assumptions in (i) we have that simultaneously $h \to 0$ as $J \to \infty$, then every accumulation point of the sequence $\big(u_h^{*,J+1/2}\big)_{J\in\mathbb{N}}$ is a harmonic map into $S^2$.*

*Proof.* (i) The inverse triangle inequality and conservation of the constraint $|u_h^{i+1}| = 1$ of iterates of Algorithm B show that for all $z \in \mathcal{N}_h$ we have

$$\begin{aligned}\big||u_h^{*,J+1/2}(z)|-1\big| &= \big||u_h^{*,J+1/2}(z)|-|u_h^{*,J+1}(z)|\big| \\ &\le \big|\big(u_h^{*,J+1/2}-u_h^{*,J+1}\big)(z)\big| = \frac{\kappa}{2}\big|\kappa^{-1}(u_h^{*,J+1}-u_h^{*,J})(z)\big| \le C(h)\kappa\varepsilon_J,\end{aligned} \tag{3.10}$$

where we used that $h$ is fixed and that owing to the stopping criterion of Algorithm B we have $\big\|\kappa^{-1}(u_h^{*,J+1} - u_h^{*,J})\big\|_h \le \varepsilon_J$. Therefore, any accumulation point $u_h \in \mathcal{S}^1(\mathcal{T}_h)^n$ of the sequence $\big(u_h^{*,J+1/2}\big)_{J\in\mathbb{N}}$ satisfies $|u_h(z)| = 1$ for all $z \in \mathcal{N}_h$. The properties of the cross product that $(a\times b)\cdot c = -b\cdot(a\times c)$ and $a\times(b\times c) = (a\cdot c)b - (a\cdot b)c$ for $a, b, c \in \mathbb{R}^3$ imply

$$\begin{aligned}
&\big(u_h^{*,J+1/2} \times (u_h^{*,J+1/2} \times \widetilde{\Delta}_{M_h} u_h^{*,J+1/2}); v_h\big)_h \\
&\quad = \big(\widetilde{\Delta}_{M_h} u_h^{*,J+1/2}; u_h^{*,J+1/2} \times (u_h^{*,J+1/2} \times v_h)\big)_h \\
&\quad = -\big(\widetilde{\Delta}_{M_h} u_h^{*,J+1/2}; |u_h^{*,J+1/2}|^2 v_h\big)_h + \big(\widetilde{\Delta}_{M_h} u_h^{*,J+1/2}; (u_h^{*,J+1/2}\cdot v_h) u_h^{*,J+1/2}\big)_h \\
&\quad = -\big(\widetilde{\Delta}_{M_h} u_h^{*,J+1/2}; v_h\big)_h - \big(\widetilde{\Delta}_{M_h} u_h^{*,J+1/2}; (|u_h^{*,J+1/2}|^2 - 1) v_h\big)_h \\
&\qquad + \big(\widetilde{\Delta}_{M_h} u_h^{*,J+1/2}; (u_h^{*,J+1/2}\cdot v_h) u_h^{*,J+1/2}\big)_h.
\end{aligned} \tag{3.11}$$

The identity in Step 2 of Algorithm B, (3.11), and properties of the discrete inner product show for $v_h = \mathcal{I}_h\big[u_h^{*,J+1/2} \times w_h\big]$ with arbitrary $w_h \in \mathcal{S}^1(\mathcal{T}_h)^3$ that

$$\begin{aligned}
&-\big(\widetilde{\Delta}_{M_h} u_h^{*,J+1/2}; u_h^{*,J+1/2} \times w_h\big)_h \\
&\quad = -\kappa^{-1}\big(u_h^{*,J+1} - u_h^{*,J}; u_h^{*,J+1/2} \times w_h\big)_h - \big(u_h^{*,J+1/2} \times \widetilde{\Delta}_{M_h} u_h^{*,J+1/2}; (|u_h^{*,J+1/2}|^2 - 1) w_h\big)_h \\
&\qquad + \big(r_h^{*,J+1} \times u_h^{*,J+1}; u_h^{*,J+1/2} \times w_h\big)_h \\
&\quad =: I + II + III.
\end{aligned}$$

Since by the stopping criterion it holds $\kappa^{-1}\big\|u_h^{*,J+1} - u_h^{*,J}\big\|_h \le \varepsilon_J$ and since $|u_h^{*,J+1/2}(z)| \le 1$ for all $z \in \mathcal{N}_h$ we deduce that

$$I \le \varepsilon_J \|w_h\|_h.$$

Lemmas 3.3.2 and 3.3.3 imply that

$$\big\|u_h^{*,J+1/2} \times \widetilde{\Delta}_{M_h} u_h^{*,J+1/2}\big\|_h \le C\kappa^{-1/2} \tag{3.12}$$

and together with $\big||u_h^{*,J+1/2}(z)| - 1\big| \le C(h)\kappa\varepsilon_J$ we verify that

$$II \le C(h)\kappa^{1/2}\varepsilon_J \|w_h\|_h.$$

The guaranteed bound $\big\|r_h^{*,J+1}\big\|_h \le \delta_J$ and $|u_h^{*,J+1/2}(z)| \le 1$ for all $z \in \mathcal{N}_h$ shows that

$$III \le \delta_J \|w_h\|_h.$$

Therefore, we verify that an accumulation point $u_h$ of the sequence $\big(u_h^{*,J+1/2}\big)_{J\in\mathbb{N}}$ satisfies

$$\big(\widetilde{\Delta}_{M_h} u_h; u_h \times w_h\big)_h = 0$$

for all $w_h \in \mathcal{S}^1(\mathcal{T}_h)^3$. For every $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$ such that $v_h(z) \in T_{u_h(z)}S^2$ there exists $w_h \in \mathcal{S}^1(\mathcal{T}_h)^3$ satisfying $v_h(z) = u_h(z) \times w_h(z)$ for all $z \in \mathcal{N}_h$. Therefore, we verify by definition of $\widetilde{\Delta}_{M_h}$ that

$$0 = -\big(\widetilde{\Delta}_{M_h} u_h; v_h\big)_h = \big(\nabla_{M_h} u_h; \nabla_{M_h} v_h\big)$$

for all such $v_h$ which, owing to Lemma 3.1.4, proves that $u_h$ is a discrete harmonic map into $S^2$.
(ii) To prove the second assertion we notice that the lifted sequences of outputs $\big(\widetilde{u}_h^{*,J+1/2}\big)$ and $\big(\widetilde{u}_h^{*,J+1}\big)$ are bounded in $W^{1,2}(M;\mathbb{R}^3)$ and owing to the estimate

$$\big\|u_h^{*,J+1/2} - u_h^{*,J+1}\big\| = \frac{1}{2}\big\|u_h^{*,J+1} - u_h^{*,J}\big\| \le \frac{1}{2}\kappa\varepsilon_J$$

have the same accumulation points. Since $|u_h^{*,J+1}(z)| = 1$ for all $z \in \mathcal{N}_h$, nodal interpolation estimates imply

$$\big\||u_h^{*,J+1}|^2 - 1\big\| \le Ch\big\|\nabla_{M_h}\big[|u_h^{*,J+1}|^2\big]\big\| \le Ch\big\|\nabla_{M_h}u_h^{*,J+1}\big\|.$$

Therefore, every accumulation point $u \in W^{1,2}(M;\mathbb{R}^3)$ satisfies $|u| = 1$ almost everywhere in $M$. For any $\phi \in C_c^\infty(M)$ we denote by $\check{\phi} \in L^\infty(M_h)$ the function whose lifting onto $M$ coincides with $\phi$ and notice

$$\begin{aligned}
\big(\widetilde{\Delta}_{M_h}u_h^{*,J+1/2}; u_h^{*,J+1/2} \times \check{\phi}\big)_h &= -\big(\nabla_{M_h}u_h^{*,J+1/2}; \nabla_{M_h}\mathcal{I}_h\big[u_h^{*,J+1/2} \times \check{\phi}\big]\big) \\
&= -\big(\nabla_{M_h}u_h^{*,J+1/2}; \nabla_{M_h}\big[u_h^{*,J+1/2} \times \check{\phi}\big]\big) \\
&\quad - \big(\nabla_{M_h}u_h^{*,J+1/2}; \nabla_{M_h}\big\{u_h^{*,J+1/2} \times \check{\phi} - \mathcal{I}_h\big[u_h^{*,J+1/2} \times \check{\phi}\big]\big\}\big).
\end{aligned}$$

Employing (3.11) and the identity in Step 2 of Algorithm B shows

$$\begin{aligned}
&\big|\big(\widetilde{\Delta}_{M_h}u_h^{*,J+1/2}; u_h^{*,J+1/2} \times \check{\phi}\big)_h\big| \\
&\quad \le \big|\big(u_h^{*,J+1/2} \times (u_h^{*,J+1/2} \times \widetilde{\Delta}_{M_h}u_h^{*,J+1/2}); u_h^{*,J+1/2} \times \check{\phi}\big)_h\big| \\
&\qquad + \big|\big(\widetilde{\Delta}_{M_h}u_h^{*,J+1/2}; (|u_h^{*,J+1/2}|^2 - 1)u_h^{*,J+1/2} \times \check{\phi}\big)_h\big| \\
&\quad \le \big|\kappa^{-1}\big(u_h^{*,J+1} - u_h^{*,J}; u_h^{*,J+1/2} \times \check{\phi}\big)_h\big| + \big|\big(u_h^{*,J+1/2} \times r_h^{J+1}; u_h^{*,J+1/2} \times \check{\phi}\big)_h\big| \\
&\qquad + \big|\big(\widetilde{\Delta}_{M_h}u_h^{*,J+1/2}; (|u_h^{*,J+1/2}|^2 - 1)u_h^{*,J+1/2} \times \check{\phi}\big)_h\big| \\
&\quad \le \varepsilon_J\|\check{\phi}\|_h + \delta_J\|\check{\phi}\|_h + \big\|u_h^{*,J+1/2} \times \widetilde{\Delta}_{M_h}u_h^{*,J+1/2}\big\|\,\big\||u_h^{*,J+1/2}|^2 - 1\big\|\,\|\check{\phi}\|_{L^\infty(M_h)} \\
&\quad \le C\varepsilon_J\|\phi\|_{L^2(M)} + C\delta_J\|\phi\|_{L^2(M)} + C\kappa^{1/2}\varepsilon\|\phi\|_{L^\infty(M)}.
\end{aligned}$$

For the last estimate we used (3.12) and $\big\||u_h^{*,J+1/2}|^2 - 1\big\| \le C\kappa\varepsilon_J$ which follows from (3.10).

Lemma 3.3.2 and the interpolation estimates of Section 1.4 imply that

$$\big|\big(\nabla_{M_h} u_h^{*,J+1/2}; \nabla_{M_h}\{u_h^{*,J+1/2} \times \check{\phi} - \mathcal{I}_h[u_h^{*,J+1/2} \times \check{\phi}]\}\big)\big|$$

$$\leq Ch\big\|D^2_{M_h}\big[u_h^{*,J+1/2} \times \check{\phi}\big]\big\|$$

$$\leq Ch\big(\|\nabla_{M_h} u_h^{*,J+1/2}\|\,\|\nabla_{M_h}\check{\phi}\|_{L^\infty(M_h)} + \|D^2_{M_h}\check{\phi}\|\big).$$

Lemma 1.3.5 guarantees

$$\big(\nabla_{M_h} u_h^{*,J+1/2}; \nabla_{M_h}\big[u_h^{*,J+1/2} \times \check{\phi}\big]\big) = \big(\big[\mathbf{I} + \mathbf{F}_h\big]\nabla_M \widetilde{u}_h^{*,J+1/2}; \nabla_M\big[\widetilde{u}_h^{*,J+1/2} \times \phi\big]\big)$$

and the property of $\mathbf{F}_h$ that $\mathbf{F}_h = o(h)$ together with a combination of the previous four estimates imply

$$\big(\nabla_M \widetilde{u}_h^{*,J+1/2}; \nabla_M\big[\widetilde{u}_h^{*,J+1/2} \times \phi\big]\big) \to 0$$

as $h \to 0$. With the help of Lemma 2.5.1 we note

$$\big(\nabla_M \widetilde{u}_h^{*,J+1/2}; \nabla_M\big[\widetilde{u}_h^{*,J+1/2} \times \phi\big]\big) = \sum_{\gamma=1}^{m}\big(\underline{D}_\gamma \widetilde{u}_h^{*,J+1/2}; \widetilde{u}_h^{*,J+1/2} \times \underline{D}_\gamma \phi\big).$$

Since $\widetilde{u}_h^{*,J+1/2} \to u$ strongly in $L^2(M;\mathbb{R}^3)$ we verify once more with Lemma 2.5.1 that

$$0 = \big(\underline{D}_\gamma u;\ u \times \underline{D}_\gamma \phi\big) = \big(\nabla_M u; \nabla_M\big[u \times \phi\big]\big)$$

which shows that $u$ is a harmonic map into $S^2$ and finishes the proof of the theorem. □

**Remark 3.3.6.** *A convergence proof for $h \to 0$ could also be based on Theorem 2.5.2 but would require to impose restrictive constraints on the discretization parameters. More precisely, for the setting of Theorem 2.5.2 we would have to employ the sequence $u_h^{*,J+1}$ which satisfies the constraints at the nodes. In the proof given here, we rather proved convergence of the averages $u_h^{*,J+1/2}$ which solve the discretized partial differential equation.*

### 3.3.4 Fully practical construction of iterates

We next discuss convergence of a fixed-point iteration that solves the nonlinear equation in each time-step of Algorithm B. Owing to a careful linearization, the proposed fixed-point iteration will preserve the unit length constraint in each iteration step. This is not clear for other iterative solvers such as a Newton iteration. Unfortunately, to guarantee convergence of the iteration, we have to impose severe constraints on the damping parameter $\kappa$.

**Lemma 3.3.7.** *There exists a constant $C > 0$ such that for all $\phi_h \in \mathcal{S}^1(\mathcal{T}_h)$ we have*

$$\big\|\widetilde{\Delta}_{M_h}\phi_h\big\|_h \leq C h_{min}^{-2}\|\phi_h\|_h$$

*and*

$$\big\|\widetilde{\Delta}_{M_h}\phi_h\big\|_{L^\infty(M_h)} \leq C h_{min}^{-2}\|\phi_h\|_{L^\infty(M_h)}.$$

*Proof.* The proof of the first estimate follows directly from the definition of $\widetilde{\Delta}_{M_h}$ and the inverse estimate $\|\nabla_{M_h}\phi_h\| \leq Ch_{min}^{-1}\|\phi_h\|$. To verify the second estimate, let $z \in \mathcal{N}_h$ be such that $\|\widetilde{\Delta}_{M_h}\phi_h\|_{L^\infty(M_h)} = |\widetilde{\Delta}_{M_h}\phi_h(z)|$. Choosing $\chi_h = \widetilde{\Delta}_{M_h}\phi_h(z)\varphi_z$ in the definition of $\widetilde{\Delta}_{M_h}\phi_h$ and recalling properties of the discrete inner product from Definition 1.4.10 yields that

$$\begin{aligned}
|\widetilde{\Delta}_{M_h}\phi_h(z)|^2 &= \beta_z^{-1}\big(\widetilde{\Delta}_{M_h}\phi_h; \chi_h\big)_h \\
&= -\big(\widetilde{\Delta}_{M_h}\phi_h(z)\big)\beta_z^{-1}\big(\nabla_{M_h}\phi_h; \nabla_{M_h}\varphi_z\big) \\
&= -\big(\widetilde{\Delta}_{M_h}\phi_h(z)\big)\beta_z^{-1}\sum_{y\in\mathcal{N}_h}\phi_h(y)\big(\nabla_{M_h}\varphi_y; \nabla_{M_h}\varphi_z\big) \\
&\leq C|\widetilde{\Delta}_{M_h}\phi_h(z)|\beta_z^{-1}\|\phi_h\|_{L^\infty(M_h)}\|\nabla_{M_h}\varphi_z\|^2 \\
&\leq C|\widetilde{\Delta}_{M_h}\phi_h(z)|\beta_z^{-1}\|\phi_h\|_{L^\infty(M_h)}h_{min}^{-2}\|\varphi_z\|^2 \\
&\leq C|\widetilde{\Delta}_{M_h}\phi_h(z)|h_{min}^{-2}\|\phi_h\|_{L^\infty(M_h)},
\end{aligned}$$

where we used that the number of nodes $y \in \mathcal{N}_h$ such that $\big(\nabla_{M_h}\varphi_y; \nabla_{M_h}\varphi_z\big) \neq 0$ is bounded $h$-independently, that $\|\nabla_{M_h}\varphi_y\| \leq C\|\nabla_{M_h}\varphi_z\|$ for such $y \in \mathcal{N}_h$, and that $\beta_z^{-1}\|\varphi_z\|^2 \leq C$. □

The following algorithm is motivated by the proof of Proposition 3.3.1 and solves the nonlinear equation in Step 2 of Algorithm B. It is based on a linearization of the equation in Step 2 of Algorithm B with $\delta = 0$ (i.e., $r_h^{i+1} = 0$). This linearization of the nonlinear part is done in such a way that the unit length constraint is preserved throughout the iteration and that the residual is of the form of the right-hand side in Step 2 of Algorithm B.

**Algorithm B$^{inner}$.** *Input:* triangulation $\mathcal{T}_h$, damping parameter $\kappa > 0$, approximation tolerance $\delta \geq 0$, starting value $u_h^i$ (approximate solution of $i$-th time-step).

1. Set $w_h^0 := u_h^i$ and $j := 0$.
2. Compute $w_h^{j+1} \in \mathcal{S}^1(\mathcal{T}_h)^3$ such that
$$\frac{2}{\kappa}\big(w_h^{j+1}; v_h\big)_h + \big(w_h^{j+1} \times (w_h^j \times \widetilde{\Delta}_{M_h} w_h^j); v_h\big)_h = \frac{2}{\kappa}\big(u_h^i; v_h\big)_h$$
for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$. Set $e_h^{j+1} := w_h^{j+1} - w_h^j$ and
$$r_h^{i+1} := w_h^{j+1} \times \widetilde{\Delta}_{M_h} e_h^{j+1} + e_h^{j+1} \times \widetilde{\Delta}_{M_h} w_h^j.$$
3. If $\|r_h^{i+1}\|_h \leq \delta$ then stop and set $u_h^{i+1} := 2w_h^{j+1} - u_h^i$.
4. Set $j := j+1$ and go to 2.

*Output:* $u_h^{i+1}$ and $r_h^{i+1}$.

The following theorem shows that all steps in Algorithm B$^{inner}$ are well-defined, that the algorithm terminates if $\kappa = O(h^2)$, and that the outputs $u_h^{i+1}$ and $r_h^{i+1}$ solve Step 2 of Algorithm B.

**Theorem 3.3.8.** *(i) Let $u_h^i \in \mathcal{S}^1(\mathcal{T}_h)^3$ such that $|u_h^i(z)| = 1$ for all $z \in \mathcal{N}_h$. Then, for all $j \geq 0$ the system in Step 2 of Algorithm $B^{inner}$ admits a unique solution $w_h^{j+1} \in \mathcal{S}^1(\mathcal{T}_h)^3$ such that $\big|w_h^{j+1}(z)\big| \leq 1$ and $\big|\big(2w_h^{j+1} - u_h^i\big)(z)\big| = 1$ for all $z \in \mathcal{N}_h$. Moreover, we have*

$$\|e_h^{j+1}\|_h \leq C\kappa h_{min}^{-2}\|e_h^j\|_h. \tag{3.13}$$

*(ii) If $C\kappa h_{min}^{-2} < 1$ and $\delta > 0$ then Algorithm $B^{inner}$ terminates within a finite number of iterations and the output $u_h^{i+1}$ and $r_h^{i+1}$ solve Step 2 of Algorithm B.*

*Proof.* The left-hand side in Step 2 of Algorithm $\mathrm{B}^{inner}$ defines a continuous bilinear form on $[\mathcal{S}^1(\mathcal{T}_h)^3]^2$ and properties of the cross product show that this bilinear form is elliptic. Hence, there exists a unique solution $w_h^{j+1}$. Upon choosing $v_h = w_h^{j+1}(z)\varphi_z$ for $z \in \mathcal{N}_h$ we verify that $|w_h^{j+1}(z)| \leq |u_h^i(z)| = 1$. Defining $\overline{u}_h^{i+1} = 2w_h^{j+1} - u_h^i$ implies that for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$ we have

$$\frac{1}{\kappa}\big(\overline{u}_h^{i+1} - u_h^i; v_h\big)_h + \big(w_h^{j+1} \times (w_h^j \times \widetilde{\Delta}_{M_h} w_h^j); v_h\big)_h = 0.$$

The choice $v_h = w_h^{j+1}(z)\varphi_z$ for $z \in \mathcal{N}_h$ and the identity $w_h^{j+1} = \big(\overline{u}_h^{i+1} + u_h^i\big)/2$ yield that $|\overline{u}_h^{i+1}(z)|^2 = |u_h^i(z)|^2 = 1$. We subtract equations from Step 2 corresponding to two successive iteration steps and choose $v_h = e_h^{j+1}$ to verify that for $j \geq 1$ we have

$$\begin{aligned}
\frac{2}{\kappa}\|e_h^{j+1}\|_h^2 &= -\big(w_h^j \times (e_h^j \times \widetilde{\Delta}_{M_h} w_h^j); e_h^{j+1}\big)_h - \big(w_h^j \times (w_h^{j-1} \times \widetilde{\Delta}_{M_h} e_h^j); e_h^{j+1}\big)_h \\
&\leq \|e_h^j\|_h \|\widetilde{\Delta}_{M_h} w_h^j\|_{L^\infty(M_h)} \|e_h^{j+1}\|_h + \|\widetilde{\Delta}_{M_h} e_h^j\|_h \|e_h^{j+1}\|_h,
\end{aligned}$$

where we used $\|w_h^j\|_{L^\infty(M_h)}, \|w_h^{j-1}\|_{L^\infty(M_h)} \leq 1$. The estimates of Lemma 3.3.7 yield (3.13). Estimate (3.13) then implies that the iteration of the algorithm converges and terminates within a finite number of iterations if $C\kappa h_{min}^{-2} < 1$. Suppose that for some $j \geq 0$ we have $u_h^{i+1} = 2w_h^{j+1} - u_h^i$, in particular $u_h^{i+1/2} = w_h^{j+1}$. Then, the system in Step 2 of Algorithm $\mathrm{B}^{inner}$ implies that for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$ we have

$$\begin{aligned}
&\kappa^{-1}\big(\big[u_h^{i+1} - u_h^i\big]; v_h\big)_h + \big(u_h^{i+1/2} \times (u_h^{i+1/2} \times \widetilde{\Delta}_{M_h} u_h^{i+1/2}); v_h\big)_h \\
&\quad = \big(u_h^{i+1/2} \times (w_h^{j+1} \times \widetilde{\Delta}_{M_h} w_h^{j+1}); v_h\big)_h - \big(u_h^{i+1/2} \times (w_h^j \times \widetilde{\Delta}_{M_h} w_h^j); v_h\big)_h \\
&\quad = \big(u_h^{i+1/2} \times (e_h^{j+1} \times \widetilde{\Delta}_{M_h} w_h^{j+1}); v_h\big)_h + \big(u_h^{i+1/2} \times (w_h^j \times \widetilde{\Delta}_{M_h} e_h^{j+1}); v_h\big)_h \\
&\quad = \big(u_h^{i+1/2} \times r_h^{i+1}; v_h\big)_h,
\end{aligned}$$

which proves (ii) and finishes the proof of the theorem. □

**Remark 3.3.9.** *Newton schemes for the approximate solution of the homogeneous equation in Step 2 of Algorithm B do in general not lead to a residual that has the structure of the right-hand side of the equation: Suppose that $u_h^i \in \mathcal{S}^1(\mathcal{T}_h)^3$ is given. Then, in order to compute an approximation of $u_h^{i+1/2}$, one is led to finding $w_h^* \in \mathcal{S}^1(\mathcal{T}_h)^3$ such $F(w_h^*) = 0$, where*

$$F(w_h) = \frac{2}{\kappa}\big(w_h - u_h^i\big) + \mathcal{I}_h\big[w_h \times (w_h \times \widetilde{\Delta}_{M_h} w_h)\big].$$

*Given an iterate $w_h^j$ (e.g., with $w_h^0 = u_h^i$), the correction $c_h \in \mathcal{S}^1(\mathcal{T}_h)^3$ in the update $w_h^{j+1} = w_h^j - c_h$ is the solution of*

$$DF(w_h^j)[c_h] = F(w_h^j),$$

*i.e., $c_h$ satisfies*

$$\begin{aligned}\frac{2}{\kappa}\big(c_h;v_h\big)_h + \big(c_h \times (w_h^j \times \widetilde{\Delta}_{M_h} w_h^j);v_h\big)_h + \big(w_h^j \times (c_h \times \widetilde{\Delta}_{M_h} w_h^j);v_h\big)_h \\ + \big(w_h^j \times (w_h^j \times \widetilde{\Delta}_{M_h} c_h);v_h\big)_h \\ = \frac{2}{\kappa}\big(w_h^j - u_h^i;v_h\big)_h + \big(w_h^j \times (w_h^j \times \widetilde{\Delta}_{M_h} w_h^j);v_h\big)_h,\end{aligned}$$

*for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$. Setting $u_h^{i+1} := 2w_h^{j+1} - u_h^i$, i.e., $u_h^{i+1/2} = w_h^{j+1}$, the equation may be rewritten as*

$$\begin{aligned}\kappa^{-1}\big([u_h^{i+1} - u_h^i];v_h\big)_h + \big(u_h^{i+1/2} \times (u_h^{i+1/2} \times \widetilde{\Delta}_{M_h} u_h^{i+1/2});v_h\big)_h \\ = \big(c_h^j \times (w_h^j \times \widetilde{\Delta}_{M_h} c_h^j);v_h\big)_h + \big(c_h^j \times (c_h^j \times \widetilde{\Delta}_{M_h} w_h^j);v_h\big)_h \\ + \big(w_h^j \times (c_h^j \times \widetilde{\Delta}_{M_h} c_h^j);v_h\big)_h + \big(c_h^j \times (c_h^j \times \widetilde{\Delta}_{M_h} c_h^j);v_h\big)_h,\end{aligned}$$

*and the right-hand side is not of the desired form $\big(u_h^{i+1/2} \times r_h^{i+1};v_h\big)$.*

## 3.4 $\theta$-Schemes for the approximation of the $L^2$ flow of harmonic maps

A family of $\theta$-schemes for the $L^2$ flow of harmonic maps into spheres and for Landau-Lifshitz-Gilbert equations has recently been proposed in [Alo07] and shown to be unconditionally convergent for $\theta > 1/2$ provided that underlying triangulations are weakly acute. The proof exploits the $H^1$ stability of the projection of updates onto the sphere, cf. Lemma 3.2.5. In this section we modify the scheme in order to approximate the harmonic map heat flow into targets $N$ which are the boundaries of convex sets. The main idea for the definition of the approximation scheme is to compute for given $\theta \in [0,1]$ and time-step size $\kappa > 0$ an approximate time-derivative $w^i$ satisfying $w^i(x) \in T_{u^i(x)}N$ for given $u^i$ with $u^i(x) \in N$ for almost every $x \in M$ such that

$$\big(w^i;v\big) + \big(\nabla_M\big[u^i + \theta\kappa w^i\big];\nabla_M v\big) = 0$$

holds for all $v$ with $v(x) \in T_{u^i(x)}N$ for almost every $x \in M$. The update $u^i + \kappa w^i$ is then projected onto $N$ to define the new iterate $u^{i+1} := \pi_N\big(u^i + \kappa w^i\big)$. This scheme may be regarded as a semi-implicit discretization of the harmonic map heat flow into convex targets. To compute stationary points of the flow, our full discretization reads as follows.

**Algorithm C.** *Input:* triangulation $\mathcal{T}_h$, parameter $\theta \in [0,1]$, time-step size $\kappa > 0$, stopping criterion $\varepsilon > 0$.

1. Choose $u_h^0 \in \mathcal{S}^1(\mathcal{T}_h)^n$ such that $u_h^0|_{\Gamma_\mathrm{D}} = u_{\mathrm{D},h}$ and $u_h^0(z) \in N$ for all $z \in \mathcal{N}_h \setminus \Gamma_\mathrm{D}$. Set $i := 0$.

2. Compute $w_h^i \in \mathcal{S}^1_\mathrm{D}(\mathcal{T}_h)^n$ such that $w_h^i(z) \in T_{u_h^i(z)}N$ for all $z \in \mathcal{N}_h$ and

$$\big(w_h^i; v_h\big) + \big(\nabla_{M_h}\big[u_h^i + \theta\kappa w_h^i\big]; \nabla_{M_h} v_h\big) = 0$$

for all $v_h \in \mathcal{S}^1_\mathrm{D}(\mathcal{T}_h)^n$ such that $v_h(z) \in T_{u_h^i(z)}N$ for all $z \in \mathcal{N}_h$.

3. Define $u_h^{i+1} \in \mathcal{S}^1(\mathcal{T}_h)^n$ by setting

$$u_h^{i+1}(z) := \pi_N\big(u_h^i(z) + \kappa w_h^i(z)\big)$$

for all $z \in \mathcal{N}_h$.

4. Stop if $\kappa^{-1}\big\|u_h^{i+1} - u_h^i\big\|_h \le \varepsilon$.

5. Set $i := i+1$ and go to 2.

*Output:* $u_h^* := u_h^i$.

If $\theta < 1/2$ then stability of Algorithm C requires that $\kappa \le Ch^{1+d/2}$, see [Alo07] for the case that $N = S^2$. The following proposition shows that the scheme is unconditionally stable for $\theta \in [1/2, 1]$ provided that $N = \partial\mathcal{C}$ for a bounded, open, convex set $\mathcal{C} \subset \mathbb{R}^n$ and $\mathcal{T}_h$ is weakly acute.

**Proposition 3.4.1.** *Suppose that $\theta \in [1/2,1]$, $\mathcal{T}_h$ is weakly acute, and $N$ is $C^2$ and satisfies $N = \partial\mathcal{C}$ for a bounded, open, convex set $\mathcal{C} \subset \mathbb{R}^n$. Then the iteration of Algorithm C is well-defined and iterates satisfy*

$$\kappa \sum_{i=0}^{J} \Big\{ \big\|w_h^i\big\|^2 + (\theta - 1/2)\kappa \big\|\nabla_{M_h} w_h^i\big\|^2 \Big\} + \frac{1}{2}\big\|\nabla_{M_h} u_h^{J+1}\big\|^2 \le \frac{1}{2}\big\|\nabla_{M_h} u_h^0\big\|^2.$$

*Moreover, Algorithm C terminates within a finite number of iterations and the output $u_h^*$ satisfies $u_h^*(z) \in N$ for all $z \in \mathcal{N}_h$, $u_h^*|_{\Gamma_\mathrm{D}} = u_{\mathrm{D},h}$, and*

$$\big(\nabla_{M_h} u_h^*; \nabla_{M_h} v_h\big) = \mathcal{R}es_h\big(v_h\big)$$

*for all $v_h \in \mathcal{S}^1_\mathrm{D}(\mathcal{T}_h)^n$ such that $v_h(z) \in T_{u_h^*(z)}N$ for all $z \in \mathcal{N}_h$, where the linear functional $\mathcal{R}es_h : \mathcal{S}^1_\mathrm{D}(\mathcal{T}_h)^n \to \mathbb{R}$ satisfies*

$$\big|\mathcal{R}es_h\big(v_h\big)\big| \le C\varepsilon\, \|\nabla_{M_h} v_h\|,$$

*$v_h \in \mathcal{S}^1_\mathrm{D}(\mathcal{T}_h)^n$ provided that $\kappa^2 h_{min}^{-1-d/2}\varepsilon \le C'$.*

*Proof.* Well posedness of the iteration is an immediate consequence of the Lax-Milgram lemma for Step 2 and the assumed convexity of $\mathcal{C}$ in Step 3. Given $i \ge 0$ set $\widehat{u}_h^{i+1} := u_h^i + \kappa w_h^i$. Then, $u_h^{i+1} = \mathcal{I}_h \pi_N\big(\widehat{u}_h^{i+1}\big)$ and the proof of Lemma 3.2.5 shows that

$$\big\|\nabla_{M_h} u_h^{i+1}\big\| \le \big\|\nabla_{M_h} \widehat{u}_h^{i+1}\big\|. \tag{4.14}$$

Notice that

$$u_h^i + \theta\kappa w_h^i = \theta\big(u_h^i + \kappa w_h^i\big) + (1-\theta)u_h^i = \theta\widehat{u}_h^{i+1} + (1-\theta)u_h^i$$

and

$$w_h^i = \kappa^{-1}\big(\widehat{u}_h^{i+1} - u_h^i\big).$$

Therefore, the choice $v_h = w_h^i$ in the equation in Step 2 of Algorithm C yields to

$$\|w_h^i\|^2 + \kappa^{-1}\big(\nabla_{M_h}\big[\theta\widehat{u}_h^{i+1} + (1-\theta)u_h^i\big]; \nabla_{M_h}\big[\widehat{u}_h^{i+1} - u_h^i\big]\big) = 0.$$

Using that for all $a, b \in \mathbb{R}^\ell$ we have

$$\big(\theta a + (1-\theta)b\big)\cdot(b-a) = \frac{1}{2}\big(|b|^2 - |a|^2\big) + \frac{2\theta-1}{2}|b-a|^2,$$

we deduce from the previous identity that

$$\|w_h^i\|^2 + \frac{\kappa^{-1}}{2}\big(\|\nabla_{M_h}\widehat{u}_h^{i+1}\|^2 - \|\nabla_{M_h}u_h^i\|^2\big) + \frac{2\theta-1}{2\kappa}\big\|\nabla_{M_h}\big[\widehat{u}_h^{i+1} - u_h^i\big]\big\|^2 = 0$$

holds for all $v_h \in \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^n$ such that $v_h(z) \in T_{u_h(z)}N$ for all $z \in \mathcal{N}_h$. Upon using (4.14) and summing over $i = 0, 1, 2, ..., J$ we deduce the first asserted estimate which also implies termination of the algorithm. If $u_h^* = u_h^{i^*+1}$ for some $i^* \geq 0$ then

$$\begin{aligned}\big(\nabla_{M_h}u_h^*; \nabla_{M_h}v_h\big) &= \big(\nabla_{M_h}\big[u_h^{i^*} + \theta\kappa w_h^{i^*}\big]; \nabla_{M_h}v_h\big) + \big(\nabla_{M_h}\big[u_h^{i^*+1} - \widehat{u}_h^{i^*+1}\big]; \nabla_{M_h}v_h\big)\\ &= -\big(w_h^{i^*}; v_h\big) + \big(\nabla_{M_h}\big[u_h^{i^*+1} - \widehat{u}_h^{i^*+1}\big]; \nabla_{M_h}v_h\big).\end{aligned}$$

We use that for all $z \in \mathcal{N}_h$ we have, cf. the proof of Lemma 3.2.4,

$$u_h^{i^*+1}(z) - \widehat{u}_h^{i^*+1}(z) = \pi_N\big(u_h^{i^*}(z) + \kappa w_h^{i^*}(z)\big) - \big(u_h^{i^*}(z) + \kappa w_h^i(z)\big) = \kappa^2\mathcal{O}\big(|w_h^{i^*}(z)|^2\big)$$

to verify with an interpolation result and an inverse estimate that

$$\big\|u_h^{i^*+1} - \widehat{u}_h^{i^*+1}\big\| \leq C\kappa^2\|w_h^{i^*}\|^2_{L^4(M_h)} \leq C\kappa^2 h_{min}^{-d/2}\|w_h^{i^*}\|^2_{L^2(M_h)}.$$

A combination of the last two equations and an inverse estimate lead to

$$\big(\nabla_{M_h}u_h^*; \nabla_{M_h}v_h\big) \leq C\big(1 + \kappa^2 h_{min}^{-1-d/2}\varepsilon\big)\varepsilon\big(\|v_h\| + \|\nabla_{M_h}v_h\|\big)$$

which finishes the proof of the proposition. □

**Remarks 3.4.2.** *(i) To approximate a solution of the time-dependent problem one has to assume that $\theta > 1/2$ or $\kappa/h \to 0$ if $\theta = 1/2$, see [Alo07]. If $\theta > 1/2$ then one can show that (the liftings of the functions) $d_t u_h^{i+1} := \kappa^{-1}\big(u_h^{i+1} - u_h^i\big)$ and $w_h^i$ always have the same weak limit in $L^2\big[0, T; W^{1,2}(M; \mathbb{R}^n)^*\big]$: with estimates from the proof of Proposition 3.4.1, interpolation of $L^4$ between $L^2$ and $W^{1,2}$ for $d \leq 3$, and Hölder's inequality we find that for $\lambda \in [0, 1]$ we have*

$$\begin{aligned}\big\|d_t u_h^{i+1} - w_h^i\big\| &= \kappa^{-1}\big\|u_h^{i+1} - \widehat{u}_h^{i+1}\big\| \leq C\kappa\|w_h^i\|^2_{L^4(M_h)} \leq C\kappa\|w_h^i\|^{1/2}_{L^2(M_h)}\|\nabla_{M_h}w_h^i\|^{3/2}_{L^2(M_h)}\\ &\leq C\kappa^{4\lambda}\|w_h^i\|^2_{L^2(M_h)} + C\kappa^{4(1-\lambda)/3}\|\nabla_{M_h}w_h^{i^*}\|^2_{L^2(M_h)},\end{aligned}$$

*where we assumed $\Gamma_{\mathrm{D}} \neq \emptyset$ for simplicity. Multiplication with $\kappa$ and summation over $i = 0, 1, 2, ..., J$ yields with the estimates of Proposition 3.4.1 that*

$$\kappa \sum_{i=0}^{J} \left\| d_t u_h^{i+1} - w_h^i \right\| \leq C\big(\kappa^{4\lambda} + \kappa^{(1-4\lambda)/3}/(2\theta - 1)\big) \left\| \nabla_{M_h} u_h^0 \right\|^2$$

*and the right-hand side tends to 0 provided $0 < \lambda < 1/4$ and $\theta > 1/2$.*
*(ii) Owing to the projection step, the scheme realized by Algorithm C can only be expected to be of first order.*
*(iii) For $\theta = 0$ Algorithm C coincides with the schemes proposed in [AJ06, BBFP07].*

## 3.5 Discussion of Newton iterations

The equivalent definition of discrete harmonic maps in (b) of Lemma 3.1.4 has the advantage that the constraint $u_h(z) \in N$ for all $z \in \mathcal{N}_h$ is formulated as an equality rather than an inclusion. This makes it possible to directly try standard Newton solvers for the solution of the nonlinear systems of equations. The idea to reformulate the harmonic map problem as a saddle-point problem has been considered in [CD03] in case that the target manifold is the unit sphere and in [HTW06] for one-dimensional target manifolds. For other attempts towards the design of higher-order schemes we refer the reader to [LL89, Mor04].

Throughout this section we assume that Assumption (O) is satisfied, i.e., $N$ is orientable and given as the intersection of the zero level sets of differentiable functions $f^{k+1}, f^{k+2}, ..., f^n : \mathbb{R}^n \to \mathbb{R}$. We recall that in this case the saddle-point formulation of Lemma 3.1.4 seeks $u_h \in \mathcal{S}^1(\mathcal{T}_h)^n$ satisfying $u_h|_{\Gamma_{\mathrm{D}}} = u_{\mathrm{D},h}$ and $\lambda_h^\ell \in \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)$, $\ell = k+1, ..., n$ such that

$$\begin{aligned} \big(\nabla_{M_h} u_h; \nabla_{M_h} v_h\big) \quad + \sum_{\ell=k+1}^{n} \big(\lambda_h^\ell; (\overline{\nu}^\ell \circ u_h) \cdot v_h\big)_h \quad &= 0, \\ \sum_{\ell=k+1}^{n} \big(\varrho_h^\ell; f^\ell \circ u_h\big)_h \quad &= 0 \end{aligned}$$

for all $\big(v_h, (\varrho_h^{k+1}, ..., \varrho_h^n)\big) \in \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^n \times \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^{n-k}$, where we incorporated boundary conditions for the multipliers $\lambda_h^\ell$ as well. Defining $X_h := \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^n \times \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^{n-k}$ we may recast the saddle-point formulation as:

*Find $x_h = \big(u_h', (\lambda_h^{k+1}, ..., \lambda_h^n)\big) \in X_h$ such that $F(x_h)[y_h] = 0$ for all $y_h = \big(v_h, (\varrho_h^{k+1}, ..., \varrho_h^n)\big) \in X_h$, where*

$$\begin{aligned} F(x_h)[y_h] := &\big(\nabla_{M_h}[u_h' + \overline{u}_{\mathrm{D},h}]; \nabla_{M_h} v_h\big) \\ &+ \sum_{\ell=k+1}^{n} \big(\lambda_h^\ell; \overline{\nu}^\ell \circ [u_h' + \overline{u}_{\mathrm{D},h}] \cdot v_h\big)_h + \sum_{\ell=k+1}^{n} \big(\varrho_h^\ell; f^\ell \circ [u_h' + \overline{u}_{\mathrm{D},h}]\big)_h, \end{aligned}$$

*with the trivial extension $\overline{u}_{\mathrm{D},h} \in \mathcal{S}^1(\mathcal{T}_h)^n$ of the discrete Dirichlet data $u_{\mathrm{D},h}$.*

Given some $x_h^i \in X_h$, a Newton iteration computes in each step a correction $c_h^i \in X_h$ such that for all $y_h \in X_h$ the identity

$$DF(x_h^i)(c_h^i)[y_h] = -F(x_h^i)[y_h]$$

is satisfied. This defines the new iterate $x_h^{i+1} := x_h^i + c_h^i$. Replacing $u_h' + \overline{u}_{D,h}$ by $u_h$, the Newton scheme may also be written as follows.

**Algorithm D.** *Input:* triangulation $\mathcal{T}_h$ and termination parameter $\varepsilon > 0$.

1. Choose $u_h^0 \in \mathcal{S}^1(\mathcal{T}_h)^n$ and $\lambda_h^{0,\ell} \in \mathcal{S}_{\rm D}^1(\mathcal{T}_h)$, $\ell = k+1, ..., n$, such that $u_h^0|_{\Gamma_{\rm D}} = u_{{\rm D},h}$. Set $i := 0$.
2. Compute $\big(w_h^i, (\mu_h^{i,k+1}, ..., \mu_h^{i,n})\big) \in X_h = \mathcal{S}_{\rm D}^1(\mathcal{T}_h)^n \times \mathcal{S}_{\rm D}^1(\mathcal{T}_h)^{n-k}$ such that

$$\begin{aligned}\big(\nabla_{M_h} w_h^i; \nabla_{M_h} v_h\big) + \sum_{\ell=k+1}^{n} \big(\lambda_h^{i,\ell}; (D\overline{\nu}^\ell \circ u_h^i)[w_h^i] \cdot v_h\big)_h + \sum_{\ell=k+1}^{n} \big(\mu_h^{i,\ell}; (\overline{\nu}^\ell \circ u_h^i) \cdot v_h\big)_h \\ = -\big(\nabla_{M_h} u_h^i; \nabla_{M_h} v_h\big) - \sum_{\ell=k+1}^{n} \big(\lambda_h^{i,\ell}; (\overline{\nu}^\ell \circ u_h^i) \cdot v_h\big)_h, \\ \sum_{\ell=k+1}^{n} \big(\varrho_h^\ell; (\overline{\nu}^\ell \circ u_h^i) \cdot w_h^i\big)_h = -\sum_{\ell=k+1}^{n} \big(\varrho_h^\ell; f^\ell \circ u_h^i\big)_h,\end{aligned}$$

   for all $\big(v_h, (\varrho_h^{k+1}, ..., \varrho_h^n)\big) \in X_h$.
3. Set
$$\big(u_h^{i+1}, (\lambda_h^{i+1,k+1}, ..., \lambda_h^{i+1,n})\big) := \big(u_h^i + w_h^i, (\lambda_h^{i,k+1} + \mu_h^{i,k+1}, ..., \lambda_h^{i,n} + \mu_h^{i,n})\big).$$
4. Stop if $\|\nabla_{M_h} w_h^i\| + \sum_{\ell=k+1}^n \|f^\ell \circ u_h^{i+1}\|_h \le \varepsilon$. Otherwise, set $i := i+1$, and go to 2.

*Output:* $u_h^*$, $\lambda_h^{*,k+1}, ..., \lambda_h^{*,n}$.

The following remarks reveal some difficulties in the analysis of Algorithm D.

**Remarks 3.5.1.** *(i) Step 2 in Algorithm D admits no solution if, e.g.,* $N = S^2$, $f(p) = |p|^2 - 1$, *and* $u_h^j(z) = 0$ *for some* $z \in \mathcal{N}_h \setminus \Gamma_{\rm D}$*: in this case we have* $\overline{\nu}(p) = p$ *for* $p \in N$ *and the choice* $\varrho_h = \varphi_z$ *in Step 2 of Algorithm D leads to*

$$\big(\varphi_z; u_h^i \cdot w_h^i\big)_h = 0 \neq \frac{1}{2}\beta_z = -\big(\varphi_z; |u_h^i|^2 - 1\big)_h$$

*for all* $w_h^i \in \mathcal{S}_{\rm D}^1(\mathcal{T}_h)^m$*. Therefore, global well-posedness and convergence of Algorithm D is false in general.*
*(ii) In case of termination of the iteration of Algorithm D, the output* $u_h^*$ *need not satisfy* $u_h^*(z) \in N$ *for all* $z \in \mathcal{N}_h$*.*
*(iii) Assuming that* $u_h^i(z) \in N$ *for all* $z \in \mathcal{N}_h$ *and defining*

$$X_h^{tan}[u_h^i] := \big\{v_h \in \mathcal{S}_{\rm D}^1(\mathcal{T}_h)^n : v_h(z) \in T_{u_h^i(z)}N \text{ for all } z \in \mathcal{N}_h\big\},$$

*Step 2 in Algorithm D is equivalent to finding* $w_h^i \in X_h^{tan}[u_h^i]$ *such that*

$$\big(\nabla_{M_h} w_h^i; \nabla_{M_h} v_h\big) + \sum_{\ell=k+1}^{n} \big(\lambda_h^{i,\ell}; (D\overline{\nu}^\ell \circ u_h^i)\big[w_h^i\big] \cdot v_h\big)_h = -\big(\nabla_{M_h} u_h^i; \nabla_{M_h} v_h\big),$$

*for all* $v_h \in X_h^{tan}[u_h^i]$*. Notice that up to the second term on the left-hand side this is the iteration of the scheme discussed in Section 3.2. Hence, Algorithm A may be regarded as a simplified Newton iteration.*
*(iv) For* $N = S^{n-1}$*, [CD03] uses the term* $\sum_{z\in\mathcal{N}_h} \lambda_h(z) u_h(z)\cdot v_h(z)$ *instead of* $\big(\lambda_h; u_h\cdot v_h\big)_h$*. This corresponds to a strong penalization of the constraint.*
*(v) A one-dimensional minimization along the correction vector can be incorporated in Algorithm D to improve the stability of the scheme.*

Standard results (see, e.g., [Pla04, Deu04]) assert that the Newton iteration converges if, e.g., there exists $x_h^* \in X_h$ such that $F(x_h^*) = 0$, $DF(x_h^*)$ is regular, and $x_h^0$ is sufficiently close to $x_h^*$. In the following example we show that the derivative $DF(x_h^*)$ may be singular for $x_h^*$ such that $F(x_h^*) = 0$, i.e., Step 2 of Algorithm B may fail to admit a unique solution and the algorithm cannot be expected to converge in general even if a good initial value is available.

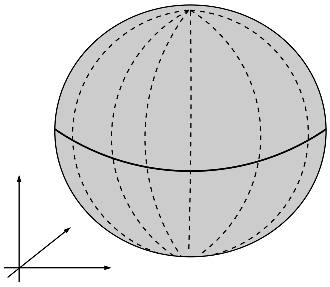

Figure 3.2: Every unit speed geodesic connecting north and south pole defines a harmonic map in Example 3.5.2.

**Example 3.5.2.** *(a) Let* $u : (0,1) \to \mathbb{R}^3$ *be a harmonic map into* $S^2$ *satisfying,* $u(0) = -u(1) = (0,0,1)$*, i.e.,* $u \in W^{1,2}(0,1;\mathbb{R}^3)$ *satisfies* $|u| = 1$ *almost everywhere in* $(0,1)$*,* $u'' - \lambda u = 0$ *in weak sense for some* $\lambda \in L^1(0,1)$ *(in fact* $\lambda = -|u'|^2$*), and* $u(0) = -u(1) = (0,0,1)$*. Then, for each* $\phi \in (-\pi,\pi)$ *the map*

$$u_\phi := \mathbf{R}_\phi u := \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} u$$

*is a harmonic map into* $S^2$ *subject to the same boundary conditions and with the same Lagrange-multiplier* $\lambda$*, i.e.,* $u_\phi$ *satisfies* $|u_\phi| = 1$ *almost everywhere in* $(0,1)$*,* $u_\phi(0) = -u_\phi(1) = (0,0,1)$*, and* $u_\phi'' - \lambda u_\phi = 0$ *in weak sense. The function*

$$w := \frac{d}{d\phi}\Big|_{\phi=0} u_\phi = \mathbf{R}_0 u := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} u$$

*satisfies* $w \neq 0$*,* $w(0) = w(1) = 0$*,* $w \cdot u = 0$ *almost everywhere in* $(0,1)$*, and*

$$w'' - \lambda w = \mathbf{R}_0 u'' + \lambda \mathbf{R}_0 u = \mathbf{R}_0\big(u'' + \lambda u\big) = 0$$

*in particular, we have* $w \in W_0^{1,2}(0,1)$*,* $u \cdot w = 0$ *almost everywhere in* $(0,1)$*, and*

$$(w'; v') + (\lambda; w \cdot v) = 0$$

*for all* $v \in W_0^{1,2}(0,1)$ *such that* $u \cdot v = 0$ *almost everywhere in* $(0,1)$.
*(b) The same example can be constructed in a discrete setting: let* $\mathcal{T}_h$ *be a partition of the interval* $(0,1)$ *and* $u_h \in \mathcal{S}^1(\mathcal{T}_h)^3$ *such that* $|u_h(z)| = 1$ *for all* $z \in \mathcal{N}_h$, $u_h(0) = -u_h(1) = (0,0,1)$, *and suppose that there exists* $\lambda_h \in \mathcal{S}^1(\mathcal{T}_h)$ *so that*

$$\big(u_h'; v_h'\big) + \big(\lambda_h; u_h \cdot v_h\big) = 0$$

*for all* $v_h \in \mathcal{S}^1(\mathcal{T}_h)^3$. *Arguing as in (a) we find that the vector field* $w_h := \mathbf{R}_0 u_h \in \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^3$ *is non-trivial, satisfies* $w_h(z) \cdot u_h(z) = 0$ *for all* $z \in \mathcal{N}_h$, *and*

$$\big(w_h'; v_h'\big) + \big(\lambda_h; w_h \cdot v_h\big) = 0.$$

*for all* $v_h \in \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^3$ *with* $v_h(z) \cdot u_h(z) = 0$ *for all* $z \in \mathcal{N}_h$. *Defining* $\mu_h \in \mathcal{S}^1(\mathcal{T}_h)$ *such that*

$$\big(\mu_h; \varphi_z\big)_h = -\big(w_h'; \varphi_z'\big) - \big(\lambda_h; w_h \cdot \varphi_z\big)_h$$

*for all* $z \in \mathcal{N}_h$ *we see that* $\big(w_h, \mu_h\big) \in X_h$ *is a non-trivial solution of the equation in Step 2 of Algorithm D.*

The above example is related to the existence of Jacobi fields along a harmonic map. A necessary condition for the existence of such, non-trivial fields is that a harmonic map fails to satisfy the so-called *cut-locus-condition*, see [JK79]. This condition requires that for every pair of points $p, q \in N$ in the image of a harmonic map there exists a unique geodesic on $N$ which connects $p$ and $q$. This is obviously not the case in the above example. However, whenever the boundary conditions are slightly perturbed in Example 3.5.2, then the cut-locus-condition is satisfied and there exists exactly one harmonic map $u$ and no non-trivial Jacobi field along $u$. We finally remark that [CD03] proposes to factor out the conformal group of $S^2$ in the set of admissible vector fields to guarantee uniqueness of harmonic maps between topological spheres even if the cut-locus condition is not satisfied.

## 3.6 Combined algorithm

Although the Newton scheme cannot be expected to converge to discrete harmonic maps in general it still performs often well in practice if a good initial value is available. The canonical idea is to use the globally convergent scheme of Algorithm A to find a reasonable starting value for Algorithm D and then monitor convergence of the local iteration. The following algorithm alternatingly iterates the global and the local strategy and is arranged in such a way that if the local strategy does not converge within a prescribed number of iterations, then the last iterate of the global strategy is used to proceed further with the global strategy. The algorithm reduces to Algorithm A or D if either $J_{global} = 0$ or $J_{local} = 0$. Figure 3.3 provides a schematic description of the algorithm. As in the previous section we assume that Assumption (O) is satisfied.

**Algorithm E.** *Input:* triangulation $\mathcal{T}_h$, damping parameter $\kappa > 0$, stopping criterion $\varepsilon > 0$, non-negative integers $J_{global}, J_{local}$ such that $\max\{J_{global}, J_{local}\} > 0$.

(I) Choose $u_h^0 \in \mathcal{S}^1(\mathcal{T}_h)^n$ such that $u_h^0|_{\Gamma_{\mathrm{D}}} = u_{\mathrm{D},h}$ and $u_h^0(z) \in N$ for all $z \in \mathcal{N}_h$. Set $i := i_{global} := 0$.

(G0) If $i_{global} = J_{global}$ then set $i_{local} := 0$ and go to (II)

(G1) Compute $w_h^i \in \mathring{\mathcal{S}}^1(\mathcal{T}_h)^n$ such that $w_h^i(z) \in T_{u_h^i(z)}N$ for all $z \in \mathcal{N}_h$ and

$$\big(\nabla_{M_h} w_h^i; \nabla_{M_h} v_h\big) = -\big(\nabla_{M_h} u_h^i; \nabla_{M_h} v_h\big)$$

for all $v_h \in \mathring{\mathcal{S}}^1(\mathcal{T}_h)^n$ such that $v_h(z) \in T_{u_h^i(z)}N$ for all $z \in \mathcal{N}_h$.

(G2) Stop if $\|\nabla_{M_h} w_h^i\| \le \varepsilon$.

(G3) Define $u_h^{i+1} \in \mathcal{S}^1(\mathcal{T}_h)^n$ by setting

$$u_h^{i+1}(z) := \pi_N\big(u_h^i(z) + \kappa w_h^i(z)\big)$$

for all $z \in \mathcal{N}_h$.

(G4) Set $i := i+1$ and $i_{global} := i_{global} + 1$ and go to (G0).

(II) Set $u_h^{old} := u_h^i$ and choose $\lambda_h^{i,\ell} \in \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)$, $\ell = k+1, ..., n$.

(L0) If $i_{local} = J_{local}$ then set $u_h^i := u_h^{old}$, $i_{global} := 0$, and go to (G0) .

(L1) Compute $\big(w_h^i, (\mu_h^{i,k+1}, ..., \mu_h^{i,n})\big) \in X_h := \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h) \times \mathcal{S}^1_{\mathrm{D}}(\mathcal{T}_h)^{n-k}$ such that

$$\begin{aligned}\big(\nabla_{M_h} w_h^i; \nabla_{M_h} v_h\big) + \sum_{\ell=k+1}^{n} \big(\lambda_h^{i,\ell}; (D\overline{\nu}^\ell \circ u_h^i)[w_h^i] \cdot v_h\big)_h + \sum_{\ell=k+1}^{n} \big(\mu_h^{i,\ell}; (\overline{\nu}^\ell \circ u_h^i) \cdot v_h\big)_h &\\ = -\big(\nabla_{M_h} u_h^i; \nabla_{M_h} v_h\big) - \sum_{\ell=k+1}^{n} \big(\lambda_h^{i,\ell}; (\overline{\nu}^\ell \circ u_h^i) \cdot v_h\big)_h&,\\ \sum_{\ell=k+1}^{n} \big(\varrho_h^\ell; (\overline{\nu}^\ell \circ u_h^i) \cdot w_h^i\big)_h = -\sum_{\ell=k+1}^{n} \big(\varrho_h^\ell; f^\ell \circ u_h^i\big)_h&,\end{aligned}$$

for all $\big(v_h, (\varrho_h^{k+1}, ..., \varrho_h^n)\big) \in X_h$.

(L2) Set $\big(u_h^{i+1}, (\lambda_h^{i+1,k+1}, ..., \lambda_h^{i+1,n})\big) := \big(u_h^i + w_h^i, (\lambda_h^{i,k+1} + \mu_h^{i,k+1}, ..., \lambda_h^{i,n} + \mu_h^{i,n})\big)$.

(L3) Stop if $\|\nabla_{M_h} w_h^i\| + \sum_{\ell=k+1}^n \|f^\ell \circ u_h^{i+1}\|_h \le \varepsilon$.

(L4) Set $i := i+1$, $i_{local} := i_{local} + 1$, and go to (L0).

*Output:* $u_h^*$, $\lambda_h^{*,k+1}, ..., \lambda_h^{*,n}$.

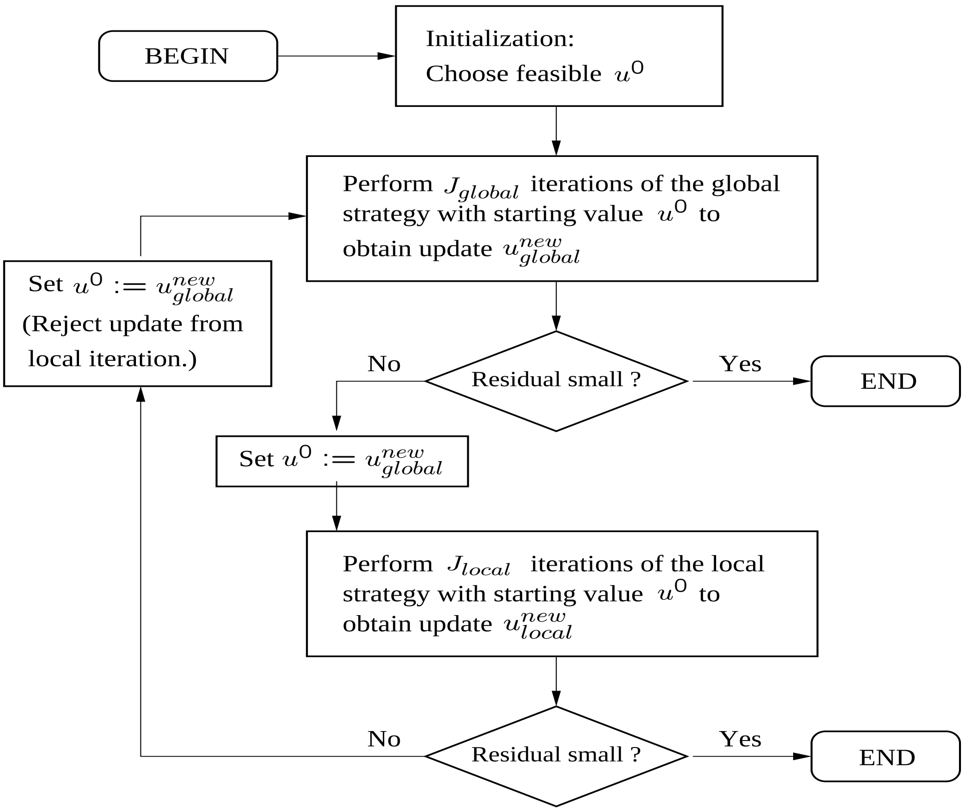


Figure 3.3: Schematic description of the combination of Algorithm E.

The only rigorous statement that we can provide for Algorithm E is that it works at least as well as Algorithm A (up to an increased number of iterations), provided that $J_{global} > 0$.

**Theorem 3.6.1.** *Suppose $J_{global} > 0$. Assume that $N$ is $C^3$ and*

$$\kappa \le C' \min\{h_{min}, \omega_N h_{min}^{d/2-1} \log\left(h_{min}\right)^{-1}\}$$

*with $C'$ from Lemma 3.2.4, or $N = \partial\mathcal{C}$ for a bounded, open, convex set $\mathcal{C} \subset \mathbb{R}^n$, $\mathcal{T}_h$ is weakly acute, and $\kappa < 2$. Then the iteration of Algorithm E terminates within a finite number of iterations.*

*Proof.* This follows since the global strategy alone, realized by Steps (G1)-(G4), converges, cf. Theorem 3.2.7. □

**Remarks 3.6.2.** *(i) The constraint $w_h^i(z) \in T_{u_h^i(z)}N$ for all $z \in \mathcal{N}_h$ provides a Lagrange multiplier which may be used to define an initial value $\lambda_h^j$ in Step (II) for the initialization of the local strategy, cf. Lemma 3.1.4.*
*(ii) Another useful stopping criterion for the (temporary) termination of the global strategy can be based on a small decrease of the Dirichlet energy.*
*(iii) The iteration of the local strategy should be terminated (e.g., by setting $j_{local} = N_{local}$) if the problem in (L1) does not admit a solution.*

# Chapter 4

# Numerical experiments

## 4.1 Implementation aspects

The algorithms introduced and analyzed in the previous chapter were implemented in the numerical computing environment MATLAB. In order to decrease the total runtime, some subroutines were programmed in C and included with the interface MEX. Besides employing standard matrix manipulation operations and sparse data structures, we always made use of MATLAB's backslash operator to solve linear systems of equations possibly including constraints via use of Lagrange multipliers. In this section we exemplify implementation issues by discussing short MATLAB implementations of the Laplace-Beltrami operator and of an iterative scheme for the practical realization of the projection operator $\pi_N$. The implementations of Algorithms A, B, C, D and E of Chapter 3 are then immediate consequences and we refer the reader to Appendix A for details.

### 4.1.1 MATLAB routine for the Laplace-Beltrami operator

Following [Dzi88], a finite element approximation of the Poisson problem on surfaces,

$$-\Delta_M u = f$$

on $M$, subject to boundary conditions $u|_{\Gamma_\mathrm{D}} = u_\mathrm{D}$ and $\partial u/\partial\sigma = g$ on $\Gamma_\mathrm{N} := \partial M \setminus \Gamma_\mathrm{D}$, where $\Gamma_\mathrm{D}$ is a possibly empty subset of $\partial M$ and $\partial u/\partial\sigma$ the co-normal derivative of $u$ on $\partial M$ (which coincides with the normal derivative if $M$ is flat), see [DDE05], reads as follows:

*Given a triangulation $\mathcal{T}_h$ of the approximation $M_h$ of $M$ find $u_h \in \mathcal{S}^1(\mathcal{T}_h)$ such that $u_h(z) = u_\mathrm{D}(z)$ for all $z \in \mathcal{N}_h \cap \Gamma_\mathrm{D}$ and*

$$\int_{M_h} \nabla_{M_h} u_h \cdot \nabla_{M_h} v_h \,\mathrm{d}s_h = \int_{M_h} \check{f} v_h \,\mathrm{d}s_h + \int_{\Gamma_{\mathrm{N},h}} \check{g} v_h \,\mathrm{d}\,\mathrm{d}t_h \tag{1.1}$$

*for all $v_h \in \mathcal{S}^1_\mathrm{D}(\mathcal{T}_h)$ and approximations $\check{f}$, $\Gamma_{\mathrm{N},h}$, and $\check{g}$ of $f$, $\Gamma_\mathrm{N}$, and $g$, respectively.*

The triangulation $\mathcal{T}_h$ is specified by providing matrices `c4n`, `n4e`, `Db`, and `Nb`. Here, `c4n` is an $L_c \times (d+1)$ array defining the coordinates of the $L_c$ nodes $z_1, z_2, ..., z_{L_c} \in \mathcal{N}_h$ of $\mathcal{T}_h$ on the $d$-dimensional hypersurface $M \subset \mathbb{R}^{d+1}$. The $L_s \times (d+1)$ array `n4e` defines the $L_s$ subsimplices contained in $\mathcal{T}_h$ by providing the corresponding row numbers in the matrix `c4n` of the vertices for each element $K \in \mathcal{T}_h$. The discrete approximations of the boundary parts $\Gamma_D$ and $\Gamma_N$ of the possibly empty boundary $\partial M$ are specified in the $M_1 \times d$ and $M_2 \times d$ arrays `Db` and `Nb`, respectively, by providing the row numbers of the nodes on each edge ($d = 2$) or face ($d = 3$) on $\Gamma_D$ and $\Gamma_N$.

To compute the stiffness matrix corresponding to the left-hand side of (1.1) we compute for each element $K \in \mathcal{T}_h$ a unit normal vector $\mu_h|_K$ and the auxiliary node $\widehat{z}_K := x_K + h_K \mu_h|_K$, where $x_K$ is the midpoint of $K$. The nodal basis functions $\big(\varphi_z|_K : z \in \mathcal{N}_h \cap K\big)$ are then linearly extended to the $(d+1)$-simplex $\widehat{K} := \operatorname{conv}\{K, \widehat{z}_K\}$ by defining $\varphi_z(\widehat{z}_K) := 0$. This allows to compute a full, $(d+1)$-dimensional gradient whose restriction to $M_h$ is then projected onto the tangent space of the discrete surface $M_h$ using the projection matrix

$$\mathbf{P}_K := \mathbf{I}_{(d+1)\times(d+1)} - \mu_h|_K \otimes \mu_h|_K.$$

If $d = 2$, $K = \operatorname{conv}\{z_0, z_1, z_2\}$ for $z_0, z_1, z_2 \in \mathcal{N}_h$, and $z_1 - z_0$ and $z_2 - z_0$ are two different edge vectors of $K$, then the property of the vector product, that

$$2\mathcal{H}^2(K)\mu_h|_K = (z_1 - z_0) \times (z_2 - z_0)$$

allows to compute all required geometric quantities related to the triangle $K$.

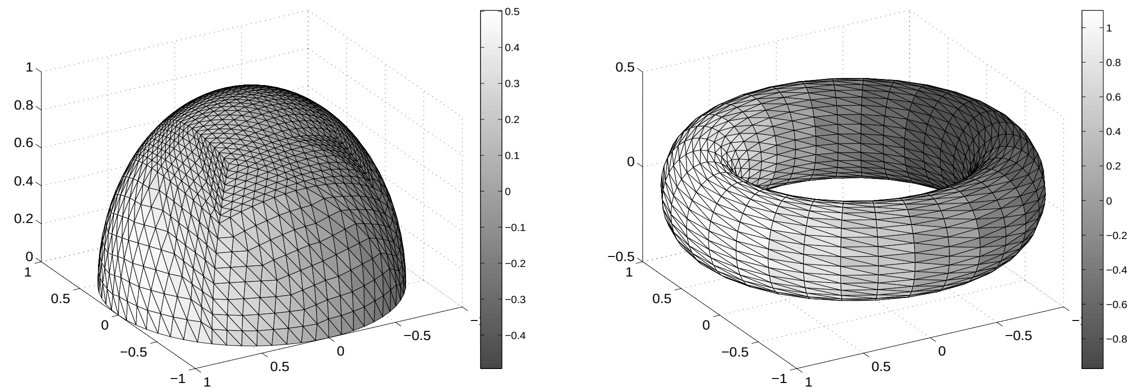


Figure 4.1: Numerical approximations of the Laplace-Beltrami problem (1.1) on the upper half sphere (left) and on a torus (right) with homogeneous Dirichlet conditions on $\partial M$ and with $f(x_1, x_2, x_3) = x_2$.

The MATLAB code shown in Figure 4.2 realizes these ideas for $d = 2$ and was developed in the spirit of [ACF99, BC04]. Dirichlet conditions are included by eliminating corresponding nodes from the linear system of equations; in case that $\Gamma_D = \emptyset$ the constraint $\int_{M_h} u_h \, ds_h = 0$ is added to the linear system of equations in order to guarantee a regular system matrix, assuming that $f$ and $g$ as well as their approximations satisfy appropriate compatibility conditions.

The computed approximate solution for (1.1) with $f(x_1, x_2, x_3) = x_2$ and $u_D = 0$ and for triangulations of the upper half unit sphere and the torus with radii $(r_1, r_2) = (1, 1/4)$ into 2560 and 2048 triangles, respectively, are displayed in left and right plots of Figure 4.1.

### 4.1.2 Numerical realization of the orthogonal projection $\pi_N$

Various approaches have been discussed in the literature to iteratively approximate the orthogonal projection $\pi_N$ onto closed surfaces $N$. For ease of presentation we consider the simplest case of a

```
function LaplaceBeltrami
load c4n.dat; load n4e.dat; load Db.dat; load Nb.dat;
DiriNodes = unique(Db);
freeNodes = setdiff([1:size(c4n,1)],DiriNodes);
for j = 1 : size(n4e,1)
    mu_K(j,:) = cross(c4n(n4e(j,2),:)-c4n(n4e(j,1),:),c4n(n4e(j,3),:)-c4n(n4e(j,2),:));
    area_K(j) = norm(mu_K(j,:))/2;
    mu_K(j,:) = mu_K(j,:) / norm(mu_K(j,:));
    mp_K(j,:) = sum(c4n(n4e(j,:),:))/3;
    diam_K(j) = norm(c4n(n4e(j,2),:)-c4n(n4e(j,1),:));
end
A = sparse(size(c4n,1),size(c4n,1));
b = zeros(size(c4n,1),1);
c = zeros(size(c4n,1),1);
x = zeros(size(c4n,1),1);
for j = 1 : size(n4e,1)
    tmp_tetra = [c4n(n4e(j,:),:);mp_K(j,:)+diam_K(j)*mu_K(j,:)];
    grads3_K = [1,1,1,1;tmp_tetra'] \ [0,0,0;eye(3)];
    P_K = eye(3) - mu_K(j,:)' * mu_K(j,:);
    for m = 1 : 3
        b(n4e(j,m)) = b(n4e(j,m)) + (1/3) * area_K(j) * f(mp_K(j,:));
        c(n4e(j,m)) = c(n4e(j,m)) + (1/3) * area_K(j);
        for n = 1 : 3
            A(n4e(j,m),n4e(j,n)) = A(n4e(j,m),n4e(j,n)) + ...
                area_K(j) * (P_K * grads3_K(m,:)')' * (P_K * grads3_K(n,:)');
        end
    end
end
for j = 1 : size(Nb,1)
    length_E = norm(c4n(Nb(j,1),:) - c4n(Nb(j,2),:));
    mp_E = (c4n(Nb(j,1),:) - c4n(Nb(j,2),:))/2;
    b(Nb(j,1)) = b(Nb(j,1)) + (1/2) * length_E * g(mp_E);
    b(Nb(j,2)) = b(Nb(j,2)) + (1/2) * length_E * g(mp_E);
end
if isempty(DiriNodes)
    A = [A,c;c',0];
    b = [b;0];
else
    for j = 1 : size(DiriNodes,1)
        x(DiriNodes(j)) = u_D(c4n(DiriNodes(j),:));
    end
    b = b - A * x;
end
x(freeNodes) = A(freeNodes,freeNodes) \ b(freeNodes);
trisurf(n4e,c4n(:,1),c4n(:,2),c4n(:,3),x(1:size(c4n,1)))

function val = f(X)
val = X(2);
function val = u_D(X)
val = 0;
function val = g(X)
val = 0;
```

Figure 4.2: MATLAB implementation of the finite element scheme for the Laplace-Beltrami operator and approximate solution of the Poisson problem on a surface.

hypersurface $N \subset \mathbb{R}^{k+1}$ defined by a continuous function $f\colon \mathbb{R}^{k+1} \to \mathbb{R}$ through

$$N = \{p \in \mathbb{R}^{k+1} : f(p) = 0\}.$$

We assume that $f$ is $C^1$ and satisfies $|\nabla f| \neq 0$ in a neighborhood of $N$. Closed formulas are available for simple geometries such as the unit sphere, tori defined through radii $r_1, r_2$, and ellipsoids with equatorial and polar radii $d_1, d_2$ and $c$, respectively.

**Example 4.1.1.** *The zero sets of the functions $f_1, f_2, f_3\colon \mathbb{R}^3 \to \mathbb{R}$, given for $q = (q_1, q_2, q_3) \in \mathbb{R}^3$, by*

$$\begin{aligned} f_1(q) &:= 1 - |q|^2, \\ f_2(q) &:= \left[\left(|(q_1, q_2)| - r_1\right)^2 + q_3^2\right]^{1/2} - r_2, \\ f_3(q) &:= \frac{q_1^2}{r_1^2} + \frac{q_2^2}{r_2^2} + \frac{q_3^2}{c^2} - 1 \end{aligned}$$

*define the unit sphere $S^2$, a torus $T^2_{r_1 r_2}$ with radii $r_1$ and $r_2$, and an ellipsoid $E^2_{r_1 r_2 c}$ with equatorial radii $r_1, r_2$ and polar radius $c$, respectively.*

We follow ideas in [DD07] and consider for given $q \in \mathbb{R}^{k+1}$ the functional

$$G(p, \lambda) := \frac{1}{2}|p - q|^2 + \lambda f(p). \tag{1.2}$$

Then, if $q \in U_{\delta_N}(N)$ we have $p = \pi_N(q)$ if and only if $|p - q| \leq \delta_N$ and there exists $\lambda \in \mathbb{R}$ such that the pair $(p, \lambda)$ is stationary for $G$. Indeed, we have

$$\partial_\lambda G(p, \lambda) = f(p) = 0 \quad \Longleftrightarrow \quad p \in N$$

and if $p \in N$, upon recalling that $\nu(p) = \nabla f(p)/|\nabla f(p)|$, then

$$\partial_p G(p, \lambda) = p - q + \lambda \nabla f(p) = 0 \quad \Longleftrightarrow \quad p - q = \alpha\, \nu(p),$$

where $|\alpha| = |p - q|$. We thus deduce that $p = \pi_N(q)$.

We employ a classical Newton iteration to compute stationary points for $G$. We note

$$\nabla_{(p,\lambda)} G(p, \lambda) = \left(p - q + \lambda \nabla f(p), f(p)\right)$$

and, if $f \in C^2$ in a neighborhood of $N$,

$$D^2_{(p,\lambda)} G(p, \lambda) = \begin{bmatrix} \mathbf{I}_{(k+1)\times(k+1)} + \lambda D^2 f(p) & \nabla f(p) \\ \left(\nabla f(p)\right)^{\mathrm{T}} & 0 \end{bmatrix}.$$

Given $(p, \lambda) \in N \times \mathbb{R}$ such that $\nabla_{(p,\lambda)} G(p, \lambda) = 0$, i.e., $|\lambda| = |p - q|/|\nabla f(p)|$, the matrix $D^2_{(p,\lambda)} G(p, \lambda)$ is invertible if the linear mapping defined by the matrix $\mathbf{I}_{(k+1)\times(k+1)} + \lambda D^2 f(p)$ restricted to $T_p N = \left(\nabla f(p)\right)^\perp$ is invertible, cf. [BF91, Bra01]. Sufficient for this is that $|\lambda| < \rho\left(D^2 f(p)\right)^{-1}$, where $\rho\left(D^2 f(p)\right)$ denotes the spectral radius of $D^2 f(p)$. The following result then follows from well-known assertions on Newton iterations, see e.g. [Pla04, Deu04].

**Proposition 4.1.2.** *Suppose that* $f \in C^3\big(U_{\delta_N}(N)\big)$. *There exists* $\delta_{N,\pi} > 0$ *such that the Newton iteration*

$$\big(p^{i+1}, \lambda^{i+1}\big) := \big(p^i, \lambda^i\big) - \big[D^2_{(p,\lambda)} G\big(p^i, \lambda^i\big)\big]^{-1} \nabla_{(p,\lambda)} G\big(p^i, \lambda^i\big).$$

*with starting value* $\big(p^0, \lambda^0\big) = \big(q, 0\big)$ *converges to* $\big(p, \lambda\big) = \big(\pi_N(q), (p-q)\cdot \nu(p)/|\nabla f(p)|\big)$ *whenever* $q \in U_{\delta_{N,\pi}}(N) := \big\{q' \in \mathbb{R}^{n+1} : \operatorname{dist}(q', N) < \delta_{N,\pi}\big\}$. ☐

A MATLAB routine that realizes the Newton scheme for two-dimensional tori but which can easily be adapted to other submanifolds is shown in Figure 4.3.

```
function pi_N(q,r)
eps_pi = 1.0E-10;
p = q;
lambda = 0;
dG = [(p-q)'+lambda*df(p,r)';f(p,r)];
res = norm(dG)
while res >= eps_pi
    d2G = [eye(3)+lambda*d2f(p,r),df(p,r)';df(p,r),0];
    v_new = [p';lambda] - d2G\dG;
    p = v_new(1:3)';
    lambda  = v_new(4);
    dG = [(p-q)' + lambda*df(p,r)';f(p,r)];
    res = norm(dG);
end

function val = f(p,r)
val = Gamma(p,r)^(1/2) - r(2);
function val = df(p,r)
val = (1/2)*Gamma(p,r)^(-1/2)*dGamma(p,r);
function val = d2f(p,r)
val = -(1/4)* Gamma(p,r)^(-3/2)*dGamma(p,r)'*dGamma(p,r) + ...
    (1/2)*Gamma(p,r)^(-1/2)*d2Gamma(p,r);

function val = Gamma(p,r)
val = (norm(p(1:2)) - r(1))^2 + p(3)^2;
function val = dGamma(p,r)
val = 2*[(norm(p(1:2))-r(1))*p(1:2)/norm(p(1:2)),p(:,3)];
function val = d2Gamma(p,r)
val = 2*[eye(2)-(r(1)/norm(p(1:2)))*(eye(2)-p(1:2)'*p(1:2)/norm(p(1:2))^2),...
     zeros(2,1);0,0,1];
```

Figure 4.3: MATLAB implementation of the Newton iteration for the approximation of $\pi_{T^2_{r_1,r_2}}$ for the torus $T^2_{r_1,r_2}$ with radii $r = (r_1, r_2)$. Here, $T^2_{r_1,r_2} = f^{-1}\big(\{0\}\big)$ for $f(p) := \Gamma(p)^{1/2} - r_2$ and $\Gamma(p) = \big(|(p_1,p_2)| - r_1\big)^2 + p_3^2$ for $p = \big(p_1, p_2, p_3\big)$.

**Remarks 4.1.3.** *(i) Let* $q \in U_{\delta_N}(N)$, *define the signed distance function* $d_N(q) := \big(\operatorname{sign} f(q)\big)\big|\pi_N(q) - q\big|$, *and abbreviate* $p := \pi_N(q)$. *The identities* $p = q - d_N(q)\nu(p)$ *and* $\nu(p) = \nabla f(p)/\big|\nabla f(p)\big|$ *imply*

$$\begin{aligned} f(q) &= f(p) + \nabla f(p)\cdot(q-p) + \mathcal{O}\big(d_N(q)^2\big) \\ &= d_N(q)\big|\nabla f(p)\big| + \mathcal{O}\big(d_N(q)^2\big). \end{aligned}$$

*Based on the resulting approximation $d_N(q) \approx f(q)/|\nabla f(q)|$, [DD07] employs the iteration*

$$(1) \qquad \widehat{p}^{i+1} := p^i - \frac{f(p^i)}{|\nabla f(p^i)|}\frac{\nabla f(p^i)}{|\nabla f(p^i)|}$$

$$(2) \qquad p^{i+1} := q - \big(\operatorname{sign} f(q)\big)\,|\widehat{p}^{i+1} - q|\frac{\nabla f(\widehat{p}^{i+1})}{|\nabla f(\widehat{p}^{i+1})|}$$

*to approximate $\pi_N(q)$.*
*(ii) If $f = d_N$ is the signed distance function of $N$ then the Newton scheme of Proposition 4.1.2 with starting value $(q,0)$ detects the correct solution $p = \pi_N(q)$ within one step: we have*

$$\begin{aligned}(p^1, \lambda^1) &= (q,0) - \begin{bmatrix} \mathbf{I}_{(k+1)\times(k+1)} & \nu(p) \\ (\nu(p))^{\mathrm{T}} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ d_N(q) \end{bmatrix} \\ &= (q,0) - \big(d_N(q)\nu(p), -d_N(q)\big) \\ &= \big(p, d_N(q)\big).\end{aligned}$$

*(iii) For submanifolds $N$ of codimension larger than one and defined through continuous functions $f^{k+1}, ..., f^n\colon \mathbb{R}^n \to \mathbb{R}$ by $N = \{p \in \mathbb{R}^n\colon f^{k+1}(p) = .... = f^n(p) = 0\}$, the projection $\pi_N(q)$ can be be approximated by computing stationary points for the functional*

$$G\colon U_{\delta_N}(N) \times \mathbb{R}^{n-k} \to \mathbb{R}, \qquad \big(p, (\lambda^{k+1}, ..., \lambda^n)\big) \mapsto \frac{1}{2}|p-q|^2 + \sum_{\ell=k+1}^{n} \lambda^\ell f^\ell(p).$$

We ran the Newton iteration and the scheme described in Remark 4.1.3 (i) with the same stopping criterion $\big|\nabla_{(p,\lambda)} G(p^i, \lambda^i)\big| \le \varepsilon_\pi := 10^{-10}$ for different starting values. Table 4.1 displays the numbers of iterations and shows that the Newton scheme works efficiently in this example. In most of the experiments the scheme described in Remark 4.1.3 (i) works as well as the Newton scheme. Our overall experience with the algorithms is that the Newton scheme is more robust and even converges to the right object when the distance of $q$ to $N$ is very large.

| $N$ | $f$ | $q$ | Newton | Demlow & Dziuk |
|---|---|---|---|---|
| $T^2_{r_1,r_2}$ | $\tan(d_N)$ | $(0,0,0) + \xi/10$ | 4 | 4 |
| $T^2_{r_1,r_2}$ | $\tan(d_N)$ | $(2,2,2) + \xi/10$ | 4 | 4 |
| $T^2_{r_1,r_2}$ | $d_N$ | $(2,2,2) + \xi/10$ | 1 | 1 |
| $T^2_{r_1,r_2}$ | $d_N$ | $30\,(1,1,1)$ | 1 | $> 100$ |
| $S^2$ | $\tan(d_N)$ | $(0,0,0) + \xi/10$ | 5 | 4 |
| $S^2$ | $d_N$ | $(0,0,0) + \xi/10$ | 1 | 1 |

Table 4.1: Numbers of iterations of the Newton method and the scheme described in Remark 4.1.3 (i) for the approximation of the orthogonal projection $\pi_N(q)$ onto a torus with radii $(r_1, r_2) = (1, 1/4)$ and the unit sphere $S^2$ (which coincides with the degenerated torus with radii $(r_1, r_2) = (0, 1)$). The vector $\xi = (\xi_1, \xi_2, \xi_3)$ denotes a random vector satisfying $\max_{i=1,2,3} |\xi_i| \le 1$.

## 4.2 Performance of Algorithms A and D for a 3D prototype singularity

In this section we report on the practical performance of Algorithms A and D for the approximation of a three-dimensional harmonic map into the two-dimensional unit sphere $S^2$ with a prototype singularity. In particular, we consider the Euclidean situation $M = (-1/2, 1/2)^3 \times \{0\} \subseteq \mathbb{R}^{3+1}$ and $N = S^2$ together with

$$u_{\mathrm{D}}(x) := \frac{x}{|x|} \qquad \text{for} \quad x \in \Gamma_{\mathrm{D}} := \partial M.$$

It is known [Lin87] that the unique energy minimizing harmonic map into $S^2$ subject to these boundary conditions is $u(x) = x/|x|$ for $x \in M$ and has a singularity at the origin. In the following we neglect the forth (vanishing) component of coordinates and write $\nabla$ for the usual three-dimensional gradient. Notice that $S^2 = \partial B_1(0)$ is the boundary of a bounded, open, convex set.

### 4.2.1 Stability of Algorithm A and necessity of weak acuteness

As proved in Lemma 3.2.5 for the case that $N = \partial\mathcal{C}$ for a convex set $\mathcal{C}$, Algorithm A is unconditionally stable and convergent for $\kappa < 2$ if the underlying triangulation is weakly acute. If $\mathcal{T}_h$ is not weakly acute, then a proper damping parameter $\kappa = O(h_{min})$ has to be employed in Algorithm A.

For each positive integer $J$ we defined a weakly acute triangulation $\mathcal{T}_J^{ac}$ with $h_{min} \approx 2^{-J}$ of $M$ into $6 \cdot 2^{3J}$ tetrahedra by first partitioning $M$ into $2^{3J}$ many cubes of side-length $2^{-J}$ and subsequently dividing each cube into six tetrahedra by dilation and translation of the following acute triangulation of the unit cube: for the reference cube $Q_{ref} := (0, 1)^3$ with vertices

$$\begin{aligned} &\widehat{z}_1 := (0,0,0), \quad \widehat{z}_2 := (1,0,0), \quad \widehat{z}_3 := (0,0,1), \quad \widehat{z}_4 := (1,0,1), \\ &\widehat{z}_5 := (0,1,0), \quad \widehat{z}_6 := (1,1,0), \quad \widehat{z}_7 := (0,1,1), \quad \widehat{z}_8 := (1,1,1) \end{aligned}$$

a partition of $Q_{ref}$ into six right-angled tetrahedra is given by

$$\begin{aligned} &\widehat{K}_1 := \operatorname{conv}\{\widehat{z}_1, \widehat{z}_2, \widehat{z}_3, \widehat{z}_6\}, \quad \widehat{K}_2 := \operatorname{conv}\{\widehat{z}_2, \widehat{z}_4, \widehat{z}_3, \widehat{z}_6\}, \quad \widehat{K}_3 := \operatorname{conv}\{\widehat{z}_3, \widehat{z}_4, \widehat{z}_8, \widehat{z}_6\}, \\ &\widehat{K}_4 := \operatorname{conv}\{\widehat{z}_3, \widehat{z}_8, \widehat{z}_7, \widehat{z}_6\}, \quad \widehat{K}_5 := \operatorname{conv}\{\widehat{z}_7, \widehat{z}_5, \widehat{z}_3, \widehat{z}_6\}, \quad \widehat{K}_6 := \operatorname{conv}\{\widehat{z}_3, \widehat{z}_5, \widehat{z}_1, \widehat{z}_6\}, \end{aligned}$$

see Figure 4.4. It is important to notice that the faces of tetrahedra on two opposite sides of $Q_{ref}$ match so that an assembly of such partitions results in a regular triangulation. We remark that such and similar partitions have also been used in [Bey95, KK01, KK03], e.g., to guarantee validity of a discrete maximum principle for a discretization of the Poisson problem.

We also defined a sequence of non-weakly acute triangulations $\mathcal{T}_J^{n-ac}$ by using the coarse triangulation $\mathcal{T}_0^{n-ac}$ (which actually is weakly acute) of $M$ into six tetrahedra shown in Figure 4.4 and recursively defining $\mathcal{T}_{J+1}^{n-ac}$ by partition of each tetrahedron $K \in \mathcal{T}_J^{n-ac}$ into eight sub-tetrahedra obtained from connecting midpoints of edges of $K$, cf. Figure 4.5. Hence, the triangulation $\mathcal{T}_J^{n-ac}$ consists of $6 \cdot 8^J = 6 \cdot 2^{3J}$ many tetrahedra. One verifies however, that even though $\mathcal{T}_0^{n-ac}$ is weakly acute, the triangulations $\mathcal{T}_J^{n-ac}$ are not weakly acute if $J \geq 1$.

We employ two different types of initial data to experimentally study the reliability and efficiency of Algorithm A. Both of them interpolate the function $x/|x|$ on the boundary of $M$. The first one is constant in the interior of $M$ while the second one is defined through random vectors on the unit

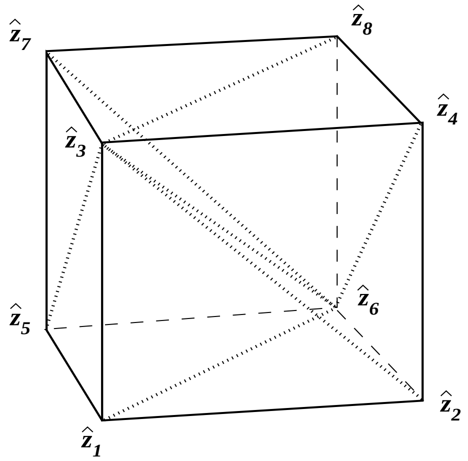


Figure 4.4: Partition of the unit cube into six tetrahedra such that each of them has a right (acute) angle.

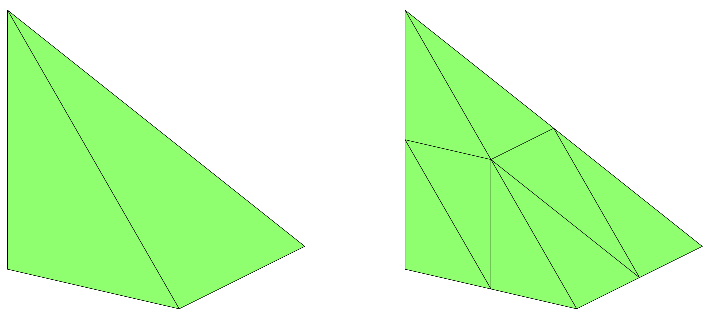

Figure 4.5: Uniform refinement of a tetrahedron into eight sub-tetrahedra. This refinement does not preserve weak acuteness of a triangulation.

sphere at interior nodes of a triangulation. More precisely, given a triangulation $\mathcal{T}_h$ with nodes $\mathcal{N}_h$ the initial vector field $u_h^0 \in \mathcal{S}^1(\mathcal{T}_h)^3$ is defined through

$$u_h^0(z) := \begin{cases} (1,0,0) & \text{for } z \in \mathcal{N}_h \setminus \Gamma_{\rm D}, \\ z/|z| & \text{for } z \in \mathcal{N}_h \cap \Gamma_{\rm D}, \end{cases} \tag{2.3}$$

or

$$u_h^0(z) := \begin{cases} \xi(z) & \text{for } z \in \mathcal{N}_h \setminus \Gamma_{\rm D}, \\ z/|z| & \text{for } z \in \mathcal{N}_h \cap \Gamma_{\rm D}, \end{cases} \tag{2.4}$$

where $\xi$ is a random vector-field with values in $S^2$.

To study the necessity of a weak acuteness property of the underlying triangulation we ran Algorithm A with $\varepsilon = 10^{-4}$ and the three pairs

$$\big(\mathcal{T}_h, \kappa\big) = \big(\mathcal{T}_4^{ac}, 1\big),\ \big(\mathcal{T}_4^{n-ac}, 1\big),\ \big(\mathcal{T}_4^{n-ac}, h_{min}/10\big),$$

where $h_{min} = 2^{-4}$. The upper plot of Figure 4.6 displays the norm of the computed correction vectors in Algorithm A, $\|\nabla w_h^i\|$, for $i = 1, 2, \ldots$ until termination, when $u_h^0$ is defined through (2.3). The iteration stopped after 43 iterations when the weakly acute triangulation was used. This is in

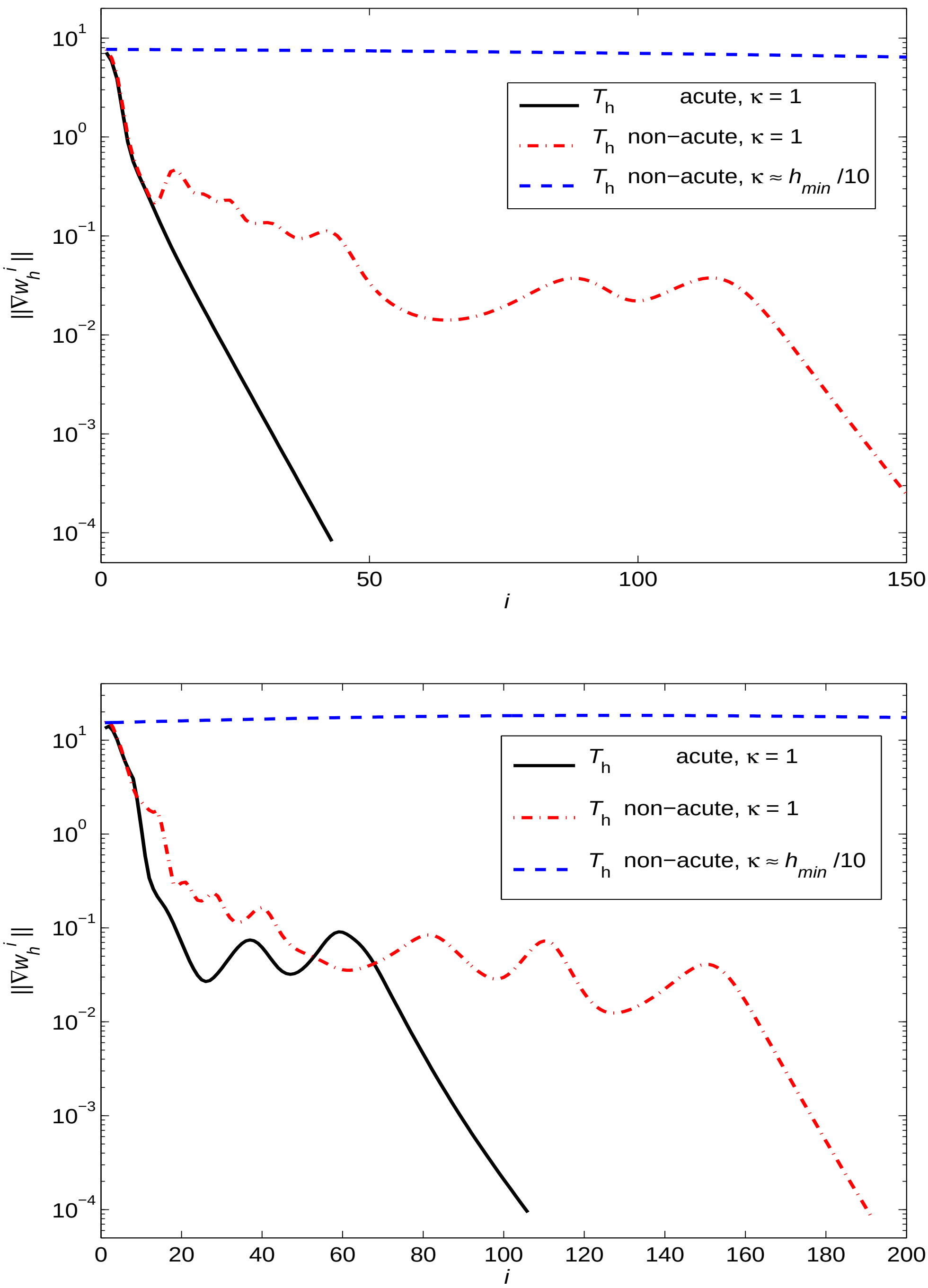


Figure 4.6: $W^{1,2}$ semi-norm of the correction $w_h^i$ in Algorithm A for a weakly acute triangulation and a non-weakly acute triangulation with 24.576 tetrahedra and $h_{min} = 1/16$ with different damping parameters $\kappa$ for initial data which are constant (upper plot) and random vectors on $S^2$ (lower plot) in the interior of $M$.

good agreement with Theorem 3.2.7, which states that Algorithm A is globally convergent on weakly acute triangulations if $N$ is the boundary of a convex set. The fact that 156 iteration steps, i.e., about four times as many, are needed to satisfy the termination criterion of Algorithm A when a non-weakly acute triangulation is used, indicates that the angle condition is not just a technical detail but has influence on the practical performance of Algorithm A. In order to guarantee convergence on arbitrary (regular) triangulations, a proper damping parameter $\kappa \in \mathcal{O}(h_{min})$ has to be used. In this case many more iterations are needed to achieve $\|\nabla w_h^i\| \leq \varepsilon$ and in this experiment the termination criterion was not satisfied within the first 1000 iterations. The advantages of weakly acute triangulations are also clearly visible when the initial $u_h^0$ is defined as in (2.4). The lower plot of Figure 4.6 shows the rapid decay of the norm of the correction vectors on $\mathcal{T}_4^{ac}$ and the slower decay on $\mathcal{T}_4^{n-ac}$. We remark that while the energy is known to decrease monotonically on weakly acute triangulations, we only know that the $W^{1,2}$ semi-norms of the corrections $(w_h^i)$ are square summable.

To illustrate the evolution defined by the $H^1$ gradient flow of Algorithm A, we plotted in Figures 4.7 and 4.8 the first two components of the restriction of the vector field $u_h^i$ to the cross section $(-1/2, 1/2)^2 \times \{0\}$ of $M$. We used the initial vector fields $u_h^0$ defined either constantly in the interior of $M$ as in (2.3) and shown in Figure 4.7 or randomly as in (2.4) shown in Figure 4.8. In both cases we used the weakly acute triangulation $\mathcal{T}_3^{ac}$. We observe that for both choices of initial data a smooth vector field away from a single singularity develops within a few iterations. Then, in the subsequent iterations the singularity is transported to the origin leading to a stable and symmetric equilibrium state. Thus, the energy decreasing iteration of Algorithm A provides good approximations of the unique global energy minimizing harmonic map $x/|x|$ in this example independently of the choice of initial data $u_h^0$.

For weakly acute triangulations with different mesh-size but with the same randomly and constantly defined initial data as above, we plotted in Figure 4.9 the decay of the energy $\|\nabla u_h^i\|^2/2$. A rapid decrease of the energy can be observed within the first ten iterations and only small energy variations occur when the singularity is transported towards the origin. As predicted by the theory, the energy of iterates is monotonically decreasing.

From the first set of numerical experiments we draw the conclusion that Algorithm A performs very reliably on weakly acute triangulations. Its efficiency seems best in the first iterations to significantly decrease the possibly large energy of a possibly discontinuous initial vector field. While leading to satisfying results, the algorithm appears to be rather slowly convergent once a low energy level is passed owing to its global character.

### 4.2.2 Performance of the combined scheme

With $M = (-1/2, 1/2)^3$, $u_{\mathrm{D}}(x) = x/|x|$, and the two types of initial data defined by (2.3) and (2.4) on the weakly acute triangulation $\mathcal{T}_4^{ac}$ of $M$ we ran Algorithm E with different choices of the input parameters $N_{global}$ and $N_{local}$. We recall that the iteration of Algorithm E reduces to the $H^1$ gradient flow of Algorithm A if $N_{local} = 0$ and to the classical Newton iteration defined in Algorithm D if $N_{global} = 0$. We employed the residual quantity

$$\|\nabla w_h^i\| + \|\,|u_h^i|^2 - 1\,\|_h \tag{2.5}$$

to measure the quality of approximation of a discrete harmonic map by iterates $(u_h^i, w_h^i)$ of Algorithm E; recall that owing to Proposition 3.1.4 the pair of iterates $(u_h^i, w_h^i)$ of Algorithm E is a discrete harmonic map into the sphere if and only if (2.5) vanishes. We notice that the second

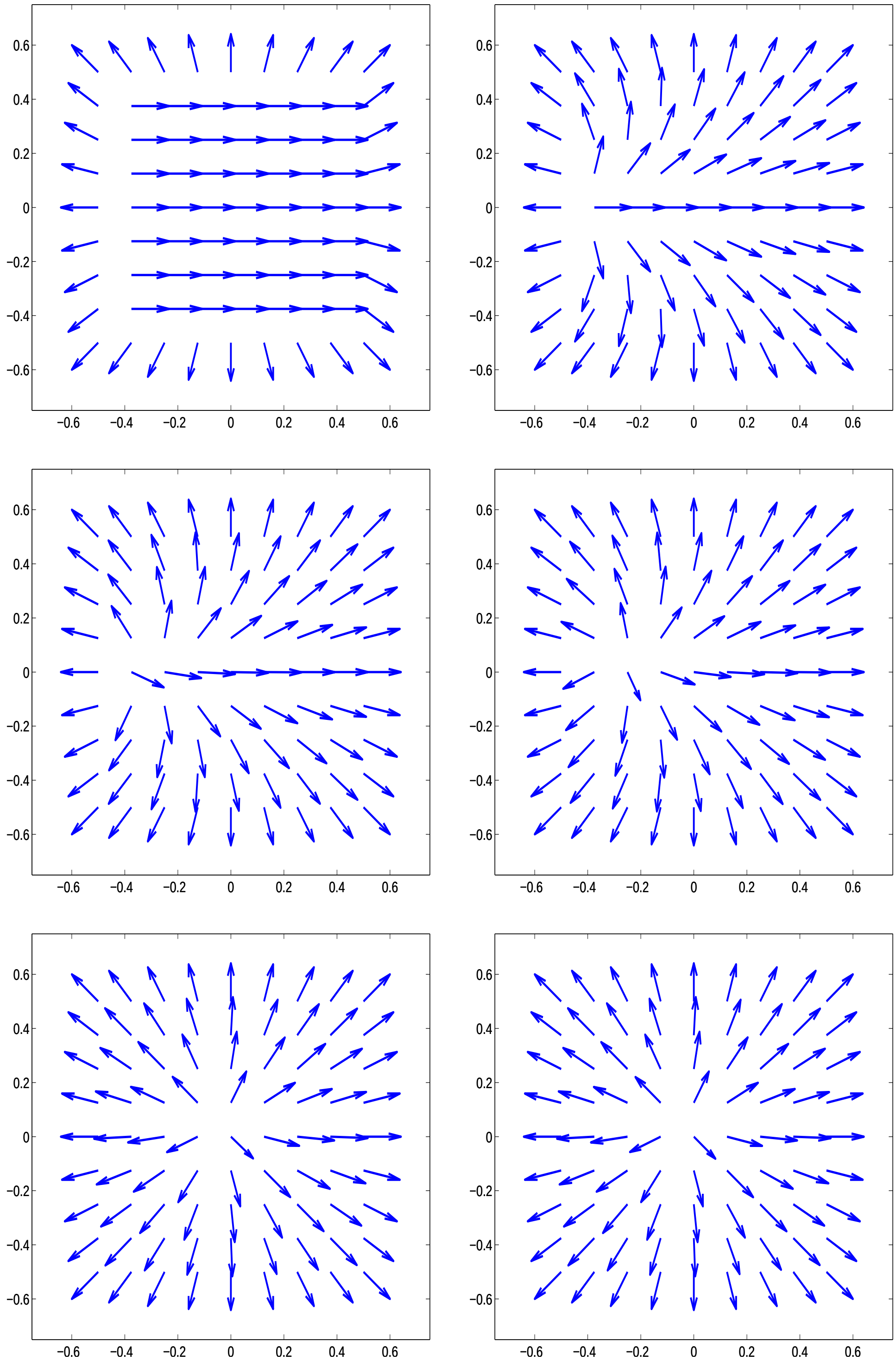


Figure 4.7: First two components of the restriction of the vector fields $u_h^i$ to $(-1/2, 1/2)^2 \times \{0\}$ for $i = 0, 5, 85, 90, 125, 275$ (from left to right and top to bottom) obtained with Algorithm A on the weakly acute triangulation $\mathcal{T}_3^{ac}$ and with constant initial data at interior nodes.

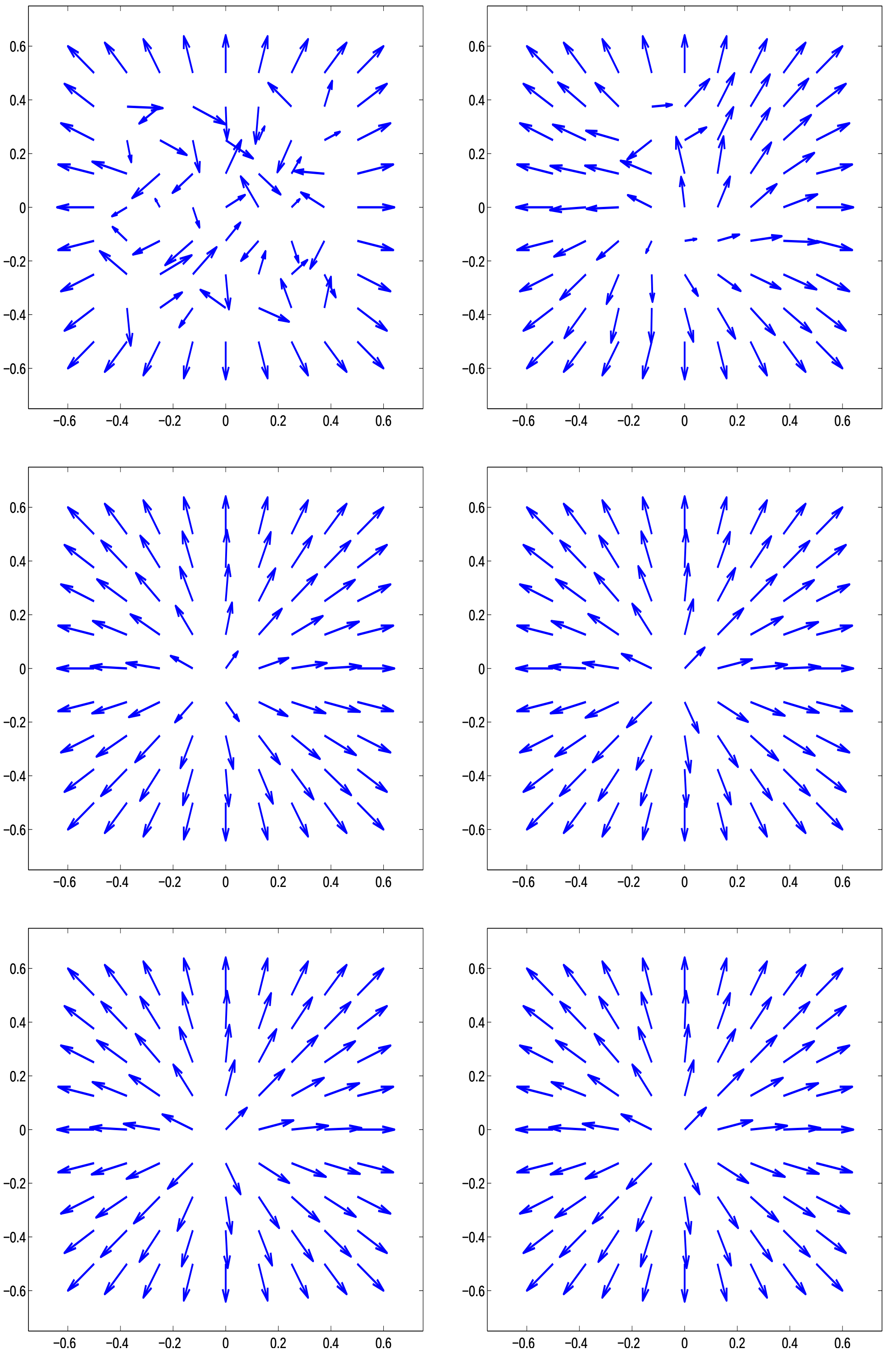


Figure 4.8: First two components of the restriction of the vector fields $u_h^i$ to $(-1/2, 1/2)^2 \times \{0\}$ for $i = 0, 5, 10, 25, 50, 185$ (from left to right and top to bottom) obtained with Algorithm A on the weakly acute triangulation $\mathcal{T}_3^{ac}$ and with random initial data at interior nodes.

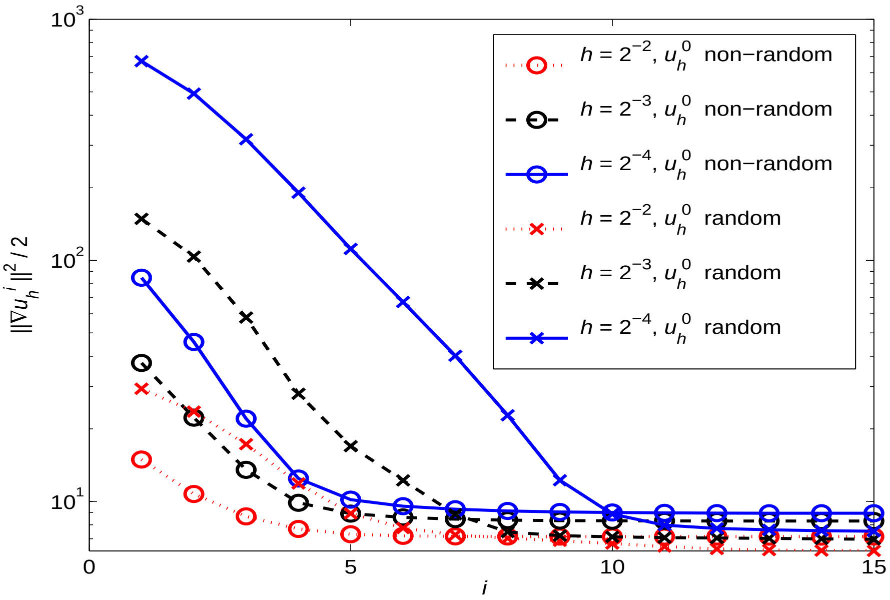


Figure 4.9: Energy decay for different choices of initial data and different mesh-sizes on weakly acute triangulations which guarantee monotone energy decay.

term in (2.5) disappears when $N_{local} = 0$ since then the constraint is satisfied exactly at the nodes of the triangulation. In the upper and lower plot of Figure 4.10 we displayed this quantity for the extreme cases

$$\big(N_{global}, N_{local}\big) = (5, 0) \quad \text{and} \quad \big(N_{global}, N_{local}\big) = (0, 5)$$

as well as the combined case

$$\big(N_{global}, N_{local}\big) = (5, 5)$$

for random and constant initial vector fields as defined in (2.3) and (2.4), respectively. Notice that in the first two cases we could have used any positive integer instead of 5 to obtain the same schemes while the values of the two numbers are important for the performance of Algorithm E when both $N_{global}$ and $N_{local}$ are positive.

From the upper plot of Figure 4.10 we deduce that Algorithm E does not converge within 45 iterations when $N_{global} = 0$, i.e., when a Newton iteration is used to solve the discrete problem. In fact, the system matrices tend to deteriorate which indicates divergence of the iteration for the actual choice of constantly defined initial data. When $N_{local} = 0$ the residual quantity decays monotonically and after 43 iterations the termination criterion is met. Convergence of the iteration in this case is indeed guaranteed by Theorem 3.2.7, however, a lot of iterations are needed to satisfy $\|\nabla w_h^i\| \le \varepsilon = 10^{-4}$. This is different for the combined scheme of Algorithm E with $N_{global} = N_{local} = 5$. We observe significantly faster termination of the combined iteration than for the other methods. In fact, only 10 iterations of the global scheme (indicated by the thicker solid line) are required to define a starting value for the local scheme (indicated by the thinner solid line) that allows for rapid convergence. The performance of Algorithm E could actually be further improved by the choice $\big(N_{global}, N_{local}\big) = (10, 5)$ in this example. In fact, the advantages of the combined scheme would become even more significant if a smaller termination parameter had been used. Similar observations can be made for the randomly defined initial vector field $u_h^0$ from (2.4)

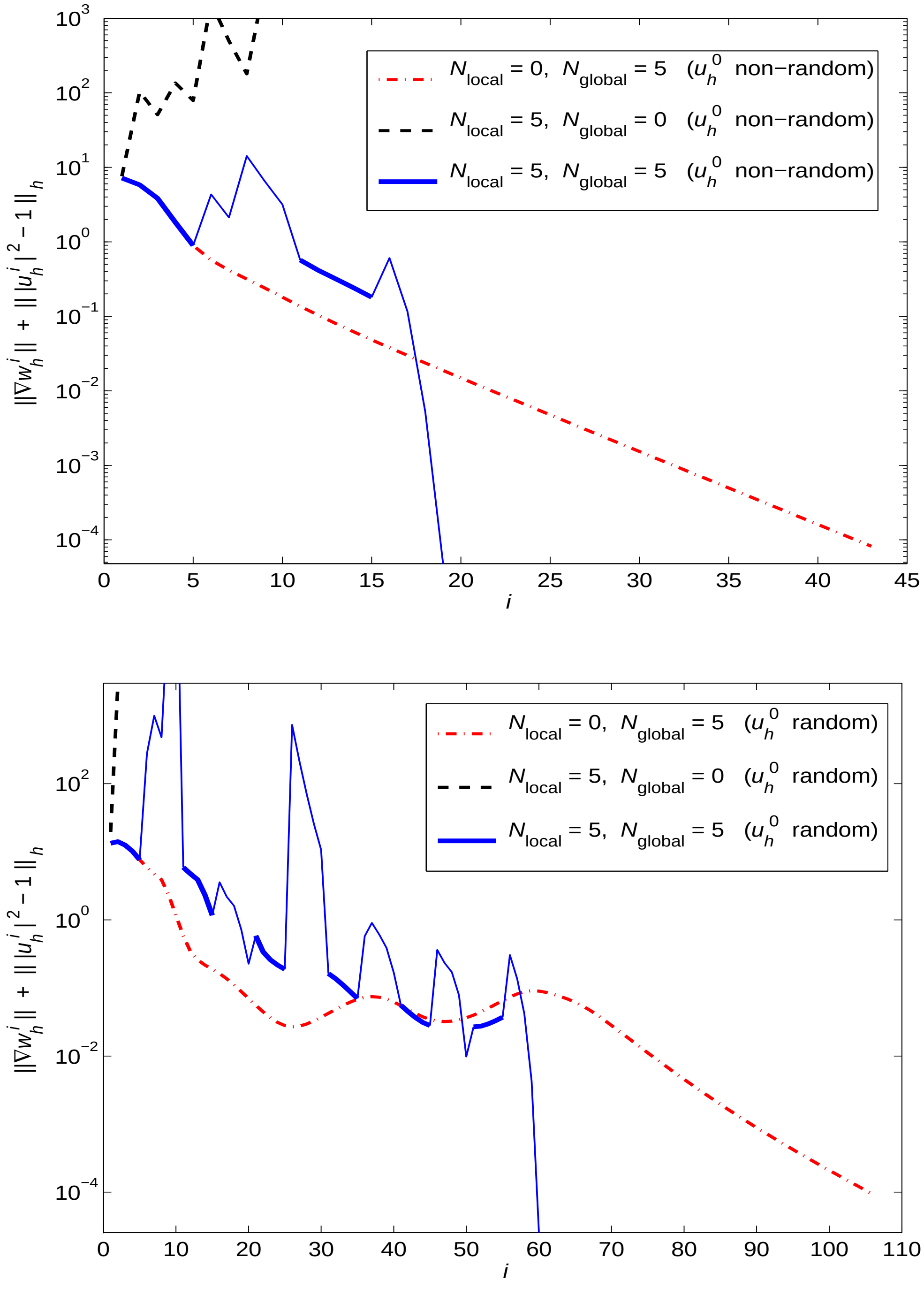


Figure 4.10: Residual in Algorithm E for a weakly acute triangulation with 24.576 tetrahedra for various choices of $N_{global}$ and $N_{local}$ and initial data which are constant (upper plot) and random vectors on $S^2$ (lower plot) in the interior of $M$. For the combined scheme with $N_{global} = N_{local} = 5$ the thick line indicates the iterations with the global scheme and the thin lines those iterations corresponding to the local scheme.

and the reader is referred to the lower plot of Figure 4.10 for details.

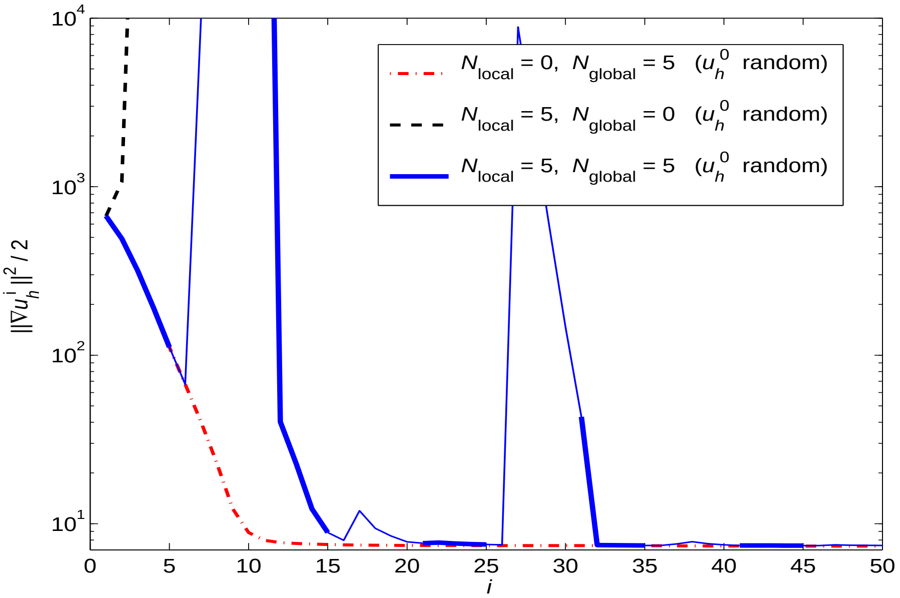


Figure 4.11: Energy decay in Algorithm E for different choices of $N_{global}$ and $N_{local}$ and a randomly defined initial vector field. Thick parts of the solid line indicate iterations with the globally convergent scheme.

We plotted in Figure 4.11 the decay of the energy for Algorithm E for randomly defined initial data and various pairs of $N_{global}$ and $N_{local}$. It is important to observe that the iteration of the Newton method does not increase the energy in the combined scheme when it is convergent. This is of essential importance but not guaranteed by theory.

### 4.2.3 Convergence of the Newton scheme to irregular discrete harmonic maps

With the one-dimensional example from Section 3.5 defined through $M = (0, 1)$, $N = S^2$, and the boundary conditions $u(0) = -u(1) = (0, 0, 1)$ we illustrate in this subsection that Newton iterations may converge to discrete harmonic maps which are unfavorable in practice when an arbitrary initial discrete vector field is used to setup the iteration. Here, arbitrary means that $u_h^0$ satisfies $|u_h^0(z)| = 1$ for all nodes $z \in \mathcal{N}_h$ and $u_h^0(0) = -u_h^0(1) = (0, 0, 1)$ but is arbitrary otherwise. This observation underlines the advantages of Algorithm E which can compute a good initial value of low energy to start the Newton iteration. For some randomly defined discrete initial vector fields with values in the unit sphere at the nodes of the uniform partition of the interval $(0, 1)$ into 25 elements of diameter $h_{min} = 1/25$, the Newton iteration converged and detected the discrete harmonic maps (up to a residual smaller that $10^{-6}$) shown in Figure 4.12. For each of the computed stationary points we also displayed the corresponding energy

$$E(u_h) = \frac{1}{2} \int_{(0,1)} |u_h'|^2 \, ds_h.$$

In all of the displayed cases we find that irregular solutions with practical discontinuities correspond to energies larger than $2/h_{min} = 50$. The experiments thus underline the necessity and importance

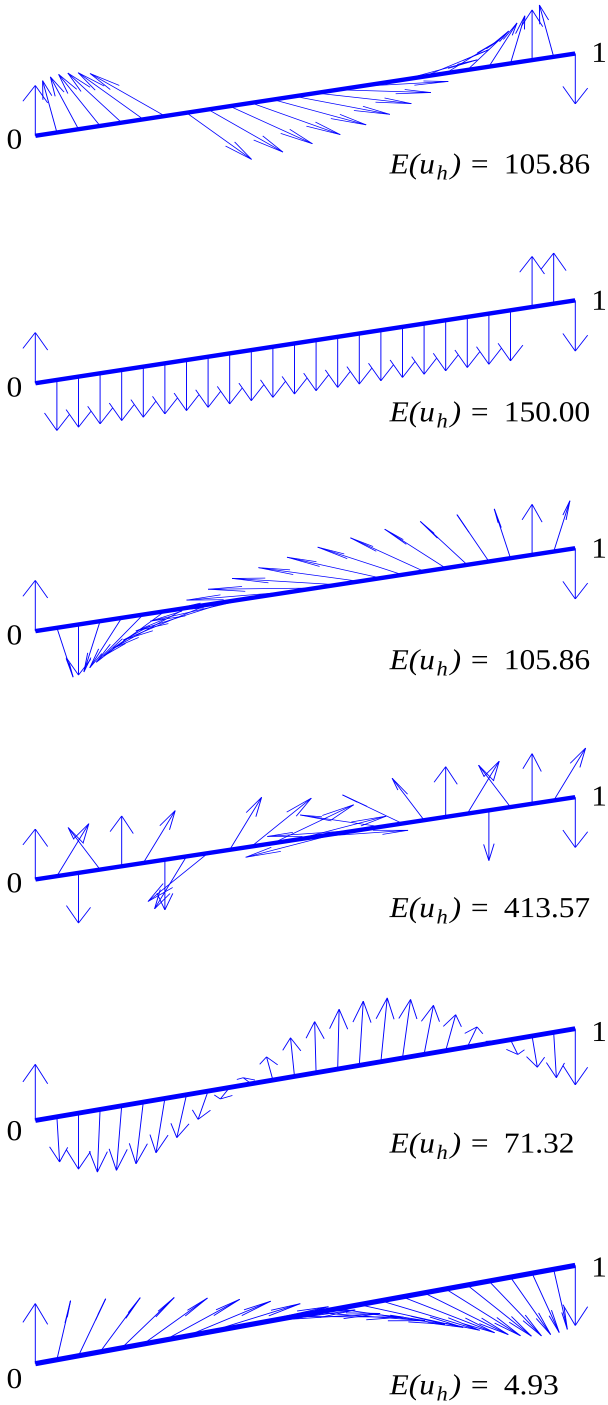


Figure 4.12: Various (almost) discrete harmonic maps into $S^2$ as approximations of (non-unique) solutions to $-u'' = |u'|^2 u$ and $|u| = 1$ in $(0,1)$ subject to $u(0) = -u(1) = (0,0,1)$. The approximations were computed with a classical Newton scheme (i.e., Algorithm E with $N_{global} = 0$) and randomly defined initial data $u_h^0$ satisfying $|u_h^0(j/25)| = 1$ for $j = 0,1,2,...,25$ and $u_h^0(0) = -u_h^0(1) = (0,0,1)$.

of approximation schemes that decrease the energy in the iteration. In these one-dimensional experiments the condition numbers of the system matrices tended to deteriorate as predicted in Example 3.5.2 in Section 3.5. Nevertheless, MATLAB provided solutions of the linear systems of equations and thereby approximations with residuals smaller than $10^{-6}$ for approximately two out of three randomly defined initial fields $u_h^0$.

## 4.3 Geometric changes and occurrence of singularities in the harmonic map heat flow into spheres

As is well understood theoretically, the $L^2$ flow of harmonic maps into spheres may develop singularities resulting in a loss of control over $\|\nabla_M u(t,\cdot)\|_{L^\infty(M)}$. In particular, [CDY92] constructs smooth initial data $u_0 : M \to S^2$ for $M = B_1(0) \subset \mathbb{R}^2$ such that there exists a weak solution of the time-dependent problem

$$\partial_t u - \Delta_M u = |\nabla_M u|^2 u, \qquad |u(t,\cdot)| = 1, \qquad u(t,\cdot)|_{\partial M} = u_0|_{\partial M}, \qquad u(0,\cdot) = u_0 \tag{3.6}$$

which satisfies $\lim_{t\to t_1} \|\nabla_M u(t,\cdot)\|_{L^\infty(M)} = \infty$ for some $t_1 > 0$. We notice that a weak solution is unique as long as $\|\nabla_M u(t,\cdot)\|_{L^\infty(M)} < \infty$. Related to this is the fact that the $L^2$ flow of harmonic maps does in general not define a topological homotopy in the sense that $u_0$ and $u_\infty = \lim_{t\to\infty} u(t,\cdot)$ are topologically equivalent, cf. [EW76]. Here, $u_\infty$ is a weak accumulation point of the sequence $\big(u(t,\cdot)\big)_{t\geq 0}$ which is bounded in $W^{1,2}(M;\mathbb{R}^3)$ as $t \to \infty$. This problem cannot be overcome by imposing additional constraints since the set of vector fields $\overline{u} \in W^{1,2}(M;\mathbb{R}^3)$ that are topologically equivalent to $u_0$ is in general not weakly closed in $W^{1,2}(M;\mathbb{R}^3)$, cf. [Lem78]. A positive partial result by [ES64] states that $u$ and $u_\infty$ are indeed topologically equivalent if the target manifold $N$ has non-positive sectional curvature and this condition is sharp, see [EW76]. In this section we study the occurrence of finite-time blow-up numerically in order to verify the reliability of our approximation schemes defined by Algorithms A and E. The experiments show that our methods can deal with singularities and topological changes appropriately and are capable of predicting singularities correctly.

### 4.3.1 Discrete finite-time blow-up in the $L^2$ flow of harmonic maps

In our first set of experiments we employ the initial data of [CDY92] and set $M := (-1,1)^2$, $N = S^2$, $\Gamma_\mathrm{D} := \partial M$, and

$$u_0(x) := \frac{1}{|x|}\Big(x_1 \sin h(|x|), x_2 \sin h(|x|), |x| \cos h(|x|)\Big), \qquad u_0(0) := \big(0,0,1\big)$$

for $x = (x_1, x_2) \in M$ and with $h : [0,\infty) \to \mathbb{R}$ defined by $h(r) := br^2$ if $r \leq 1$ and $h(r) := b$ otherwise. It is proved in [CDY92] that for $M = B_1(0)$, a singular solution of the $L^2$ flow occurs if and only if $b \geq \pi$; below we choose $b = 3\pi/2$.

We approximate solutions of the time dependent problem with the following modification of Algorithm A which approximates the $L^2$ flow rather than the $H^1$ flow of harmonic maps. The resulting algorithm coincides with Algorithm E for $\theta = 0$. Notice that the main change occurs in the bilinear form on the left-hand side of the equation in Step 2.

**Algorithm A'.** *Input:* triangulation $\mathcal{T}_h$, time horizon $T > 0$, time-step size $\kappa > 0$.

1. Choose $u_h^0 \in \mathcal{S}^1(\mathcal{T}_h)^n$ such that $u_h^0(z) = u_0(z)$ for all $z \in \mathcal{N}_h$. Set $i := 0$.
2. Compute $w_h^i \in \mathcal{S}_{\mathrm{D}}^1(\mathcal{T}_h)^n$ such that $w_h^i(z) \in T_{u_h^i(z)}N$ for all $z \in \mathcal{N}_h$ and
$$\big(w_h^i; v_h\big)_h = -\big(\nabla_{M_h} u_h^i; \nabla_{M_h} v_h\big)$$
for all $v_h \in \mathcal{S}_{\mathrm{D}}^1(\mathcal{T}_h)^n$ such that $v_h(z) \in T_{u_h^i(z)}N$ for all $z \in \mathcal{N}_h$.
3. Define $u_h^{i+1} \in \mathcal{S}^1(\mathcal{T}_h)^n$ by setting
$$u_h^{i+1}(z) := \pi_N\big(u_h^i(z) + \kappa w_h^i(z)\big)$$
for all $z \in \mathcal{N}_h$.
4. Stop if $t_{i+1} := \kappa(i+1) \geq T$.
5. Set $i := i+1$ and go to 2.

*Output:* $\big(u_h^i\big)_{i=0,1,2,\dots,J_T+1}$.

Here, $\kappa$ is a time-step size and $u_h^i$ approximates an exact solution at time $t_i := i\kappa$. The algorithm terminates when the time-horizon $T > 0$ is reached. The discrete inner product $(\cdot;\cdot)_h$ on the left-hand side of Algorithm A' can be replaced by the standard inner product in $L^2(M_h)$, resulting in a slightly more expensive and less stable scheme, see [BBFP07] for a discussion. Following the lines of the proof of Lemma 3.2.4 one can show that for $J = 0, 1, 2, \dots, J_T$

$$(1 - C''\kappa h_{min}^{-1-d/2})\kappa \sum_{i=0}^{J} \big\|d_t u_h^{i+1}\big\|_h^2 + \frac{1}{2}\big\|\nabla_{M_h} u_h^{J+1}\big\|^2 \leq \frac{1}{2}\big\|\nabla_{M_h} u_h^0\big\|^2,$$

where $d_t u_h^{i+1} := \kappa^{-1}\big(u_h^{i+1} - u_h^i\big) \approx w_h^i$ for $i = 1, 2, \dots, J+1$. Based on this estimate, weak sub-convergence of appropriate interpolations of iterates to a weak solution of the $L^2$ flow of harmonic maps into spheres, i.e., for $N = S^2$, can be verified as in [BBFP07] provided that $\kappa = o\big(h_{min}^{1+d/2}\big)$.

We ran Algorithm A' for uniform, right-angled triangulations $\mathcal{T}_J$ of $M$ into $2\cdot 2^{2J}$ triangles which are squares halved along the direction $(1,1)$ and with edge-length $h_{min} = 2 \cdot 2^{-J}$ for $J = 5, 6, 7$. The discrete initial data $u_h^0$ was defined by nodal interpolation of $u_0$ and we employed $\kappa = h_{min}^2/10$.

Figure 4.13 displays the $W^{1,\infty}$ semi-norm and the energy

$$E\big(u_h(t,\cdot)\big) = \frac{1}{2}\big\|\nabla_{M_h} u_h(t,\cdot)\big\|^2$$

of numerical approximations as functions of $t \in (0, 0.35)$ for different discretization parameters. Here, the function $u_h : (0,T) \times M \to S^2$ is the function which is piecewise constant on $(0,T)$ and satisfies

$$u_h(t,\cdot) = u_h^i$$

for $t \in \kappa[i, i+1)$. For all three employed triangulations $\mathcal{T}_J$, $J = 5, 6, 7$, we observe in Figure 4.13 that the energy is monotonically decreasing, indicating good stability properties of our method,

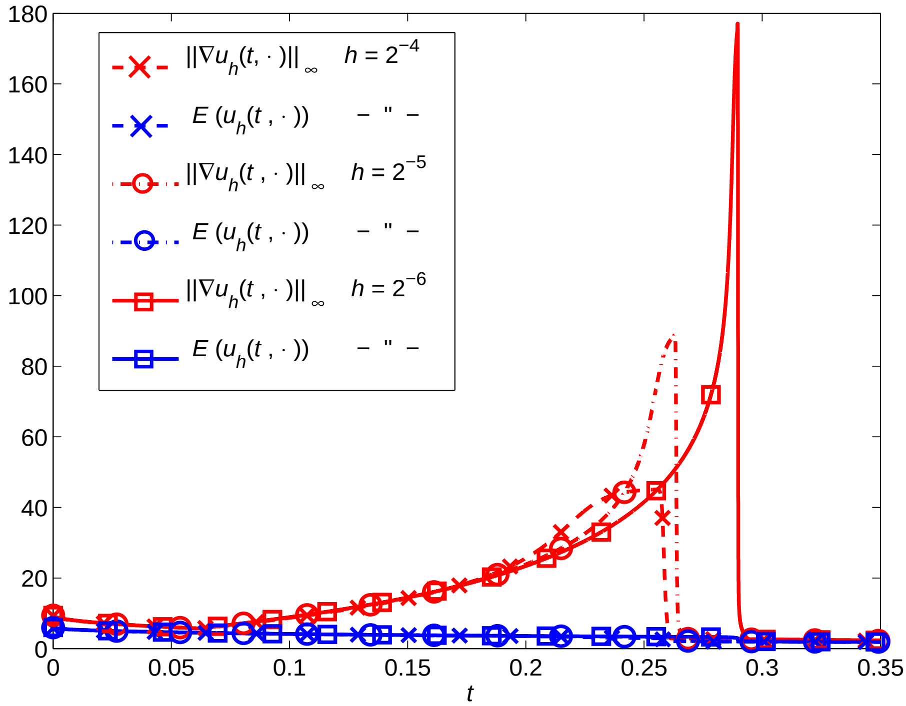


Figure 4.13: $W^{1,\infty}$ semi-norm and energy as functions of $t \in (0, 0.35)$ of numerical approximations of the harmonic map heat flow problem into the sphere for different discretization parameters for initial data leading to blow-up.

and assumes a constant, positive value when an equilibrium state is attained. Owing to the non-constant Dirichlet boundary conditions employed in this example, the energy does not decrease to zero. While the energy is decaying, the maximum norms of the gradients of the approximations become extremely large and attain different maximal values for different triangulations indicating the occurrence of so-called finite-time blow-up in this example. In fact, for each fixed triangulation, the function $t \mapsto \|\nabla_{M_h} u_h(t,\cdot)\|_{L^\infty(M)}$ assumes the maximum possible value among functions in $v_h \in \mathcal{S}^1(\mathcal{T}_J)^3$ satisfying $|v_h(z)| = 1$ for all $z \in \mathcal{N}_h$, namely

$$\max_{\substack{v_h \in \mathcal{S}^1(\mathcal{T}_J)^3, \\ |v_h(z)|=1 \text{ f.a. } z\in\mathcal{N}_h}} \|\nabla_{M_h} v_h\|_{L^\infty(M)} = 2\sqrt{2}\, h_{min}^{-1} = 2\sqrt{2}\, 2^{J-1},$$

which can be verified by defining for a right-angled triangle $K \in \mathcal{T}_h$ with vertices $z_0, z_1, z_2$ the function $v_h|_K$ by $v_h(z_0) := (-1, 0, 0)$ and $v_h(z_1) = v_h(z_2) := (1, 0, 0)$. After attainment of the maximum $W^{1,\infty}$ semi-norm, it drops to an $h$-independent value corresponding to a smooth equilibrium defined by the boundary data.

The displayed vector fields in Figures 4.15 and 4.16 illustrate the evolution of the discrete solution on a criss-cross triangulation $\mathcal{T}_3^{cc}$ with 145 free nodes in this example. The choice of such a triangulation is only for presentational convenience owing to its higher symmetry. We observe that for $t \approx 0.39$ the vector at the origin points in a different direction than the surrounding vectors resulting in large, maximal gradients. Then, at $t \approx 0.4$ the vector at the origin changes its direction and for $t \geq 0.43$ a smooth vector field with moderate energy and $W^{1,\infty}$ semi-norm has developed.

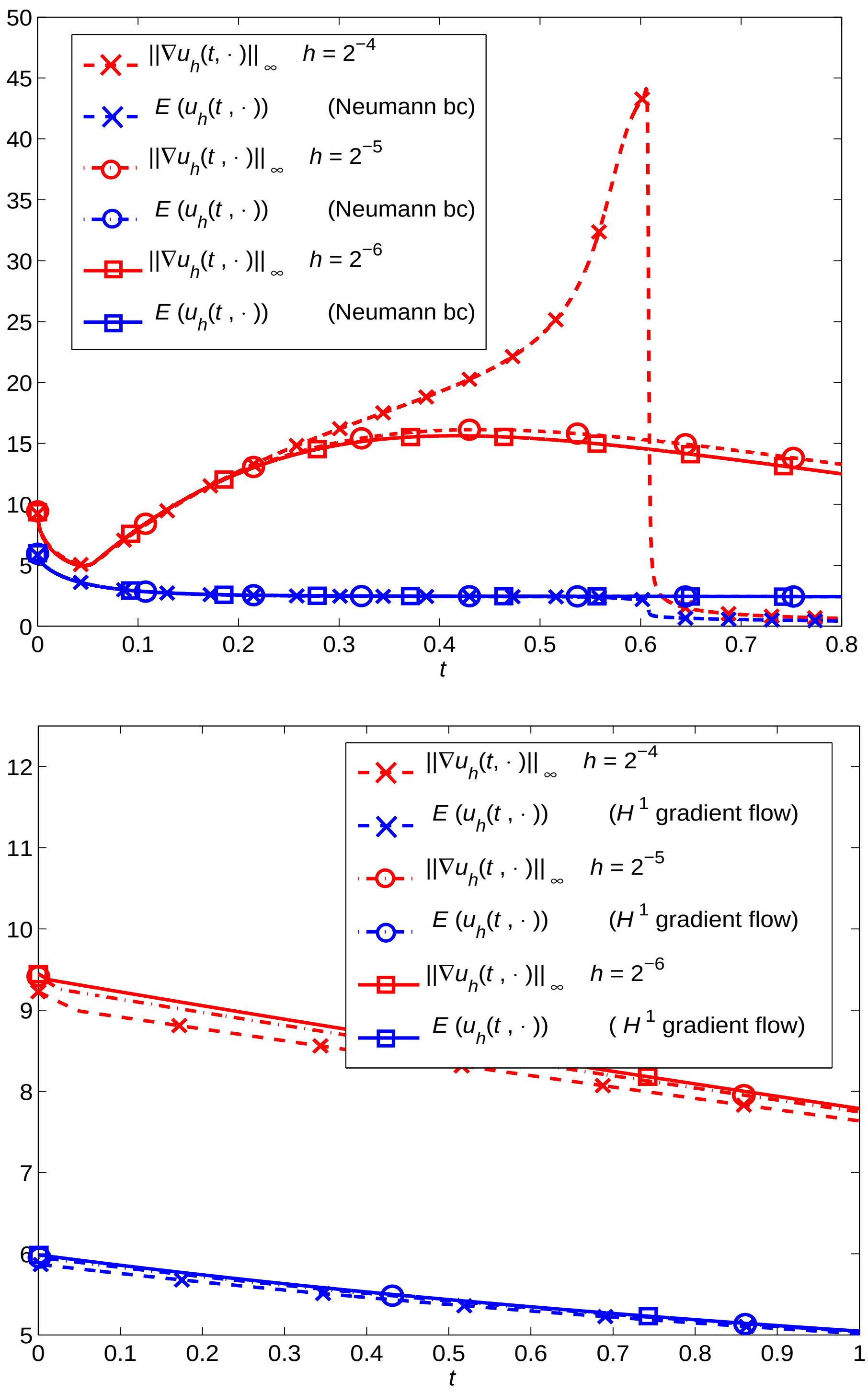


Figure 4.14: $W^{1,\infty}$ semi-norm and energy of numerical approximations of the harmonic map heat flow problem into the sphere for different discretization parameters and subject to Neumann boundary conditions (upper plot). Same quantities for the $H^1$ flow of harmonic maps subject to Dirichlet conditions (lower plot). In both cases no large (maximal) gradients occur.

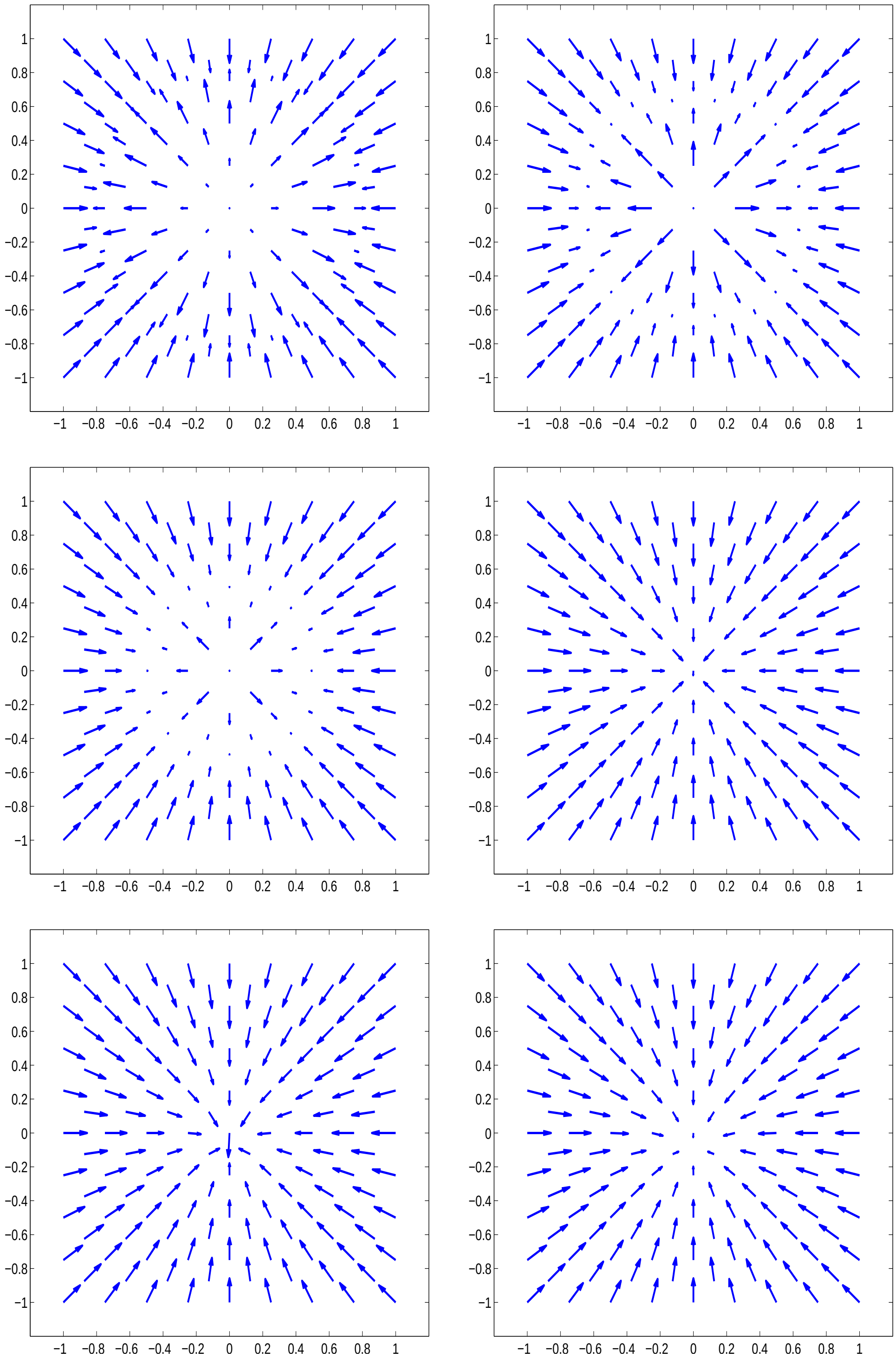

Figure 4.15: First two components of the vector field $u_h(t, \cdot)$ for $t = 0$, 0.1078, 0.2016, 0.3891, 0.4047, 0.4359 (from left to right and top to bottom). Vectors are scaled by a factor 1/7 for graphical purposes. At $t \approx 0.40$ the vector at the origin changes its direction leading to large gradients.

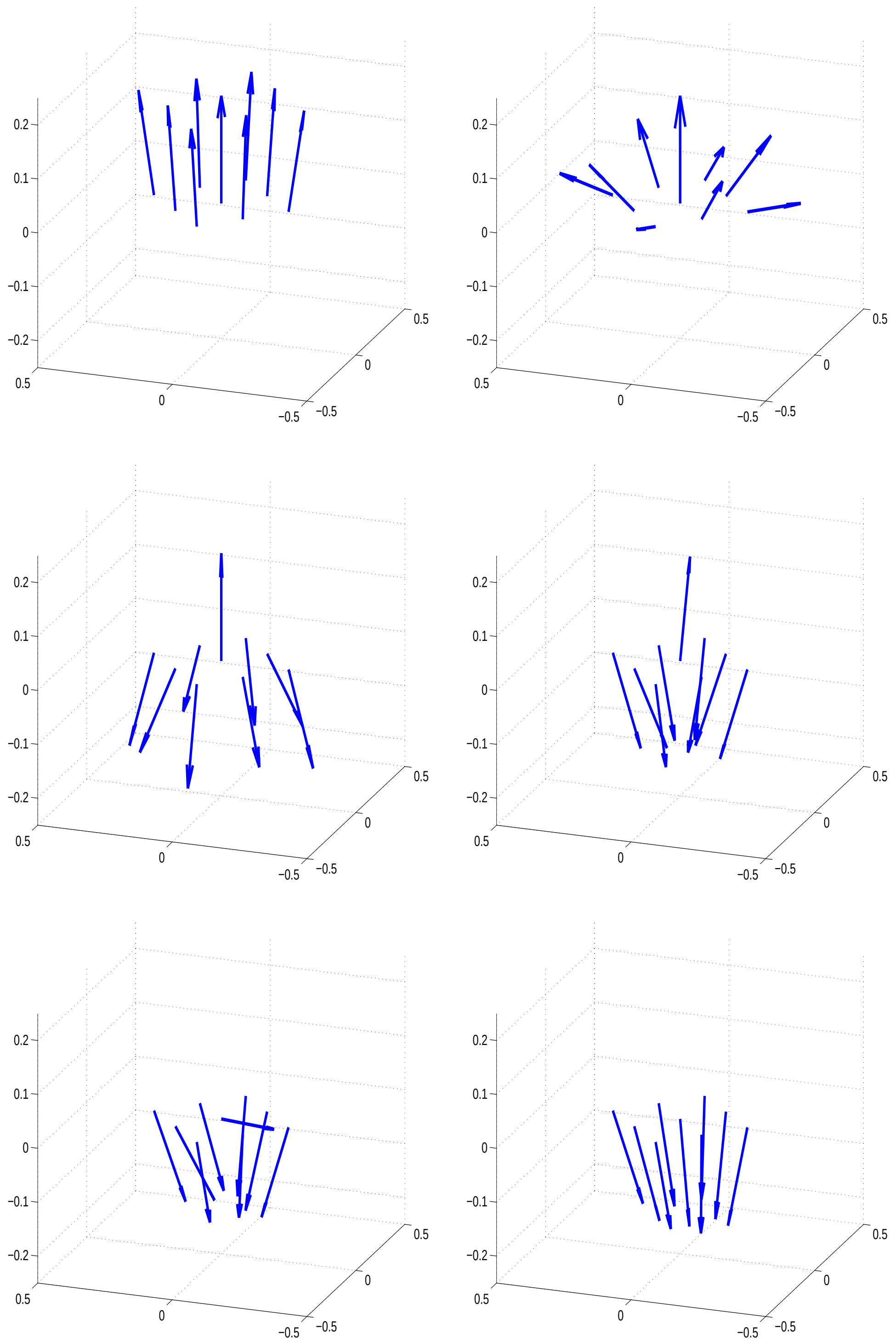


Figure 4.16: Zoom of the vector field $u_h(t,\cdot)$ to a neighborhood of 0 for $t = 0$, 0.1078, 0.2016, 0.3891, 0.4047, 0.4359 (from left to right and top to bottom). A maximal $W^{1,\infty}$ norm is attained, when the vector at the origin points into another direction than the surrounding ones. Vectors are scaled by a factor 1/5 for graphical purposes.

The evolution is significantly different when homogeneous Neumann type boundary conditions are employed to define the time-dependent problem. The upper plot in Figure 4.14 shows the evolution of the energy and of the $W^{1,\infty}$ semi-norm. While for $h = 2^{-4}$ the results are qualitatively similar to the ones obtained with Dirichlet type boundary conditions, the $W^{1,\infty}$ semi-norms seem to be uniformly bounded by the moderate value 18 for $h = 2^{-5}$ and $h = 2^{-6}$. Thus, we may conclude that finite-time blow-up only occurs for Dirichlet type boundary conditions for the initial data considered here. A rigorous proof for this statement is however missing and beyond the scope of this work. We remark that for Neumann type boundary conditions and a sufficiently large $T > 0$, the evolution assumes a constant state in this example.

We also applied the $H^1$ gradient flow approximation defined by the original Algorithm A in this example. For the choice $\kappa = h_{min}/10$, the results obtained for the triangulations $\mathcal{T}_J$, $J = 4, 5, 6$, and with Dirichlet boundary conditions are displayed in the lower plot of Figure 4.14. We observe that no large gradients occur. It is interesting to remark that the (semi-) discretized $H^1$ gradient flow preserves the topological degree of the initial $u_0$ owing to a result in [Mor04]. This result asserts that if $M = B_1(0) \subset \mathbb{R}^2$ then the topological degree

$$\deg \frac{u + t\phi}{|u + t\phi|}$$

is independent of $t \in \mathbb{R}$ for $u, \phi \in W^{1,2}(M; \mathbb{R}^3)$ satisfying $\phi|_{\partial M} = 0$ and $|u| = 1$, $u \cdot \phi = 0$ almost everywhere in $M$.

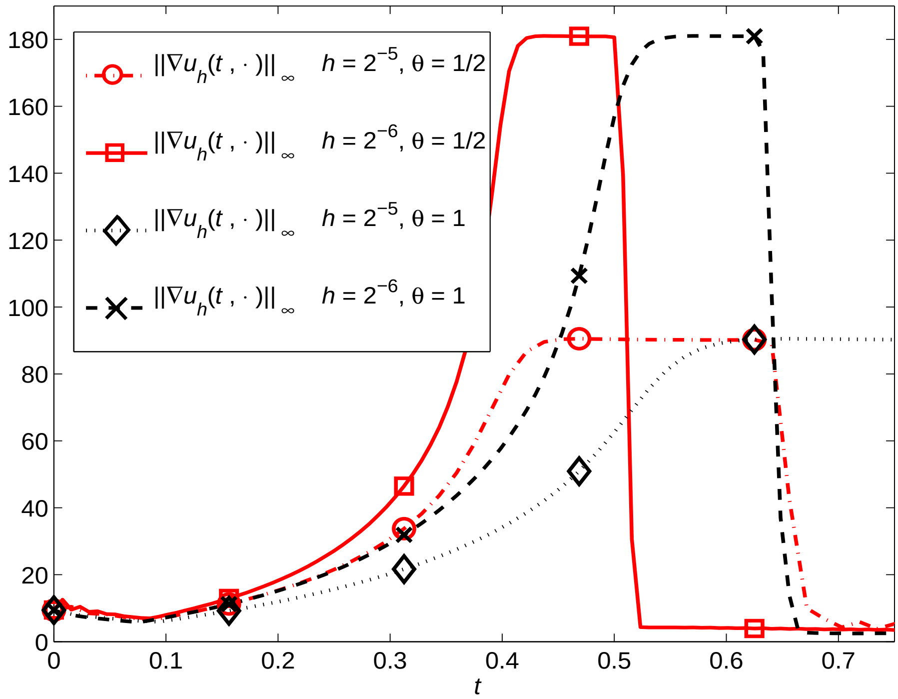


Figure 4.17: $W^{1,\infty}$ semi-norm and energy as functions of $t \in (0, 0.75)$ of numerical approximations obtained with Algorithm C for $\theta = 1/2$ and $\theta = 1$ with the blow-up initial data.

Finally, we ran the unconditionally stable $\theta$-scheme of Algorithm C with $\theta = 1/2$ and $\theta = 1$

for the blow-up initial data $u_0$. The results obtained for the regular triangulations $\mathcal{T}_J$, $J = 4, 5, 6$ and time-step size $\kappa = h/4$ are displayed in Figure 4.17. We observe that the discrete finite-time blow-up occurs at a later time than in the approximation with the explicit scheme of Algorithm A' corresponding to $\theta = 0$. Nevertheless for small mesh-sizes the blow-up time seems to approach the same value as the results in Figure 4.13 do. For $\theta = 1$ this phenomenon is most likely due to the dissipative character and strong damping property of the implicit scheme.

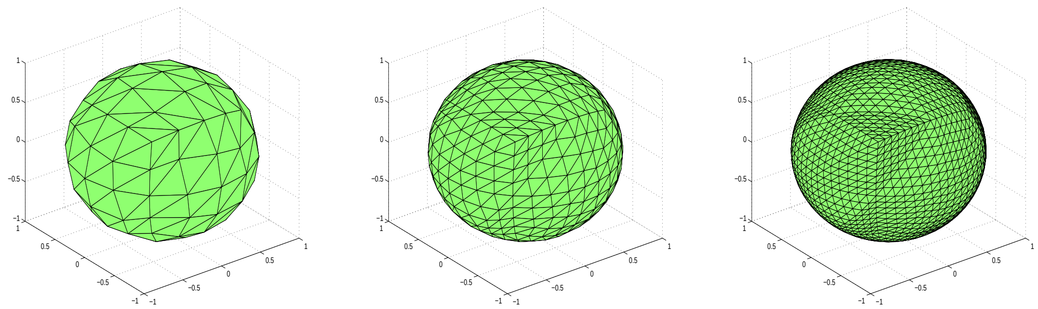

Figure 4.18: Triangulations of the unit sphere into 192, 768, and 3072 triangles. These triangulations are not weakly acute.

### 4.3.2 Discrete collapse of geometries in the $L^2$ flow of harmonic maps

To illustrate problems related to topological changes and occurrence of singularities in the $L^2$ flow of harmonic maps we choose $M = N = S^2$ and let $u_0 : S^2 \to S^2$ be a perturbation of the identity on $S^2$. The identity map on $S^2$ is a harmonic map into $S^2$ as can be deduced from the identity

$$-\Delta_M x = H(x)\mu(x)$$

for $x \in M$ which holds on a large class of smooth surfaces with $H$ denoting the (scalar) mean curvature and $\mu$ a unit normal vector field on $M$, see [DDE05]. With the triangulation $\mathcal{T}_h$ of $S^2$ into 192 triangles shown in Figure 4.18, $\kappa = h_{min}^2/10$, $\varepsilon = 10^{-10}$, and $\delta = h^4$ we ran Algorithm B for the initial data displayed in the left, upper plot of Figure 4.19. Snapshots of the evolution are displayed in the remaining plots of Figure 4.19, where for various times $t$ we plotted the deformed triangulations

$$\mathcal{T}_h^t := \{u_h(t, K) \colon K \in \mathcal{T}_h\}.$$

We see that within a very small time interval, the mapping $u_h(t, \cdot)$ becomes smooth and seems to approach the nodal interpolant of the identity map on $S^2$. However, after a very long period in time with almost no spatial changes, the discrete sphere starts to collapse and vanishes at $t \approx 47$. It is not clear to the author whether this collapse is a numerical artifact or due to the strength of the perturbation. For smaller time-steps a similar behaviour can be observed but for more moderate perturbations the collapse does not seem to occur. In the experiment at hand, we found that the discrete time derivative satisfied $\|d_t u_h(t, \cdot)\| \geq 10^{-4}$ for $0 \leq t \leq 46.484$. As soon as the sphere collapses to a single point we have that $u_h(t, \cdot)$ is constant and stationary so that $\|d_t u_h(t, \cdot)\| = 0$ for sufficiently large times. In Figure 4.21 we displayed the $W^{1,\infty}$ semi-norm, the energy, and the $L^2$ norm of the discrete time derivative in this example and observe the late finite-time blow-up with accompanying drop of the energy. We remark that similar observations can be made for the $H^1$ flow of harmonic maps if the perturbation of the identity is strong enough for the employed triangulation. We ran the same experiment with the finer triangulation of $S^2$ consisting of 3072 triangles and with a perturbation of the nodal interpolant of the identity of similar strength. Snapshots of the evolution determined by Algorithm B are shown in Figure 4.20. Here, the sphere does not collapse before $t = 50$ and appears to be a stable configuration. We may thus indeed conclude that the collapse occurring on the coarser discretization is an effect resulting from the discretization and the corresponding underestimation of the exact energy. We also tried Algorithm C in this experiment and obtained similar results. Since the employed triangulations are not weakly acute those results are however not supported by the theory.

## 4.4 Approximation of harmonic maps into a torus

To test the performance of Algorithm A for other target manifolds than $S^2$ we use Algorithm A to approximate a function $u\colon T_{1,1/4} \to T_{1,1/4}$ with low Dirichlet energy. For a uniform triangulation of the torus $T_{1,1/4}$ into 1024 triangles of diameter $h = 0.006$ we defined an initial deformation $u_h^0 \in \mathcal{S}^1(\mathcal{T}_h)^3$ satisfying $u_h^0(z) \in T_{1,1/4}$ for all $z \in \mathcal{N}_h$ by perturbing the identity on $T_{1,1/4}$, i.e., we set

$$u_h^0(z) := \pi_{T_{1,1/4}}\big(z + \xi_h(z)/3\big)$$

for all $z \in \mathcal{N}_h$ and random vectors $\xi_h(z)$ with $|\xi_h(z)| \leq 1$. The damping parameter $\kappa$ was chosen as $\kappa = h/2$. As in Figure 4.19, we displayed in Figure 4.22 the deformed triangulations

$$\mathcal{T}_h^i := \big\{u_h^i(K) \colon K \in \mathcal{T}_h\big\}$$

after various numbers of iterations. We see that the flow selects the an approximation of the identity map on $T_{1,1/4}$ yielding a smooth regularization of the rough initial data. We stress that in this setting we could not observe a collapse or change of the geometry also for different discretization parameters.

## 4.5 Practical stability and performance of Algorithms A' and B

To obtain a better understanding of the practical reliability and performance of Algorithms A' and B for the approximation of the $L^2$ flow of harmonic maps into spheres, we employed

$$M = N = S^2 \qquad \text{and} \qquad u_h^0(x) := \mathcal{I}_h\big[\pi_{S^2}\big(x + \xi_h(x)\big)\big]$$

for $x \in M_h$ and a random vector-field $\xi$ whose components satisfy $\|\xi^i\|_{L^\infty(M_h)} \leq 5/2$. The interpolation operator $\mathcal{I}_h$ is defined through three uniform triangulations of $S^2$ into 3072, 12288, and 49152 triangles corresponding to maximal mesh-sizes $h = 2^{-J}$ for $J = 4, 5, 6$, respectively, see Figure 4.18. We approximated the $L^2$ flow in the time-interval $t \in (0, 1/4)$ with Algorithms A' and B as well as the Algorithm A" which is obtained by replacing the discrete inner product $(\cdot,\cdot)_h$ in Step 2 of Algorithm A' by the standard $L^2$ inner product on $M_h$. For the fixed-point iteration defining the inner loop of Algorithm B and specified in Algorithm B$^{inner}$ we chose the termination criterion $\delta = h^2$.

Table 4.2 displays the total CPU times for the three different algorithms and the three different triangulations. The table also indicates stability of the iteration through "$\surd$" and occurrence of an instability by the symbol "$\times$". The results shown in the table imply that the choice $\kappa = h^2/2$ is not

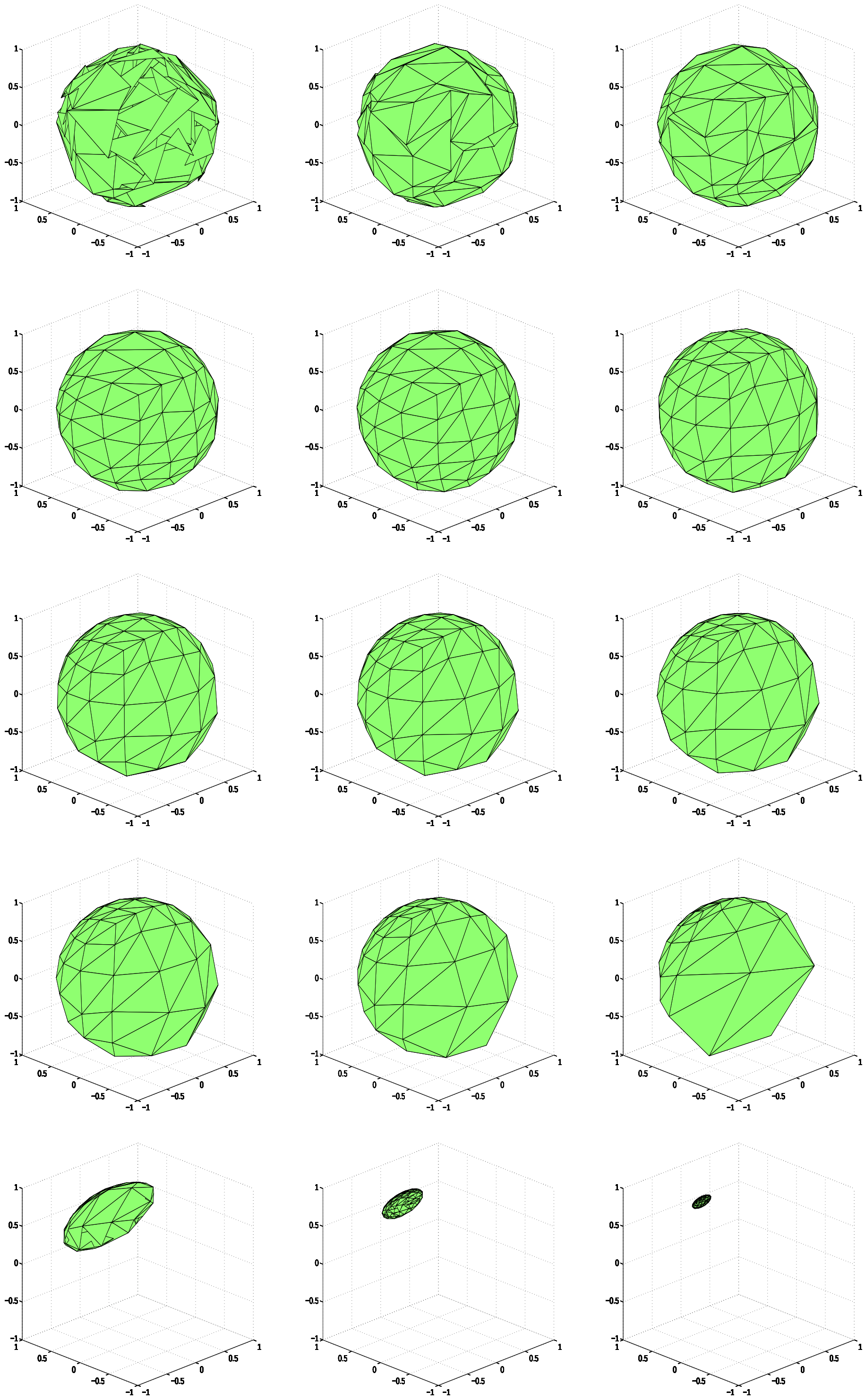

Figure 4.19: Deformed triangulations $\mathcal{T}_h^t$ of $\mathcal{T}_h$ under $u_h(t,\cdot)$ for $t = 0$, 0.016, 0.031, 0.078, 0.156, 30.078, 35.547, 37.109, 40.234, 41.016, 42.969, 45.313, 45.703, 46.094, 46.484 (from left to right and top to bottom). Here, $u_h$ is defined by the $L^2$ flow approximation with the implicit scheme of Algorithm B.

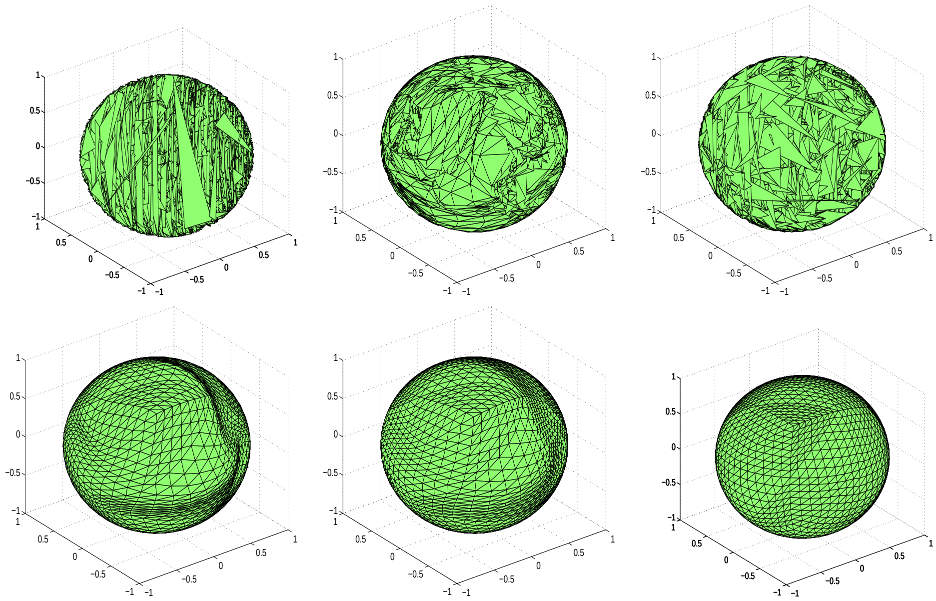

Figure 4.20: Deformed triangulations $\mathcal{T}_h^t$ of $\mathcal{T}_h$ under $u_h(t,\cdot)$ for a triangulation $\mathcal{T}_h$ of $S^2$ into 3072 triangles for $t = 0$, 0.016, 0.004, 0.051, 0.070, 50.0 (from left to right and top to bottom) obtained with Algorithm B.

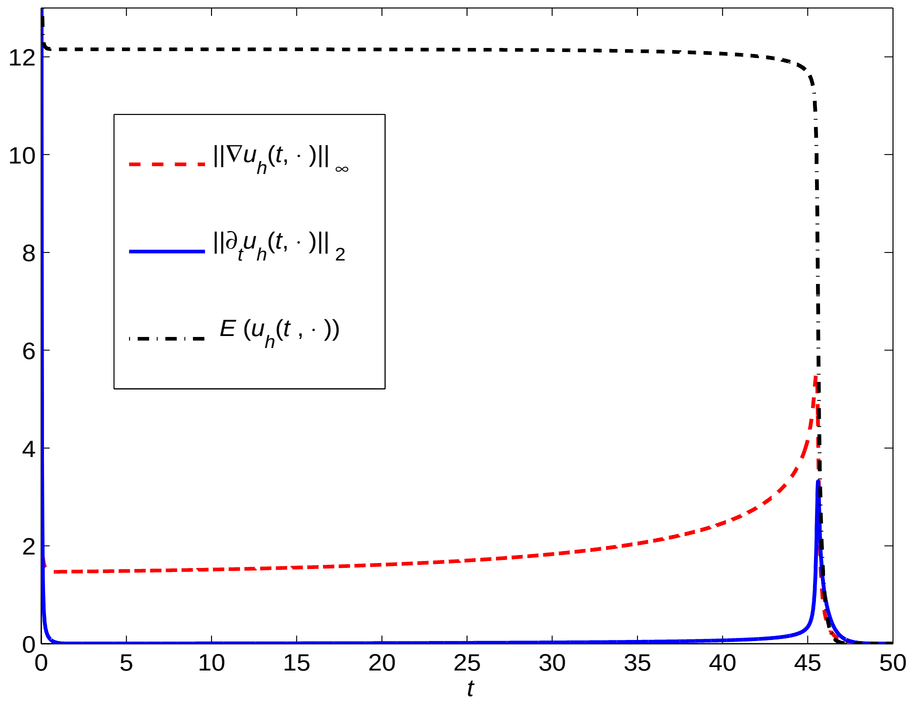


Figure 4.21: $W^{1,\infty}$ semi-norm, energy, and norm of the discrete time derivative for the collapse of the sphere displayed in the sequence of plots in Figure 4.19.

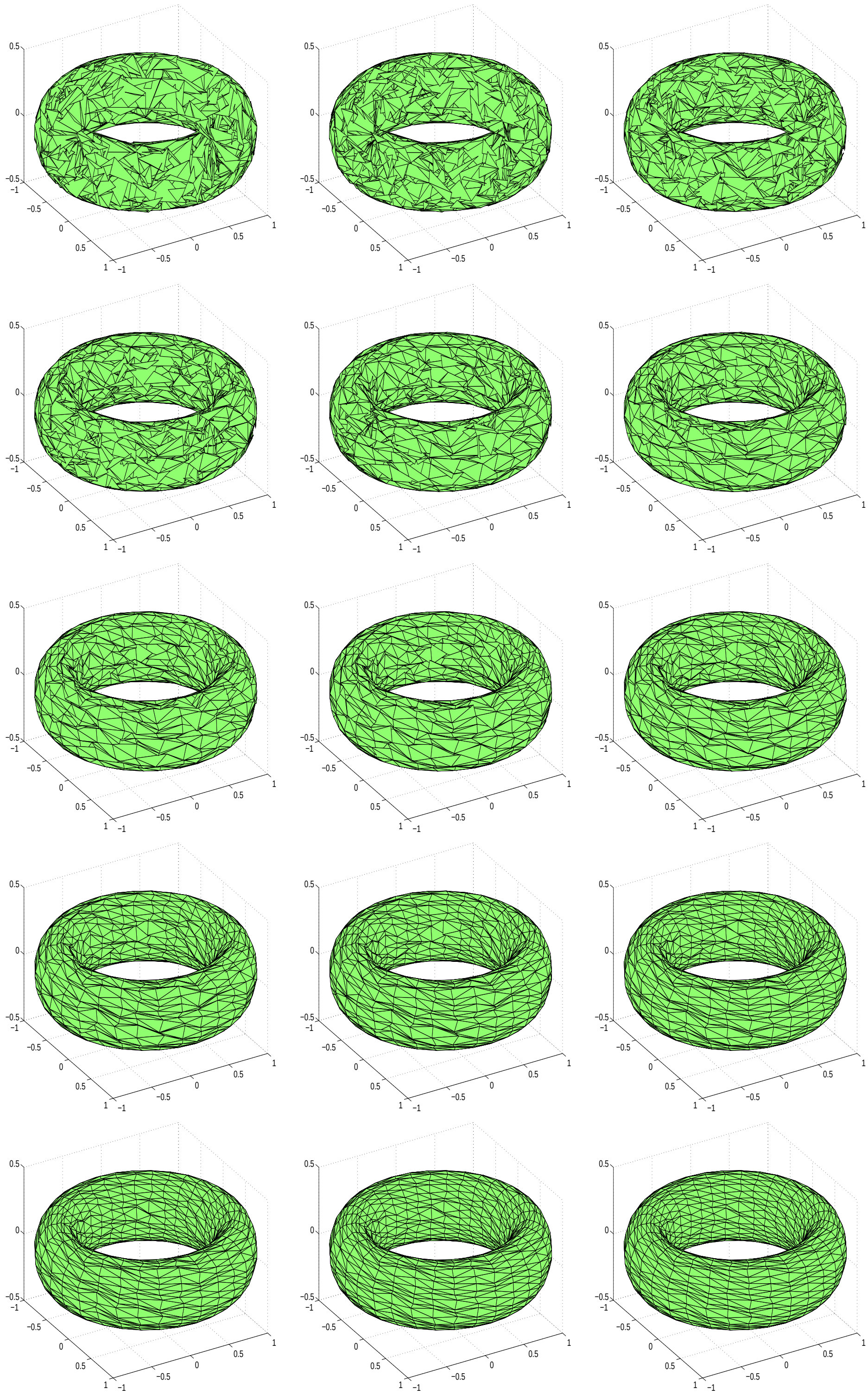

Figure 4.22: Deformations of the torus $T_{1,1/4}$ defined through deformed triangulations $\mathcal{T}_h^i$ of $\mathcal{T}_h$ under $u_h^i(\cdot)$ after 0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, and 140 iterations with Algorithm A (from left to right and top to bottom).

| $\kappa = h^2/2$ | $h = 2^{-4}$ | $h = 2^{-5}$ | $h = 2^{-6}$ |
|---|---|---|---|
| Alg. A' | 0.4 min ($\times$) | 8.5 min ($\times$) | 2.9 h ($\times$) |
| Alg. A" | 4.4 min ($\surd$) | 2.7 h ($\surd$) | 134.3 h ($\surd$) |
| Alg. B | 2.7 min ($\surd$) | 47.5 min ($\surd$) | 14.7 h ($\surd$) |

Table 4.2: CPU times and stability of various algorithms for the approximation of the harmonic map heat flow into the unit sphere. The initial $u_0$ was a perturbation of the identity on $S^2$ in each example.

sufficient to guarantee stability of Algorithm A". The results and numbers also show that reduced integration has a stabilizing effect, cf. Remark 1.4.11, and significantly reduces the total CPU times. The results for the implicit scheme always indicated stability for $\kappa = h^2/2$, as predicted by the theory in Section 3.3. The CPU times for Algorithm B are between the corresponding ones for Algorithms A' and A". We note however, that we could have employed a larger termination criterion in Algorithm $\mathrm{B}^{inner}$ such as $\delta = o(1)$, which would have presumably led to significantly smaller CPU times. In this experiment, the explicit scheme with reduced integration, i.e., Algorithm A', performs best but we stress that in contrast to Algorithm B, in each time step a saddle-point formulation has to be solved (which is done efficiently by MATLAB's backslash operator) and that the time-step size $\kappa = h^2/2$ is not sufficient to guarantee convergence of iterates of Algorithm A' to a weak solution of the $L^2$ flow of harmonic maps into spheres, see [BBFP07]. Table 4.5 displays the results for Algorithm A" with $\kappa = h^2/10$. For this choice of the time-step size, the algorithm provided stable approximations. Not surprisingly, the CPU times are then about 20 and 100 larger than those for Algorithms B and A', respectively (with the larger time-step sizes).

| $\kappa = h^2/10$ | $h = 2^{-4}$ | $h = 2^{-5}$ | $h = 2^{-6}$ |
|---|---|---|---|
| Alg. A' | 23.0 min ($\surd$) | 13.4 h ($\surd$) | 521.3h ($\surd$) |

Table 4.3: Stable numerical results for Algorithm A' for smaller time-step sizes.

# Appendix A

# Implementations

In this appendix we display short MATLAB implementations of Algorithms A, B, and E as well as a C-Routine for a realization of the nearest-neighbor projection $\pi_N$ for two-dimensional tori. The routines can easily be modified to realize Algorithms C and D or to implement $\pi_N$ for other hypersurfaces given by a level set function. Most of the codes employ the following subroutine which computes the finite element stiffness and mass matrices on a given triangulated surface.

```
function [S,M,m,area_K] = surf_fe_matrices(n4e,c4n);
for j = 1 : size(n4e,1)
    mu_K(j,:) = cross(c4n(n4e(j,2),:)-c4n(n4e(j,1),:),c4n(n4e(j,3),:)-c4n(n4e(j,2),:));
    area_K(j) = norm(mu_K(j,:))/2;
    mu_K(j,:) = mu_K(j,:) / norm(mu_K(j,:));
    mp_K(j,:) = sum(c4n(n4e(j,:),:))/3;
    diam_K(j) = norm(c4n(n4e(j,2),:)-c4n(n4e(j,1),:));
end
s = sparse(size(c4n,1),size(c4n,1)); m = s;
for j = 1 : size(n4e,1)
    tmp_tetra = [c4n(n4e(j,:),:);mp_K(j,:)+diam_K(j)*mu_K(j,:)];
    grads3_K = [1,1,1,1;tmp_tetra'] \ [0,0,0;eye(3)];
    P_K = eye(3) - mu_K(j,:)' * mu_K(j,:);
    for k = 1 : 3
        for ell = 1 : 3
            s(n4e(j,k),n4e(j,ell)) = s(n4e(j,k),n4e(j,ell)) ...
                + area_K(j) * (P_K * grads3_K(k,:)')' * (P_K * grads3_K(ell,:)');
        end
        m(n4e(j,k),n4e(j,k)) = m(n4e(j,k),n4e(j,k)) + (1/3) * area_K(j);
    end
end
S = sparse(3*size(c4n,1),3*size(c4n,1)); M = S;
S(1:3:end,1:3:end) = s; S(2:3:end,2:3:end) = s; S(3:3:end,3:3:end) = s;
M(1:3:end,1:3:end) = m; M(2:3:end,2:3:end) = m; M(3:3:end,3:3:end) = m;
```

Figure A.1: MATLAB routine to compute the stiffness matrix and the mass matrix for reduced integration on a given triangulated surface.

```
function Algorithm_A  % for closed surfaces
eps = 1.0E-3;
r = 1/4; R = 1;
[c4n,n4e] = triang_torus(r,R,4);
Nb = []; Db = [];
[S,M,m,areas] = surf_fe_matrices(n4e,c4n);
h = sqrt(max(areas));
kappa = h;
I1 = reshape(repmat([1:size(c4n,1)]',1,3)',3*size(c4n,1),1);
I2 = [1:3*size(c4n,1)];
X_int = sparse(3,3*size(c4n,1));
aver = ones(1,size(c4n,1)) * m;
X_int(1,1:3:end) = aver;
X_int(2,2:3:end) = aver;
X_int(3,3:3:end) = aver;
u = u_0(c4n);
[u,nu] = projection(u,'Torus',[r,R],eps);
xx = zeros(4*size(c4n,1),1);
norm_corr = inf;
while norm_corr > eps
    X_nu = sparse(I1,I2,nu);
    X_nu = [X_nu;X_int];
    AA = [S,X_nu';X_nu,sparse(size(X_nu,1),size(X_nu,1))];
    bb = [S*u;zeros(size(c4n,1)+3,1)];
    xx = AA \ bb;
    v = xx(1:3*size(c4n,1));
    u = u - kappa * v;
    norm_corr = sqrt(v'*S*v);
    [u,nu] = projection(u,'Torus',[r,R],eps);
end
trisurf(n4e,u(1:3:end),u(2:3:end),u(3:3:end),zeros(size(c4n,1),1));

function [u,nu] = projection(u,geom,r,eps)
U = [u(1:3:end),u(2:3:end),u(3:3:end)];
[U,Nu] = projection(U,geom,r,eps);
u = reshape(U',size(u,1),1);
nu = reshape(Nu',size(u,1),1);

function val = u_0(X);
pert_fac = .25;
val = zeros(3*size(X,1),1);
for j = 1 : size(X,1)
    val(3*j-[2,1,0],1) = X(j,:)' + pert_fac*(rand(3,1)-.5);
end
```

Figure A.2: MATLAB implementation of Algorithm A for closed surfaces. The routine `projection` realizes the projection onto the employed surface, see Figure A.5 for an example.

```
function Algorithm_B
load surf_triang.dat -mat;
[S,M,m,areas] = surf_fe_matrices(n4e,c4n);
h = sqrt(max(areas));
kappa = h^2/10;
delta = h^4;
eps = 10^-6;
inv_Laplace = -inv(M) * S;
u = u_0(c4n);
norm_dtu = inf;
while norm_dtu > eps
    w = u;
    diff = inf;
    while diff > delta
        psi = inv_Laplace * u;
        X = skew_sym(n4e,c4n,areas,P1_cross(w,psi));
        A = M + (kappa/2) * X;
        b = M * u;
        w_new = A \ b;
        diff = h^(-2) * sqrt((w - w_new)' * M * (w - w_new));
        w = w_new;
    end
    norm_dtu = (2/kappa) * sqrt((u - w)' * M * (u - w));
    u = 2 * w - u;
end
trisurf(n4e,u(1:3:end),u(2:3:end),u(3:3:end),zeros(size(c4n,1),1));

function X = skew_sym(n4e,c4n,areas,u);
X = sparse(3*size(c4n,1),3*size(c4n,1));
for j = 1 : size(n4e,1)
    for k = 1 : 3
        X_loc = zeros(3,3);
        uu = u(3*n4e(j,k)-[2,1,0]);
        for a = 1 : 3
            phi_a = zeros(3,1); phi_a(a) = 1;
            for b = 1 : 3
                phi_b = zeros(3,1); phi_b(b) = 1;
                X_loc(a,b) = areas(j)/3 * ...
                    ((uu(2) * phi_a(3) - uu(3) * phi_a(2)) * phi_b(1) ...
                    + (uu(3) * phi_a(1) - uu(1) * phi_a(3)) * phi_b(2)  ...
                    + (uu(1) * phi_a(2) - uu(2) * phi_a(1)) * phi_b(3)) ;
            end
        end
        I = [3*n4e(j,k)-[2,1,0]];
        X(I,I) = X(I,I) + X_loc;
    end
end

function u = P1_cross(v,w)
u = zeros(size(w));
u(1:3:end) = v(2:3:end).*w(3:3:end) - v(3:3:end).*w(2:3:end);
u(2:3:end) = v(3:3:end).*w(1:3:end) - v(1:3:end).*w(3:3:end);
u(3:3:end) = v(1:3:end).*w(2:3:end) - v(2:3:end).*w(1:3:end);
```

Figure A.3: MATLAB implementation of Algorithm B for the implicit approximation of the harmonic map heat flow into $S^2$.

```
function Algorithm_E   % for N = S^2
N_local = 5; N_global = 10;
eps = 1.0E-9;
load triang_surf.dat -mat
[S,M,m,areas] = surf_fe_matrices(n4e,c4n);
tmpDiriNodes = unique(Db);
DiriNodes = [3*tmpDiriNodes-2;3*tmpDiriNodes-1;3*tmpDiriNodes-0;3*size(c4n,1)+tmpDiriNodes];
freeNodes = setdiff(1:4*size(c4n,1),DiriNodes)';
I1 = reshape(repmat([1:size(c4n,1)]',1,3)',3*size(c4n,1),1);
I2 = [1:3*size(c4n,1)];
u = u_0(c4n); lambda = zeros(size(c4n,1),1); xx = zeros(4*size(c4n,1),1);
norm_res = inf;
while norm_res > eps
    ctr_global = 0;
    while ctr_global < N_global & norm_res > eps
        X_u = sparse(I1,I2,u);
        AA = [S,-2*X_u'*m;-2*m*X_u,sparse(size(X_u,1),size(X_u,1))];
        bb = [S*u;zeros(size(c4n,1),1)];
        xx(freeNodes) = AA(freeNodes,freeNodes) \ bb(freeNodes);
        v = xx(1:3*size(c4n,1)); lambda = xx(3*size(c4n,1)+1:end);
        u = u - v;
        norm_u = sqrt(u(1:3:end).^2 + u(2:3:end).^2 + u(3:3:end).^2);
        u = u ./reshape(ones(3,1) * norm_u',3*size(c4n,1),1);
        norm_res = comp_norm_res(u,lambda,S,M,m,c4n,freeNodes)
        ctr_global = ctr_global + 1;
    end
    u_old = u;
    ctr_local = 0;
    while ctr_local < N_local & norm_res > eps
        res = zeros(4*size(c4n,1),1);
        vec_lambda = reshape(repmat(lambda,1,3)',3*size(c4n,1),1);
        D_lambda = spdiags(vec_lambda,0,3*size(c4n,1),3*size(c4n,1));
        F = [u'*S + 2*u'*(D_lambda*M),(norm_u.^2-1)'*m]';
        X_u = sparse(I1,I2,u);
        DF = [S + 2*D_lambda*M , 2*X_u'*m ; ...
            2*m*X_u , sparse(size(c4n,1),size(c4n,1))];
        res(freeNodes) =  -DF(freeNodes,freeNodes) \ F(freeNodes);
        v = res(1:3*size(c4n,1)); mu = res(3*size(c4n,1)+1:end);
        u = u + v; lambda = lambda + mu;
        norm_u = sqrt(u(1:3:end).^2 + u(2:3:end).^2 + u(3:3:end).^2);
        norm_res = comp_norm_res(u,lambda,S,M,m,c4n,freeNodes)
        ctr_local = ctr_local + 1;
    end
    if norm_res >= eps
        u = u_old;
    end
end

function norm_F = comp_norm_res(u,lambda,S,M,m,c4n,freeNodes);
vec_lambda = reshape(repmat(lambda,1,3)',3*size(c4n,1),1);
D_lambda = spdiags(vec_lambda,0,3*size(c4n,1),3*size(c4n,1));
norm_u = sqrt(u(1:3:end).^2 + u(2:3:end).^2 + u(3:3:end).^2);
F = [u'*S + 2*u'*(D_lambda*M) , (norm_u.^2-1)'*m]';
norm_F = norm(F(freeNodes));
```

Figure A.4: MATLAB implementation of Algorithm E for $N = S^2$.

```c
#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#include "mex.h"

double Gamma(double p[3], double r[2])
{
    return pow(sqrt(pow(p[0],2.0) + pow(p[1],2.0))-r[1],2.0) + pow(p[2],2.0);
}
void dGamma(double val[3], double p[3], double r[2])
{
    val[0]= 2.0*( sqrt(pow(p[0],2.0) + pow(p[1],2.0) )-r[1])*p[0]/
                   sqrt(pow(p[0],2.0) + pow(p[1],2.0));
    val[1]= 2.0*( sqrt(pow(p[0],2.0) + pow(p[1],2.0) )-r[1])*p[1]/
                   sqrt(pow(p[0],2.0) + pow(p[1],2.0));
    val[2]= 2.0* p[2];
}
void d2Gamma(double val[3][3], double p[3], double r[2])
{
    val[0][0] = 2.0* (1.0 -  r[1]/sqrt(pow(p[0],2.0) + pow(p[1],2.0)) *(1.0- p[0]* p[0])/
                   (pow(p[0],2.0) + pow(p[1],2.0)));
    val[0][1] = 2.0* (r[1]/sqrt(pow(p[0],2.0) + pow(p[1],2.0)) *(p[0]* p[1])/
                   (pow(p[0],2.0) + pow(p[1],2.0)));
    val[1][0] = val[0][1];
    val[1][1] = 2.0* (1.0 -  r[1]/sqrt(pow(p[0],2.0) + pow(p[1],2.0)) *(1.0- p[1]* p[1])/
                    (pow(p[0],2.0) + pow(p[1],2.0)));
    val[0][2] = 0.0;  val[1][2] = 0.0;
    val[2][0] = 0.0;  val[2][1] = 0.0;  val[2][2] = 2.0;
}

double f(double p[3],double r[2])
{
    return sqrt(Gamma(p,r)) - r[0];
}
void df(double val[3], double p[3], double r[2])
{
    double grad[3];
    dGamma(grad,p,r);
    val[0] = 1.0/(2.0*sqrt(Gamma(p,r)))*grad[0];
    val[1] = 1.0/(2.0*sqrt(Gamma(p,r)))*grad[1];
    val[2] = 1.0/(2.0*sqrt(Gamma(p,r)))*grad[2];
}
void d2f(double val[3][3], double p[3],double r[2])
{
    double grad[3],hesse[3][3];
    int m,k;
    dGamma(grad,p,r);
    d2Gamma(hesse,p,r);
    for (m=0;m<3;m++){
        for (k=0;k<3;k++){
            val[m][k] = -1.0/4.0 * pow(Gamma(p,r),-3.0/2)* grad[m]* grad[k] +  hesse[m][k]/
                          (2.0 * sqrt(Gamma(p,r)));
        }
    }
}
```

```
void projection(double points[],int nr_points, double r[], double *eps_pi,
      double points_new[], double nu[], double res[]){
    int ell,n,m,k,i,i1,i2,j;
    double lambda, norm;
    double d2G[4][4], hesse[3][3];
    double dG[4], grad[3], v_new[4], cur_point[3];
    for(n =0;n<nr_points;n++){
        lambda = 0;
        for(k=0;k<3;k++){
            cur_point[k] = points[n+ k* nr_points];
            dG[k]        = 0;
        }
        d2f(hesse,cur_point,r);
        df(grad,cur_point,r);
        norm = sqrt(pow(grad[0],2.0) + pow(grad[1],2.0) + pow(grad[2],2.0));
        for(k=0;k<3;k++){
            nu[n+ k* nr_points] = grad[k]/norm;
        }
        dG[3]   = f(cur_point,r);
        res[n]  = sqrt(pow(dG[3],2.0));
        while(res[n]>eps_pi[0]){
            d2f(hesse,cur_point,r);
            df(grad,cur_point,r);
            for(i1=0;i1<3;i1++){
                for(i2=0;i2<3;i2++){
                    d2G[i1][i2] = lambda*hesse[0][1];
                    if(i1==i2){
                        d2G[i1][i2] = d2G[i1][i2] + 2.0;
                    }
                }
            }
            for(ell=0;ell<3;ell++){
                d2G[3][ell] = grad[ell];
                d2G[ell][3] = grad[ell];
            }
            d2G[3][3] = 0.0;
            solve_lin_sys(v_new, d2G,dG);
            lambda       = lambda - v_new[3];
            for(m=0;m<3;m++){
                cur_point[m] = cur_point[m] - v_new[m];
            }
            df(grad,cur_point,r);
            for(i=0;i<3;i++){
                dG[i] = 2*(cur_point[i]-points[n+ i* nr_points])+lambda*grad[i];
            }
            dG[3] = f(cur_point,r);
             res[n] = sqrt(pow(dG[0],2.0)+pow(dG[1],2.0)+pow(dG[2],2.0)+pow(dG[3],2.0));
        }
        for(k=0;k<3;k++){
            points_new[n+ k* nr_points] = cur_point[k];
        }
     }
}
```

```
void mexFunction(int nlhs, mxArray *plhs[], int nrhs, const mxArray *prhs[])
{
    double *points, *r, *eps_pi, *dummy;
    double *points_new, *res, *D_f, *D2_f, *nu/*, *D_nu*/;
    int nr_points,i,j;
    if (nrhs!=3)
         mexErrMsgTxt("3 input variables required!");
    points            = mxGetPr(prhs[0]);
    r                 = mxGetPr(prhs[1]);
    eps_pi            = mxGetPr(prhs[2]);
    nr_points         = mxGetM(prhs[0]);
    if(nlhs<2)
        mexErrMsgTxt("At least 2 output variables required!");
    plhs[0] = mxCreateDoubleMatrix(nr_points,3,mxREAL);
    plhs[1] = mxCreateDoubleMatrix(nr_points,3,mxREAL);
    plhs[2] = mxCreateDoubleMatrix(nr_points,1,mxREAL);
    if(nlhs>3)
        mexErrMsgTxt("At most 3 output variables defined");
    points_new       = mxGetPr(plhs[0]);
    nu               = mxGetPr(plhs[1]);
    res              = mxGetPr(plhs[2]);
    projection(points, nr_points,r, eps_pi, points_new, nu, res);
}
```

Figure A.5: Implementation of a Newton iteration for the approximation of the nearest-neighbor projection $\pi_N$ for $N = T_{r_1 r_2}$ in the programming language C using the MATLAB to C interface MEX. The routine `solve_lin_sys` solves a linear system of equations.

# Appendix B

# Frequently used notation

### Abbreviated Assumptions

| | |
|---|---|
| (T) | existence of a transfer operator from $M_h$ to $M$ |
| (O) | orientability of $N$ |

### Real Numbers, Vectors, and Matrices

| | |
|---|---|
| $\mathbb{N}$ | non-negative integers |
| $\mathbb{R}$ | real numbers |
| $[s,t]$, $(s,t)$ | closed and open interval |
| $\mathbb{R}^n$ | $n$-dimensional Euclidean vector space |
| $\mathbb{R}^{n\times m}$ | vector space of $n$ by $m$ matrices |
| $a$, $\mathbf{A}$ | (column) vector and matrix |
| $a^{\mathrm{T}}$, $\mathbf{A}^{\mathrm{T}}$ | transpose of a vector or matrix |
| $\lvert\cdot\rvert$ | Euclidean length of a vector or Frobenius-norm of a matrix |
| $a \cdot b = a^{\mathrm{T}} b$ | scalar product of vectors $a$ and $b$ |
| $a \otimes b = ab^{\mathrm{T}}$ | dyadic product of vectors $a$ and $b$ |
| $a \times b$ | cross product of vectors $a, b \in \mathbb{R}^3$ |
| $a \perp b$ | $a$ is perpendicular to $b$ |
| $\mathbf{I}_{n\times n}$ | $n$ by $n$ identity matrix |
| $SO(n)$ | group of special orthogonal matrices |
| $so(n)$ | $n$ by $n$ skew-symmetric matrices |
| $\Lambda^\ell(\mathbb{R}^n)$ | space of alternating $\ell$-linear forms |
| $*$ | Hodge duality operator |
| $\wedge$ | wedge product |
| $\begin{bmatrix} x \\ y \end{bmatrix}$, $\begin{pmatrix} x \\ y \end{pmatrix}$, $(x,y)$ | vectors with entries $x$ and $y$ |
| $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$, $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$ | matrices with entries $x_1, x_2, y_1, y_2$ |

### Continuous Submanifolds

| | |
|---|---|
| $M$ | compact, connected, orientable, $d$-dimensional submanifold in $\mathbb{R}^m$, $d = m-1$, either without boundary or $M \subset \mathbb{R}^d \times \{0\}$ with polyhedral boundary |
| $N$ | compact, parallelizable, $k$-dimensional $C^\ell$ submanifold in $\mathbb{R}^n$, $\ell \geq 2$, without boundary |
| $\mu$ | smooth unit normal on $M$ |
| $\nu$, $\nu^\ell$ | unit normal (possibly discontinuous) on $N$ |
| $A_N(p)[\cdot,\cdot]$ | second fundamental form on $N$ at $p$ |
| $\operatorname{dist}(q,N)$, $d_N(q)$ | distance and signed distance of $q$ to $N$ |
| $U_{\delta_N}(N)$ | tubular neighborhood of $N$ |
| $\pi_N$ | orthogonal (or nearest-neighbor) projection onto $N$ defined in $U_{\delta_N}(N)$ |
| $\Gamma_{\mathrm{D}} \subseteq \partial M$ | possibly empty Dirichlet part of the boundary of $M$ |
| $u_{\mathrm{D}}$ | Dirichlet data |
| $T_pN$ | tangent space of $N$ at $p$ |
| $X_M$ | Euler characteristic of $M$ |
| $\overline{\nu}^\ell$, $\overline{e}^i$ | extended unit normals and tangents to a neighborhood of $N$ |
| $\mathrm{d}s$ | surface area element on $M$ |
| $S^\ell$ | $\ell$-dimensional unit sphere |
| $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ | 2-dimensional torus |

### Discretized Submanifolds and Transfer Operators

| | |
|---|---|
| $h > 0$ | parameter $h$ that ranges over a countable set of positive numbers that accumulate at 0 |
| $M_h$ | Lipschitz-continuous, orientable approximation of $M$ |
| $\mathcal{P}_h$ | continuous bijection from $M_h$ to $M$ |
| $\mu_h$ | unit normal on $M_h$ (discontinuous, defined almost everywhere) |
| $\mathcal{T}_h$ | set of $d$-simplices, triangulation defining $M_h$ |
| $K$ | element in $\mathcal{T}_h$ |
| $\tau_K$, $\tau_E$ | unit tangents along $\partial K$ and $E$ |
| $\widehat{K}$ | scaled reference element |
| $\mathcal{F}_K$ | affine bijection from $\widehat{K}$ to $K$ (parametrization of $K$) |
| $\widetilde{K}$ | image of $K$ under $\mathcal{P}_h$ |
| $\mathcal{X}_K := \mathcal{P}_h \circ \mathcal{F}_K$ | bijection from $\widehat{K}$ to $\widetilde{K}$ (parametrization of $\widetilde{K}$) |
| $\mathrm{d}s_h$ | surface area element on $M_h$ |
| $Q$, $Q_h$ | Gramian determinants on $M$ and $M_h$ defined through $\mathcal{F}_K$ and $\mathcal{X}_K$ |
| $\widetilde{v}$ | lifting of $v\colon M_h \to \mathbb{R}$ onto $M$, i.e., $\widetilde{v} = v \circ \mathcal{P}_h^{-1}\colon M \to \mathbb{R}$ |
| $\check{w}$ | inverse of lifting, $\widetilde{\check{w}} = w$ |

## Differential Operators and Function Spaces

| | |
|---|---|
| $\nabla_M$ | tangential gradient on $M$ (column vector for scalar functions and matrix otherwise); subscript is skipped if $M$ is flat |
| $\underline{D}_\gamma$ | component of $\nabla_M$ |
| $C_c^\infty(M)$ | smooth, compactly supported functions on $M$ |
| $L^p(M;\mathbb{R}^n)$ | space of $p$-integrable vector fields on M |
| $W^{1,p}(M,\mathbb{R}^n)$ | weakly differentiable vector fields on $M$ with $p$-integrable tangential gradient |
| $W_0^{1,p}(M)$, $W_\mathrm{D}^{1,p}(M)$ | functions in $W^{1,p}(M)$ that vanish on $\partial M$ and $\Gamma_\mathrm{D}$ |
| $(\cdot;\cdot)$ | scalar product in $L^2(M;\mathbb{R}^n)$ |
| $\\|\cdot\\|$ | $L^2$ norm on $M$, induced by $(\cdot;\cdot)$ |
| $\\|\cdot\\|_{L^p(M)}$ | $L^p$ norm on $M$ |
| $\\|\cdot\\|_{W^{1,p}(M)}$ | norm in $W^{1,p}(M;\mathbb{R}^n)$ |
| $Df$ | Jacobian of $f$ |
| $\partial_t$ | partial derivative with respect to $t$ |

## Finite Element Spaces

| | |
|---|---|
| $h$, $h_{min}$ | maximal and minimal diameter of elements in $\mathcal{T}_h$ |
| $h_K$, $h_E$, $h_z$ | local mesh-sizes |
| $\mathcal{N}_h$, $\mathcal{E}_h$ | sets of vertices and edges (also called faces if $d=3$) in $\mathcal{T}_h$ |
| $z$, $E$ | node (also called vertex) and edge in $\mathcal{T}_h$ |
| $\varphi_z$ | nodal basis function ($P1$ hat function) |
| $\omega_z$, $\widehat{\omega}_z$ | patch and enlarged patch of a node |
| $\mathcal{L}^0(\mathcal{T}_h)$ | $\mathcal{T}_h$-elementwise constant functions on $M_h$ |
| $\mathcal{S}^1(\mathcal{T}_h)$ | continuous, $\mathcal{T}_h$-elementwise affine functions on $M_h$ |
| $\mathcal{S}_\mathrm{D}^1(\mathcal{T}_h)$ | functions in $\mathcal{S}^1(\mathcal{T}_h)$ vanishing on $\Gamma_D$ |
| $\mathring{\mathcal{S}}^1(\mathcal{T}_h)$ | functions in $\mathcal{S}^1(\mathcal{T}_h)$ vanishing on $\Gamma_D$ or having zero integral mean |
| $\mathcal{S}^{1,NC}(\mathcal{T}_h)$ | Crouzeix-Raviart non-conforming finite element space on $\mathcal{T}_h$ |
| $\mathcal{H}(\mathcal{T}_h;\mathbb{R}^{d+1})$ | space of $\mathcal{T}_h$-elementwise constant, discrete harmonic vector fields |
| $\mathcal{I}_h$ | nodal interpolation operator on $\mathcal{T}_h$ |
| $\mathcal{A}_h$ | averaging operator $\mathcal{A}_h\colon L^1(M_h)\to\mathcal{S}^1(\mathcal{T}_h)$ |
| $\mathcal{G}_h$ | projection onto gradients of functions in $\mathcal{S}^1(\mathcal{T}_h)$ |
| $\mathbf{K}$ | finite element stiffness matrix defined through the nodal basis of $\mathcal{S}^1(\mathcal{T}_h)$ |
| $\mathbf{A}(v_h)$ | correction factor for discrete product rule |
| $\mathcal{S}_\#^1(\mathcal{T}_h)$, $\mathcal{S}_\#^{1,NC}(\mathcal{T}_h)$ | subspaces of periodic functions |
| $\mathcal{H}_\#(\mathcal{T}_h;\mathbb{R}^2)$ | discrete, periodic harmonic fields |
| $\widehat{\mathcal{I}}_h$ | nodal interpolant on reference element $\widehat{K}$ |
| $\widehat{\nabla}$ | gradient on reference element $\widehat{K}$ |

## Differential Operators and Other Quantities on $M_h$

| | |
|---|---|
| $\nabla_{M_h}$ | elementwise tangential gradient on $M_h$ |
| $\underline{D}_{h,\gamma}$ | components of $\nabla_{M_h}$ |
| $D^2_{M_h}$ | second partial derivatives along $M_h$ |
| $(\cdot;\cdot)_h$, $\|\cdot\|_h$ | discrete inner product on $M_h$ and induced norm |
| $\beta_z$ | integral of nodal basis function $\varphi_z$ |
| $\mathrm{Curl}_{M_h}$ | tangential curl operator on $M_h$, $\mathrm{Curl}_{M_h} = \mu_h \times \nabla_{M_h}$ |
| $\widetilde{\Delta}_{M_h}$ | discrete Laplace operator |
| $\kappa$ | damping parameter or time-step size |
| $d_t u^{i+1}$ | backward difference operator, $d_t u^{i+1} := \big(u^{i+1} - u^i\big)/\kappa$ |
| $u^{i+1/2}$ | average of $u^{i+1}$ and $u^i$, $u^{i+1/2} := \big(u^{i+1} + u^i\big)/2$ |
| $\overline{u}_{\mathrm{D},h}$ | nodal interpolant of Dirichlet datum $u_{\mathrm{D}}$ extended trivially to $M_h$ |
| $\big[b_h\big]$ | jump of $\mathcal{T}_h$-elementwise continuous function across edges |
| $\partial\phi/\partial t$ | tangential derivative of $\phi$ along an edge or face |
| $\Lambda_1, \Lambda_2, \Lambda_3, \Theta_1, \Theta_2$ | error functionals |
| $\mathcal{R}es_h$ | residual of an approximate solution |
| $(\cdot;\cdot)$, $\|\cdot\|$ | $L^2$ inner product and induced norm on $M_h$ |

## Frames and Connection Forms

| | |
|---|---|
| $u^{-1}TN$ | pullback bundle, vector fields satisfying $v(x) \in T_{u(x)}N$ |
| $\big(e^i\big)_{i=1,2,\dots,k}$ | orthonormal frame for $u^{-1}TN$ (pointwise orthonormal basis of $T_{u(x)}N$) |
| $\omega^{ij}$ | connection form, $\omega^{ij} = e^{j,\mathrm{T}}\nabla_M e^i$ |
| $\vartheta^i$ | coefficients of the projection of $\nabla_M u$ onto the span of $e^i$, $\vartheta^i = e^{i,\mathrm{T}}\nabla u$ |
| $u_h^{-1}TN$ | discrete pullback bundle, vector fields $v_h \in \mathcal{S}^1(\mathcal{T}_h)^n$ satisfying $v_h(z) \in T_{u_h(z)}N$ for all $z \in \mathcal{N}_h$ |
| $\big(e_h^i\big)_{i=1,2,\dots,k}$ | discrete orthonormal frame for $u_h^{-1}TN$ (nodewise basis of $T_{u_h(z)}N$) |
| $\omega_h^{ij}$, $\overline{\omega}_h^{ij}$ | discrete connection forms |
| $\vartheta_h^i$, $\overline{\vartheta}_h^{ij}$ | coefficients for expansion of $\nabla_{M_h} u_h$ in $\big(e_h^i\big)_{i=1,2,\dots,k}$ |

## Other Notation

| | |
|---|---|
| $c, C, C', C''$ | mesh-size independent, generic constants |
| $\mathcal{H}^s$ | $s$-dimensional Hausdorff measure |
| card | cardinality of a set |
| $\delta_x$ | Dirac measure supported at $x$ |
| $\mathcal{O}(t), o(t)$ | Landau symbols |
| $\mathrm{supp}\, f$ | support of the function $f$ |
| $\mathrm{diam}(A)$ | diameter of the set $A$ |
| id | identity map |
| $B_1(0)$ | open ball of radius one centered at the origin |

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