Functional Analysis

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Contents

Introduction

- Functional Analysis: Analysis in infinite dimensional spaces.
- Linear Functional Analysis: Linear Algebra in infinite dimensional spaces.
- Typical question: X, Y vector spaces; $A : X \to Y$ linear continuous. Does A have a continuous inverse? (If not, can we somehow measure the defect?)

  This leads to the so-called Fredholm-Alternative

Example. $U \subset \mathbb{R}^n$ open, with smooth boundary,

$$\begin{aligned} X &:= \left\{u \in C^2(U) \cap C\left(\overline{U}\right) \mid u = 0 \text{ on } \partial U\right\}, \\ Y &:= C(U), \\ A &:= -\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}. \end{aligned}$$

A invertible $\Leftrightarrow$ The equation

$$\begin{aligned} -\Delta u(x) &= f(x), \quad x \in U, \\ u(x) &= 0, \qquad x \in \partial U. \end{aligned} \tag{0.1}$$

has a solution for any $f \in Y$.

Classically: The equation is a pointwise condition on u.

Modern: u is a point (or vector) in a function space, $-\Delta$ is a linear mapping between function spaces.

Fact: A is continuous and injective. However, A is *not* surjective.

In particular, there exist $f \in C(U)$: no $u \in Y$ satisfies $-\Delta u = f$. We will work out this example in the first tutorial.

Problem: C^2 is a terrible function space for the Laplacian. Much better: Hölder spaces ($C^{2,\alpha}$) or Sobolev spaces.

Topics of the class

- Spaces, norms, topologies,
- Linear operators and their properties,
- General theory $\leftrightarrow$ concrete spaces, e.g. L^p.

Outline

- Basic structures and spaces,
- Continuous linear operators,
- Hahn-Banach,
- Baire Lemma,
- Weak topology,
- L^p and Sobolev spaces,
- Hilbert space theory,
- Spectral theory.

1 Basic structures

1.1 Topological spaces

Definition 1.1 (Topology)**.** Let X be a set and let $\mathcal{T}$ be a set of subsets of X (Notation: $\mathcal{T} \subset 2^X$) with the properties

1. $\emptyset, X \in \mathcal{T}$
2. $\mathcal{T}' \subset \mathcal{T} \Rightarrow \bigcup_{U \in \mathcal{T}'} U \in \mathcal{T}$
3. $U_1, U_2 \in \mathcal{T} \Rightarrow U_1 \cap U_2 \in \mathcal{T}$.

Then, $(X, \mathcal{T})$ is called a *topological space*, $\mathcal{T}$ is called *topology* and sets $U \in \mathcal{T}$ are called *open*.

If in addition to that, we have that for all $x_1, x_2 \in X, x_1 \neq x_2$ there exist $U_1, U_2 \in \mathcal{T}$ with the property that

$$x_1 \in U_1, \;\; x_2 \in U_2, \quad U_1 \cap U_2 = \emptyset, \tag{separation axiom}$$

then $(X, \mathcal{T})$ is called a *Hausdorff space*.

Definition 1.2. A subset $A \subset X$ of a topological space is called *closed* if

$$\exists U \in \mathcal{T} \quad \text{such that} \quad A = X \backslash U =: U^c.$$

Let X henceforth be a topological space.

Definition 1.3. Let $A \subset X$.

1. $A^\circ := \{x \in X \mid \exists U \subset A, U \in \mathcal{T} \text{ s.t. } x \in U\}$ is the *interior* of A.
2. $\overline{A} := \{x \in X \mid \forall U \in \mathcal{T}, x \in U \text{ we have } U \cap A \neq \emptyset\}$ is the *closure* of A.
3. $\partial A := \overline{A} \backslash A^\circ$ is the *boundary* of A.

Definition 1.4. 1. A set $A \subset X$ is called *dense* in X, if $\overline{A} = X$.

2. X is called *separable*, if there exists $A \subset X$ that is dense and countable.

Proposition 1.5. *1. $A^\circ \subset A \subset \overline{A}$.*

2. $A = A^\circ \Leftrightarrow A$ open, $\quad A = \overline{A} \Leftrightarrow A$ closed.

3. A° is open, $\quad \overline{A}$ is closed.

4. $X \backslash \overline{A} = (X \backslash A)^\circ$.

Proof. Analysis 2 or exercise. □

Proposition 1.6. *Let $A \subset X, (X, \mathcal{T})$ be a topological space. Then $(A, \mathcal{T}_A)$ with*

$$\mathcal{T}_A := \{U \cap A \mid U \in \mathcal{T}\}$$

is a topological space. $\mathcal{T}_A$ is called a subspace topology *(relative topology).*

Proof. Should be pretty clear. □

Definition 1.7. Let $(X, \mathcal{T})$ be a topological space.

- $B \subset \mathcal{T}$ is called *basis* of $\mathcal{T}$, if any set in $\mathcal{T}$ can be written as a union of sets in B.
- $S \subset \mathcal{T}$ is called a *subbasis* of $\mathcal{T}$, if the set of all finite intersections of sets in S is a basis of $\mathcal{T}$.

Proposition 1.8. *Let X be a set, $S \subset 2^X$ a collection of subsets of X. Let now $B \subset 2^X$ be the set of subsets of X generated by taking any finite intersections between sets in S. Then the set generated by taking arbitrary unions of sets in B is a topology.*

Proof. Exercise. □

Definition 1.9. Let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on X. $\mathcal{T}_2$ is called *stronger* (or *finer*) than $\mathcal{T}_1$ and $\mathcal{T}_1$ *weaker* (or *coarser*) than $\mathcal{T}_2$, if

$$\mathcal{T}_1 \subset \mathcal{T}_2.$$

Example.

1. The indiscrete topology: $\mathcal{T} = \{\emptyset, X\}$,
2. The discrete topology: $\mathcal{T} = 2^X := \mathcal{P}(X)$,
3. The cofinite topology: $X = \mathbb{N}, \mathcal{T} := \left\{U \in 2^{\mathbb{N}} \mid \#(X \backslash U) < \infty\right\} \cup \emptyset$,
4. Topologies induced by a metric.

Definition 1.10 (Neighborhood)**.** Let $A \subset X$. A set N is called a neighborhood of A if there exists an open set U such that $A \subset U \subset N$. Note that A can be a singleton. We write $N(A)$.

Definition 1.11 (Continuity)**.** Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces.

1. A mapping $f : X \to Y$ is called *continuous* if

$$\forall V \in \mathcal{T}_Y \text{ we have } \quad f^{-1}(V) \in \mathcal{T}_X.$$

2. A mapping $f : X \to Y$ is called *continuous in* $x \in X$, if for any neighborhood N_Y of $f(x)$ there exists a neighborhood N_X of x such that $f(N_X) \subset N_Y$.

Proposition 1.12. *$f : X \to Y$ is continuous, if and only if $f : X \to Y$ is continuous in all $x \in X$.*

Proof. "⇒" Fairly easy, just take the preimage of the open set in N_Y as N_X.

"⇐" Consider $V \subset Y$ open. If the preimage is empty, we are done. Otherwise, take $x \in f^{-1}(V) =: A$. By the definition of continuity in a point, there exists an open set U containing x such that $U \subset A$. So $A = A^\circ$ and thus open.

□

Definition 1.13 (Convergence)**.** Let $(X, \mathcal{T})$ be a topological space, let $x \in X$. We say that a sequence $(x_k)_{k \in \mathbb{N}}$ *converges to* x, if

for any neighborhood $N(x)$ we have that $\exists k_0 \in \mathbb{N}$ such that $\forall k > k_0$ we have $x_k \in N$.

We write

$$x_k \xrightarrow{\mathcal{T}} x.$$

Proposition 1.14. *If $(X, \mathcal{T})$ is Hausdorff, then the limit x is uniquely determined. We then write*

$$x = \lim_{k \to \infty} x_k \ \textit{w.r.t. (with respect to)}\ \mathcal{T}.$$

Proof. Assume $x_k \to x_1, x_k \to x_2 \neq x_1$. Then

$$\exists U_1 \ni x_1, U_2 \ni x_2 \text{ with } U_1 \cap U_2 = \emptyset.$$

But: $\exists k_0 : \forall k > k_0$ we have $x_k \in U_1$ and at the same time $x_k \in U_2$. This is a contradiction. □

Example.
1. $\mathcal{T}_X = 2^X$. Any function $f : X \to Y$ is continuous.
   However, $x_k \to x$ only if $\exists k_0$ s.t. $\forall k > k_0$ we have $x_k = x$.
2. $\mathcal{T}_X = \{\emptyset, X\}$, $\mathcal{T}_Y$ Hausdorff. Then we have: f is continuous only if f is constant.
   However, any sequence in X converges to every element in X.

Remark (Direct method of the calculus of variations)*.* Goal: Find $x \in X$:

$$f(x) = \inf_{y \in X} f(y) \qquad \text{for some} \qquad f : X \to [0, \infty).$$

1. $\exists (x_k)_{k \in \mathbb{N}} : f(x_k) \to \inf f$. We have $f(x_k) < C$ for $k > k_0$.
2. We want that a subsequence of $(x_k)_{k \in \mathbb{N}}$ converges to some $x \in X$.
3. Then we need to show that $f(x) = \inf f$. ($\Leftarrow x_j \to x \Rightarrow \underbrace{f(x) \leq \liminf f(x_j)}_{\text{lower semicontinuity}}$)

Remark 1.15*.* i) One topic we left untouched is the *product topology.* For finite products it suffices to consider the *box topology* on $\prod_{i=1}^n X_i$, namely the topology to the basis consisting of all sets of the form

$$\left\{ \prod_{i=1}^n U_i \mid U_j \in \mathcal{T}_j \right\}.$$

ii) A further point we have not discussed here is the issue of compactness. In purely topological spaces, this is somewhat tricky and we will treat compactness in more detail later.

1.2 Metric

Definition 1.16. Let X be a set and $d : X \times X \to \mathbb{R}$, such that for all $x, y, z \in X$ we have

1. $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

Then we call (X, d) a *metric space*, d a *metric* or a *distance.*

Remark.
- Without the requirement that $d(x, y) = 0 \Leftrightarrow x = y$ we call d a pseudo metric.
- By taking a modulo
  $$x \hat{=} y \quad \Leftrightarrow \quad d(x, y) = 0$$
  we can transform a pseudo metric space into a metric space.
- The only thing that is to show for this claim is
  $$x_1, y_1, x_2, y_2 \in X, x_1 \hat{=} x_2, y_1 \hat{=} y_2 \quad \Rightarrow \quad d(x_1, y_1) = d(x_2, y_2).$$
  This, however follows from the triangular inequality [3. of the definition of a (pseudo) metric].

Example.
1. $X = \mathbb{R}, d(x, y) = |x - y|$.
2. $X = \mathbb{R}, d(x, y) = \frac{|x-y|}{1+|x-y|}$.

3. (X, d) metric space, $h : Y \to X$ injective,

$$d_Y(y_1, y_2) := d(h(y_1), h(y_2))$$

is called the *pullback metric*.

4. *Discrete metric*:

$$d(x, y) := \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}.$$

Definition 1.17. Let (X, d) be a metric space,

$$B_r(x) := \{y \in X \mid d(x, y) < r\} \qquad \text{for } r > 0, x \in X.$$

We call $U \subset X$ *open with respect to the metric* d, if $\forall x \in U$ there exists $r > 0$

$$B_r(x) \subset U.$$

The empty set is open.

Proposition 1.18. *Let* (X, d) *be a metric space, and let*

$$\mathcal{T} := \{U \subset X \mid U \text{ is open w.r.t. } d\}.$$

Then $(X, \mathcal{T})$ *is a Hausdorff space. We call* $\mathcal{T}$ *the* by d induced topology.

Proof. 1. $\emptyset, X \in \mathcal{T}$: clear.

2. Let $\mathcal{T}' \subset \mathcal{T}$ and $W := \bigcup_{U \in \mathcal{T}'} U$. We need to show that for all $x \in W$ there exists $r > 0$ with $B_r(x) \subset W$. This is obvious, though, since there is some $U \in \mathcal{T}'$ with $x \in U$ and we can just take the r for that U.

3. Let $U_1, U_2 \in \mathcal{T}, x \in U_1 \cap U_2$. We have $r_1, r_2 > 0$:

$$B_{r_1}(x) \subset U_1 \quad \text{and} \quad B_{r_2}(x) \subset U_2.$$

It follows by the triangle inequality that

$$B_{\min(r_1, r_2)}(x) \subset U_1 \cap U_2.$$

4. Let $x \neq y$. We have that $d(x, y) = c > 0$.

$$B_r(x) \cap B_r(y) = \emptyset, \quad r = \frac{c}{2} > 0.$$

□

Definition 1.19. Let d_1, d_2 both be a metric on X.

- d_1 is called *stronger* than d_2 or d_2 *weaker* than d_1, if the same holds for the induced topologies.
- d_1 is called *equivalent* to d_2, if the induced topologies are equal.

Proposition 1.20 (Continuity in metric spaces)**.** *Let* $(X, d_X), (Y, d_Y)$ *be metric spaces and let* $f : X \to Y$. *Then* f *is continuous in* $x \in X$, *if and only if*

$$\forall \varepsilon > 0 : \exists \delta > 0 : \quad d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon.$$

Proof. Analysis 2. □

Proposition 1.21 (Convergence in metric spaces)**.** *Let* $(X, d_X), (Y, d_Y)$ *be metric spaces.*

1. *Let* $(x_j)_{j \in \mathbb{N}}$ *be a sequence in* X, *then*

$$x_k \xrightarrow{d} x \quad \Leftrightarrow \quad \forall \epsilon\, \exists k_\epsilon : d(x_k, x) \leq \epsilon \;\; (\textit{ for } k \geq k_\epsilon \;).$$

2. *A mapping* $f : X \to Y$ *is continuous in* x*, iff for all sequences* $(x_k)_{k\in\mathbb{N}}$ *with* $x_k \to x$ *we have*

$$f(x_k) \to f(x).$$

Remark. The second condition is called *sequential continuity.*

Warning. This is generally not true in purely topological spaces.

Proof. 1. Clear.

2. "$\Rightarrow$" Clear (somewhat).

"$\Leftarrow$" Assume that f is not continuous. Then $\exists\varepsilon > 0 : \forall\delta > 0 \exists x_\delta :$

$$d_X(x, x_\delta) < \delta, \quad \text{but} \quad d_Y(f(x), f(x_\delta)) > \varepsilon.$$

$\delta := \frac{1}{k} \rightsquigarrow \exists x_k : d_X(x, x_k) < \frac{1}{k} \to 0.$

$$d_Y(f(x), f(x_k)) \geq \varepsilon, x_k \to x, \quad \text{but} \quad f(x_k) \not\to f(x).$$ □

Definition 1.22. Let (X, d) be a metric space.

1. $(x_k)_{k\in\mathbb{N}}$ is called a *Cauchy sequence*, if $\forall\varepsilon > 0 \exists k_0 \in \mathbb{N} : \forall k, l > k_0$, we have $d(x_k, x_l) < \varepsilon$.
2. The space (X, d) is called *complete*, if every Cauchy sequence admits a limit in X.

Proposition 1.23 (sequential criterion)**.** *Let* $A \subset (X, d)$ *(metric space). A is closed, iff for all sequences* $(x_j)_{j\in\mathbb{N}}$ *with* $x_j \in A$ *and* $x_j \to x \in X$*, we have* $x \in A$*.*

Proof. $\Rightarrow$ Assume there was a limit point x in A^c, which is open. Thus there exists a neighborhood $N(x) \subset A^x$. Then points in the sequence would have to lie in this neighborhood and outside of A, which is a contradiction.

$\Leftarrow$ Assume A is not closed and consider $x \in A^c \setminus A$. By the definition of the closure, for any $r > 0$ we have $B_r(x) \cap A \neq \emptyset$. Now take a sequence of radii going to zero and pick as x_k any point in the intersection of the respective ball with A.

□

Proposition 1.24. *Let* (X, d) *be a complete metric space, and* $A \subset X$ *be closed. Then* (A, d) *is a complete metric space.*

Proof. It's clear that (A, d) is a metric space. For completeness, consider a Cauchy sequence in A. By completeness of (X, d), it admits a limit in X, this limit must lie in A by the above sequential criterion. □

Example. 1. $\mathbb{Q}$ (the rationals) with the usual distance are a not complete metric space.

2. Take $I = [0, 1]$ and

$$\begin{aligned} P_n &:= \{f : I \to \mathbb{R} \mid f \text{ is a ploynomial of degree } \deg(f) \leq n\}, \\ P &:= \bigcup_{n\in\mathbb{N}} P_n, \\ d(f, g) &:= \sup_{x\in I} |f(x) - g(x)|. \end{aligned}$$

(P, d) is a metric space. We take

$$f(x) := \exp(x) \quad \text{and} \quad f_n(x) := \sum_{k=0}^{n} \frac{x^k}{k!},$$

noting that $f \notin P$. We have

$$\sup_{x\in I} |f_n(x) - f(x)| \to 0.$$

(And also $\sup_{x\in I}|f_n(x)-f_k(x)| \xrightarrow[n,k\to\infty]{} 0$.) However, if

$$g=\lim_{k\to\infty} f_k$$

would exist in P, then necessarily, we would need

$$|f_n(x)-g(x)|\leq d(f_n,g)\to 0 \quad \forall x\in I,$$

so we would have

$$g=f=\exp .$$

But again, $f\notin P$.

Theorem 1.25 (Completion)**.** *Let (X,d) be a metric space and consider the set $X^{\mathbb{N}}$ of all sequences in X. Let*

$$\tilde{X}:=\left\{\tilde{x}=(x_j)_{j\in\mathbb{N}}\in X^{\mathbb{N}} \mid (x_j)_{j\in\mathbb{N}} \text{ is a Cauchy-sequence}\right\}\Big/_{=}$$

endowed with the equivalence relation

$$(x_j)_{j\in\mathbb{N}}=(y_j)_{j\in\mathbb{N}} \quad :\Leftrightarrow \quad d(x_j,y_j)\to 0.$$

Then, $\left(\tilde{X},\tilde{d}\right)$ is a complete *metric space with*

$$\tilde{d}\left((x_j)_{j\in\mathbb{N}},(y_j)_{j\in\mathbb{N}}\right):=\lim_{j\to\infty} d(x_j,y_j).$$

Furthermore, the mapping

$$J:X\to\tilde{X},\quad J(x):=(x)_{j\in\mathbb{N}} \quad \textit{(constant sequence)}$$

defines an (therefore injective) isometry, *that is*

$$\tilde{d}(J(x),J(y))=d(x,y) \quad \forall x,y\in X,$$

and for any $(x_j)_{j\in\mathbb{N}}\in\tilde{X}$, we have

$$\tilde{d}\left((x_j)_{j\in\mathbb{N}},J(x_k)\right)\xrightarrow[k\to\infty]{} 0,$$

thus $J(X)$ is dense in $\tilde{X}$.

Proof.

- *Well-definedness* of $\tilde{d}$: Let $\tilde{x}=(x_j)_{j\in\mathbb{N}},\tilde{y}=(y_j)_{j\in\mathbb{N}}$ be in $\tilde{X}$. We have

$$\begin{aligned}|d(x_j,y_j)-d(x_i,y_i)| &\leq |d(x_j,y_j)-d(x_i,y_j)|+|d(x_i,y_j)-d(x_i,y_i)|\\ &\leq d(x_jx_i)+d(y_j,y_i)\xrightarrow[i,j\to\infty]{} 0.\end{aligned}$$

  So this was a Cauchy sequence in $\mathbb{R}$ and the limit

$$\tilde{d}(\tilde{x},\tilde{y})=\lim_{j\to\infty} d(x_j,y_j)$$

  exists. By analogous arguments,

$$d\left(x_i^2,y_i^2\right)-d\left(x_i^1,y_i^1\right)\to 0,$$

  if $\tilde{x}^1=\tilde{x}^2\in\tilde{X}$ and $\tilde{y}^1=\tilde{y}^2\in\tilde{X}$.

- Is $\tilde{d}$ a *metric*? Consider the axioms

  1. $\tilde{d}\geq 0$ clear.
     $\tilde{d}(x,y)=0\Leftrightarrow x=y$ (clear by definition of "=" on $\tilde{X}$).

2. Symmetry carries over in the limit.
3. The triangle inequality also carries over in the limit.

- *Completeness*: Consider $\left(x^k\right)_{k\in\mathbb{N}}$ a Cauchy sequence in $\tilde{X}$, $x^k = \left(x_j^k\right)_{j\in\mathbb{N}} \in \tilde{X}$. For $k \in \mathbb{N}$, we pick a j_k, such that
$$d\left(x_i^k, x_j^k\right) \leq \frac{1}{k} \quad \forall i, j \geq j_k.$$
Now for $j \geq j_k, j_l$, $k, l \in \mathbb{N}$, we have
$$\begin{aligned} d\left(x_{j_k}^k, x_{j_l}^l\right) &\leq d\left(x_{j_k}^k, x_j^k\right) + d\left(x_j^k, x_j^l\right) + d\left(x_j^l, x_{j_l}^l\right) \\ &\leq \frac{1}{k} + d\left(x_j^k, x_j^l\right) + \frac{1}{l} \\ &\xrightarrow[j\to\infty]{} \frac{1}{k} + \tilde{d}\left(x^k, x^l\right) + \frac{1}{l} \\ &\xrightarrow[k,l\to\infty]{} 0. \end{aligned}$$
So we define
$$x^\infty := \left(x_{j_l}^l\right)_{l\in\mathbb{N}} \in \tilde{X}.$$
Claim. $x^l \to x^\infty$.

Proof. $\tilde{d}\left(x^l, x^\infty\right) \xleftarrow[k\to\infty]{} d\left(x_k^l, x_k^\infty\right)$, however for $k \geq j_k$:
$$\begin{aligned} d\left(x_k^l, x_k^\infty\right) &\leq d\left(x_k^l, x_{j_l}^l\right) + d\left(x_{j_l}^l, x_{j_k}^k\right) \\ &\leq \frac{1}{l} + d\left(x_{j_l}^l, x_{j_k}^k\right) \\ &\xrightarrow[k,l\to\infty]{} 0. \end{aligned}$$
□

- The statements about J are easy to verify. □

1.3 Vector spaces and norms

We only consider vector spaces over the fields $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and consider those fields metric (or topological) spaces with the usual distance.

Definition 1.26. Let X be a vector space over $\mathbb{K}$ and let X at the same time a topological space. If vector addition and scalar multiplication are continuous, then X is called a *topological vector space.*

Remark. Note that we need to use the box topology where appropriate here.

Definition 1.27 (Norm)**.**

- Let X be a vector space over $\mathbb{K}$. A mapping $\|\cdot\| : X \to \mathbb{R}$ is called a *norm*, if we have
  1. $\|x\| \geq 0 \quad \forall x \in X$.
  2. $\|x\| = 0 \quad \Rightarrow \quad x = 0$.
  3. $\|\alpha x\| = |\alpha| \, \|x\|$.
  4. $\|x + y\| \leq \|x\| + \|y\|$.
- If $\|\cdot\|$ is a norm on X, then $(X, \|\cdot\|)$ is called a *normed space.*
- A mapping with properties 1., 3., 4. is called *semi-norm.*

Remark. Again, we can turn a space with a semi-norm into a normed space by taking the modulo.

Proposition 1.28. *Let $(X, \|\cdot\|)$ be a normed space. By taking*

$$d : X \times X \to \mathbb{R}, \quad d(x, y) := \|x - y\|,$$

we get a metric space (X, d).

Proof. Analysis II. $\square$

Definition 1.29. A complete normed space is called *Banach space.*

Proposition 1.30. *A normed vector space is a topological vector space and the topology is Hausdorff.*

Proof. Exercise. $\square$

Proposition 1.31. *Let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on X, d_1, d_2 the induced metrics and $\mathcal{T}_1, \mathcal{T}_2$ the induced topologies. We have*

1. *d_2 is stronger than d_1, iff $\exists C > 0$:*

$$\|x\|_1 \leq C \|x\|_2 \quad \forall x \in X.$$

2. *Two norms are equivalent ($:\Leftrightarrow \mathcal{T}_1 = \mathcal{T}_2$), iff $\exists C, c > 0$:*

$$c \|x\|_1 \leq \|x\|_2 \leq C \|x\|_1.$$

Proof. $\Leftarrow$ Take U open w.r.t. $\mathcal{T}_1$, and take $x \in U$. Then there exists a d_1-r-ball around x which still lies in U. The d_2-$\frac{r}{C}$-ball then also lies in U.

$\Rightarrow$ Consider $T_1 \ni B_r^{d_1}$, an open d_1-r-ball around the origin and, for any $\delta > 0$, the d_2-δ-ball $B_\delta^{d_2}$. Since the inequality is not satisfied for any C, we can always find an $x \in B_\delta^{d_2}$ with $x \notin B_r^{d_1}$, which is therefore not open wrt. d_2.

$\square$

Theorem 1.32. *All norms on a finite dimensional vector space are equivalent.*

Proof. Use the basis, Luke. $\square$

Example. The norms

$$\|f\|_\infty := \sup_{t \in [0,1]} |f(t)| \quad \text{and} \quad \|f\|_1 := \int_0^1 |f(t)| \, dt$$

on the vector space $C^\infty([0, 1])$ are not equivalent. Take e.g. $f_n(t) := t^n$. We have

$$\|f_n\|_\infty = 1, \quad \text{however}$$
$$\|f_n\|_1 = \frac{1}{n+1} \to 0.$$

Theorem 1.33. *A finite dimensional subspace of a normed space is complete and closed.*

Proof. Completeness follows from the completeness of $\mathbb{K}^n$, closedness from the sequence-criterion for closedness from before. $\square$

Remark. • This does not generally hold for infinite dimensional subspaces.

Remark. If X, Y are normed spaces, $Z = X \times Y$, then

$$\|\xi\|_{Z,p} := (\|\zeta\|_X^p + \|\eta\|_Y^p), \quad \xi = (\zeta, \eta) \in Z$$

and

$$\|\xi\|_{Z,\infty} := \max(\|\zeta\|_X, \|\eta\|_Y)$$

are all equivalent norms on Z. Z is a Banach space, iff X and Y are Banach.

1.4 Scalar product

Definition 1.34. Let X be a $\mathbb{K}$-vector space. A mapping

$$(\cdot,\cdot): X \times X \to \mathbb{R}$$

is called a *sesquilinear form*, if $\forall x, x_1, x_2, y, y_1, y_2 \in X, \alpha \in \mathbb{R}$ we have

1. $(\alpha x, y) = \alpha\,(x, y) = (x, \bar{\alpha} y)$
2. $\begin{cases} (x_1 + x_2, y) = (x_1, y) + (x_2, y) \\ (x, y_1 + y_2) = (x, y_1) + (x, y_2) \end{cases}$

A sesquilinear form is *symmetrical*, if

$$(x, y) = \overline{(y, x)}.$$

A symmetrical sesquilinear is called *positive semidefinite*, if

$$(x, x) \geq 0 \quad \forall x \in X.$$

A positive semidefinite sesquilinear form is called *positive definite*, if

$$(x, x) = 0 \quad \Rightarrow \quad x = 0.$$

Remark. If $\mathbb{K} = \mathbb{R}$, we use the phrase *bilinear.*

Definition 1.35. A positive definite (symmetrical) sesquilinear form is called *scalar product.* The pair $(X, (\cdot,\cdot))$ is called *pre-Hilbert space.*

Lemma 1.36. *Let* $(\cdot,\cdot)$ *be a scalar product and define*

$$\|x\| := \sqrt{(x, x)} \quad \forall x \in X.$$

Then $\|x\| := \sqrt{(x, x)}$ *is a norm on* X*. Furthermore we have*

1. $|(x, y)| \leq \|x\| \cdot \|y\|$ *(Cauchy-Schwarz inequality)*
2. $\|x + y\|^2 + \|x - y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2$ *(Parallelogram identity)*

Proof. Linear Algebra □

Remark. Central Property: Orthogonality.

Definition 1.37.
1. Two vectors x, y are called *orthogonal*, if $(x, y) = 0$.
2. Two subspaces $U, V \subset X$ are called *orthogonal*, if $(x, y) = 0 \quad \forall x \in U, y \in V$.

Definition 1.38. A complete pre-Hilbert space is called *Hilbert space.*

Example. $X = C^1([0, 1])$ with $(f, g) := \int_0^1 fg + \int_0^1 f'g'$ is a pre-Hilbert space.
The completion of X with respect to $\|x\| = \sqrt{(x, x)}$ is called *Sobolev space* H^1.

1.5 Example spaces

1.5.1 The finite dimensional vector spaces $\mathbb{K}^n$

A finite dimensional vector space, in fact the canonical finite dimensional vector space over the field $\mathbb{K}$. Becomes a Banach space with any of the usual norms $||x||_p = (\sum_{i=1}^n |x_j|^p)^{1/p}$ and a Hilbert space with the Euclidean norm. Do you remember the triangle inequality?

1.5.2 The sequence spaces

We consider $\mathbb{K}^{\mathbb{N}}$, the space of sequences with values in $\mathbb{K}$. The i-th canonical unit vector is denoted by e_i, the vector where the i-th entry is 1, all others are zero.

Theorem 1.39. *We have*

1. $\mathbb{K}^{\mathbb{N}}$ *is a metric space with the Fréchet-Metric* $d(x, y) = \rho(x - y)$ *with* $\rho(x) = \sum_{j=1}^{\infty} 2^{-j} \frac{|x_i|}{1+|x_i}$.
2. *Entrywise convergence is the same as convergence with respect to this metric*
3. *The space* $\mathbb{K}^{\mathbb{N}}$ *is complete.*
4. ℓ^p, $1 \leq p \leq \infty$ *are Banach-spaces.*
5. ℓ^2 *is a Hilbert-space.*

Proof. See Alt, Lineare Funktionalanalysis. □

1.5.3 Bounded Functions

For a set S and a $\mathbb{K}$-Banach-space Y we consider the set of all bounded maps from S to Y,

$$B(S, Y) = \{f : S \to Y; f(S) \text{ is a bounded subset of Y}\}.$$

Theorem 1.40. *With the supremumsnorm* $||f||_{B(S,Y)} = \sup_{x\in S} ||f(x)||_Y$*, this space becomes a Banach-space.*

Proof. The properties of being a norm are easy to verify. For any x in S, every Cauchy-sequence in $B(S, Y)$ is a Cauchy-sequence in Y and thus admits a limit (by assumption of Y being Banach) $f(x)$. We have

$$|f(x) - f_k(x)| = \lim_{l\to\infty} |f_l(x) - f_k(x)| \leq \liminf_{l\to\infty} ||f_l - f_k||_{B(S,Y)}.$$

Thus, $f - f_k$ is a bounded function and so is f. Furthermore, for $k \to \infty$, the right hand side of the above goes to zero, so f_k converges to f with respect to the supremums-norm, as the bound is independent of x. □

1.5.4 Continuous and Differentiable Functions

Theorem 1.41. *For* $S \subset \mathbb{R}^n$ *closed and bounded,* Y *a* $\mathbb{K}$*-Banach-space, the space of continuous functions from* S *to* Y *endowed with the supremums-norm is a Banach-space.*

Proof. Identification of the limit as in Theorem 1.40. We also know that uniform limits of continuous functions are continuous. □

Remark.

1. On unbounded or not closed sets $S \subset \mathbb{R}^n$, since then continuous functions may not be bounded, one can either simply look at the bounded continuous functions and derive the same theorem as above, or for compact sets $(K_j)_{j\in\mathbb{N}}$ such that $\bigcup_j K_j = S$ define a Fréchet-metric by taking $\rho(x) = \sum_{j=1}^{\infty} 2^{-j} \frac{||f||_{C(K_i,Y)}}{1+||f||_{C(K_i,Y)}}$. This results in a complete metric space.
2. On closed and bounded sets S, we consider m-times continuously differentiable functions to be the functions whose derivatives, which are naturally defined on the interior of S, can be continuously extended to the boundary. These spaces form Banach-spaces with the supremums-norm on all derivatives (proof is the same as for continuous functions, the uniform convergence also ensures that the limits of the derivatives are again the respective derivatives of the limits).
3. There is no norm such that induces convergence on each derivative on the space of infinitely-differentiable functions (see exercises).
4. We also know the spaces of continuous (or differentiable, etc.) functions with compact support.

1.6 Compactness

Theorem 1.42 (Compactness in metric spaces)**.** *Let A be a subset of a metric space (X, d). Then the following are equivalent:*

1. *A is (covering) compact, that is, every open cover of A contains a finite subcover.*
2. *A is sequentially compact, that is, every sequence in A admit a converging subsequence.*
3. *(A, d) is complete and precompact, that is, for all $\varepsilon > 0$ there is a finite cover of A with ε-balls.*

Remark. compact $\hat{=}$ covering compact.

Proof. 1. $\Rightarrow$ 2. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence without accumulation point. Then, $\forall y \in A : \exists r_y > 0 :$

$$N_y := \left\{k \in \mathbb{N} \mid x_k \in B_{r_y}(y)\right\} \text{ is finite.}$$

The balls $B_{r_y}(y)$ comprise an open cover of A. Therefore $\exists \{y_k\}_{k=1}^N :$

$$\bigcup_{k=1}^N B_{r_{y_k}}(y_k) \supset A.$$

This is a contradiction to our assumption, since otherwise there would only be finitely many elements in the sequence.

2. $\Rightarrow$ 3. Fist, we show completeness: By 2., we have that every Cauchy sequence in A admits an accumulation point. However, a Cauchy sequence admits *at most* one accumulation point. By 2., the accumulation point is *in* A. Therefore, the sequence converges to an element of A.

For compactness, we argue by contradiction: Assume $\exists\varepsilon > 0 :$ no finite cover with ε-balls exists. Therefore there exists a sequence $(x_k)_{k\in\mathbb{N}} \subset A :$

$$x_{k+1} \in A\setminus \bigcup_{j=1}^k B_\varepsilon(x_j).$$

So (x_k) has no accumulation point.

3. $\Rightarrow$ 1. Let $(U_i)_{i\in I}$ be an open cover of A. We define

$$\mathcal{B} := \left\{B \subset A \mid J \subset I, B \subset \bigcup_{i\in J} U_i \Rightarrow |J| = \infty\right\}.$$

We want to show $A \notin \mathcal{B}$. A is precompact, so $\forall B \in \mathcal{B}, \varepsilon > 0$ there is a cover

$$B \subset \bigcup_{i=1}^{n_\varrho} B_\varepsilon(x_i).$$

Hence, for some i (depending on ε):

$$B_\varepsilon(x_i) \cap B \in \mathcal{B}.$$

Assume $A \in \mathcal{B}$. So inductively, we can take $\varepsilon = \frac{1}{k}, k \in \mathbb{N}$ and get existence of $x_k \in X$ and sets:

$$B_1 := A, \quad B_k := B_{1/k}(x_k) \cap B_{k-1} \in \mathcal{B} \;\; \forall k \geq 2.$$

Take $y_k \in B_k, k \in \mathbb{N}$. For $k \leq l$, we have

$$y_k, y_l \in B_{1/k}(x_k) \Rightarrow d(y_k, y_l) \leq \frac{2}{k}.$$

$\Rightarrow (y_k)_{k\in\mathbb{N}}$ is a Cauchy sequence. Since A is complete, $\exists y \in A$:

$$\varepsilon_k := d(y_k, y) \xrightarrow[k\to\infty]{} 0.$$

But we have $y \in U_{i_0}$ for some i_0 and hence for k large enough

$$B_k \subset B_{1/k}(x_k) \subset B_{2/k}(y_k) \subset B_{\frac{2}{k}+\varepsilon_k}(y) \subset U_{i_0}.$$

Therefore $B_k \notin \mathcal{B}$, which is a contradiction. □

Theorem 1.43 (Riesz)**.** *Let X be a Banach space. then $\overline{B_1(0)}$ is compact, if and only if X is finite dimensional.*

Proof. "$\Leftarrow$" Heine-Borel.

"$\Rightarrow$" By precompactness of $\overline{B_1(0)}$, there exists a cover

$$\overline{B_1(0)} \subset \bigcup_{k=1}^{m} B_{1/2}(y_k).$$

Take

$$Y := \operatorname{span}\left(\{y_k\}_{k=1}^m\right).$$

Y is finite dimensional, so by Theorem 1.33 it is closed in X. We assume $Y \neq X$.

Claim. We have $\forall \theta \in (0,1) : \exists x_\theta \in X, \|x_\theta\| = 1$:

$$\operatorname{dist}(x_\theta, Y) \geq \theta.$$

Proof. Take $x \in X \backslash Y$. Then

$$\operatorname{dist}(x, Y) > 0.$$

There exists $y_0 \in Y$:

$$0 < \|x - y_0\| \leq \frac{1}{\theta} \operatorname{dist}(x, Y).$$

Take

$$x_\theta := \frac{x - y_0}{\|x - y_0\|}.$$

Then, for all $y \in Y$:

$$\begin{aligned}
\|x_\theta - y\| &= \frac{1}{\|x - y_0\|} \|x - \overbrace{(y_0 + \|x - y_0\| y)}^{\in Y}\| \\
&\geq \frac{1}{\|x - y_0\|} \operatorname{dist}(x, Y) \\
&\geq \frac{\operatorname{dist}(x, Y)}{\frac{1}{\theta} \operatorname{dist}(x, Y)} \\
&= \theta.
\end{aligned}$$

This proves the claim. □

Remark. Such an x_θ is called *almost orthogonal element.*)

However, for some $j \in \{1, ..., n\}$:

$$x_\theta \in B_{1/2}(y_j).$$

With $\frac{1}{2} < \theta < 1$ there is a contradiction to

$$\operatorname{dist}(x_\theta, Y) \geq \theta.$$

□

2 Lebesgue-Spaces, Part 1

In short, these are the spaces of functions f, such that

$$\int |f|^p \, d\mu < \infty.$$

2.1 A reminder of theorems from measure theory

Proofs and theorems for this paragraph and further read on measure theory:

- Fonseca & Leoni: Modern Methods in the calculus of of variations: L^p spaces
- Brokate & Kersting: Maß & Integral
- Evans & Gariepy: Measure Theory and Fine Properties of Functions ("tough read")

We consider $(\Omega, \mathcal{M}, \mu)$ be a *σ-finite measure space*, i.e. Ω is a set and $\mathcal{M}$ is a σ-algebra in Ω, μ is a measure and $(\Omega, \mathcal{M}, \mu)$ is *σ-finite*, i.e. $\exists \left(\Omega_n\right)_n$ countable family of sets in $\mathcal{M}$ such that

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n \quad \text{and} \quad \mu\left(\Omega_n\right) < \infty \quad \forall n.$$

Sets $E \in \mathcal{M}$ such that $\mu(E) = 0$ are called *null sets*. We say that the same property holds *almost everywhere (a.e.)* on Ω, if the property holds on all $x \in \Omega \backslash E$ for some null set E. In this chapter, we identify functions that agree almost everywhere. We often write $\int f$ instead of $\int_\Omega f \, d\mu$. for the integral over a measurable function.

We will need the following facts about integration.

Definition. $L^1(\mu) := \left\{ f : \Omega \to K, f \text{ measurable} : \int_\Omega |f| \, d\mu < \infty \right\}.$

Theorem (Monotone convergence theorem (Beppo-Levi))**.** *Let $\left(f_n\right)_{n\in\mathbb{N}}$ be a sequence in L^1 :*

1. $f_1 \le f_2 \le ... \le f_n \le ...$ *a.e. on* Ω.
2. $\sup_n \int f_n < \infty$.

Then $f_n(x)$ converges a.e. on Ω to a finite limit $f(x)$. We have $f \in L^1$ and

$$\|f_n - f\|_{L^1} := \int |f_n - f| \xrightarrow[n\to\infty]{} 0.$$

Theorem (Dominated convergence (Lebesgue))**.** *Let $\left(f_n\right)_{n\in\mathbb{N}}$ be a sequence of functions in L^1, such that*

1. $f_n(x) \to f(x)$ *a.e. on* Ω.
2. $\exists g \in L^1 : |f_n(x)| \le g(x) \quad \forall n \in \mathbb{N},$ *a.e. on* Ω.

Then, $f \in L^1$ and

$$\|f_n - f\|_{L^1} \xrightarrow[n\to\infty]{} 0.$$

Lemma (Fatou's lemma)**.** *Let $\left(f_n\right)_{n\in\mathbb{N}}$ be a sequence in L^1 :*

1. $\forall n : f_n \ge 0$ *a.e. on* Ω.
2. $\sup_n \int f_n < \infty$.

For almost all $x \in \Omega$, set

$$f(x) := \liminf_{n\to\infty} f_n(x) \le \infty.$$

Then, $f \in L^1$ and

$$\int f \le \liminf_{n\to\infty} \int f_n.$$

Now let $(\Omega_1, \mathcal{M}_1, \mu_1)$ and $(\Omega_2, \mathcal{M}_2, \mu_2)$ be two measure spaces. There is a canonical way to define a measure on the product space $\Omega = \Omega_1 \times \Omega_2$:

Theorem (Tonelli)**.** *Let $F : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a measurable function, such that*

1. $\int_{\Omega_2} |F(x,y)| \, d\mu_2 < \infty$ *for a.e.* $x \in \Omega_1$ *and*
2. $\int_{\Omega_1} \left(\int_{\Omega_2} |F(x,y)| \, d\mu_2 \right) d\mu_1 < \infty.$

Then $F \in L^1(\Omega_1 \times \Omega_2)$.

Theorem (Fubini)**.** *Let $F \in L^1(\Omega_1 \times \Omega_2)$. Then*

$$F(x,\cdot) \in L^1(\Omega_2) \quad \text{for a.e. } x \in \Omega_1$$

and

$$\int_{\Omega_2} F(x,y) \, d\mu_2 \in L^1(\Omega_1)$$

end vice versa. Moreover

$$\int_{\Omega_1} \left(\int_{\Omega_2} F(x,y) \, d\mu_2 \right) d\mu_1 = \int_{\Omega_2} \left(\int_{\Omega_1} F(x,y) \, d\mu_1 \right) d\mu_2 = \iint_{\Omega_1 \times \Omega_2} F(x,y) \, d(\mu_1 \otimes \mu_2).$$

Notation. A basic example is the case, where $\Omega = \mathbb{R}^n$ and μ is the Lebesgue measure on $\mathbb{R}^n$. We denote by $C_c(\mathbb{R}^n)$ the continuous functions with compact support on $\mathbb{R}^n$. The support of a (continuous) function f is the closure of the set $\{f \neq 0\}$

Theorem (density of continuous functions)**.** *The space $C_c(\mathbb{R}^n)$ lies dense in $L^1(\mathbb{R}^n)$, i.e. $\forall f \in L^1(\mathbb{R}^n), \varepsilon > 0 \exists \bar{f} \in C_c(\mathbb{R}^n)$ such that*

$$\left\| f - \bar{f} \right\|_{L^1} \leq \varepsilon.$$

2.2 Definition and basic properties of L^p

Definition 2.1. Let $p \in (1, \infty)$. Set

$$L^p(\mu) := \left\{ f : \Omega \to \mathbb{K} \mid f \text{ is measurable and } |f|^p \in L^1(\mu) \right\}$$

with

$$\|f\|_{L^p} := \|f\|_p := \left(\int_\Omega |f|^p \, d\mu \right)^{1/p}.$$

Definition 2.2. We set

$$L^\infty(\mu) := \{ f : \Omega \to \mathbb{K} \mid f \text{ is measurable and } \exists C > 0 : |f(x)| \leq C \text{ a.e. on } \Omega \},$$

where

$$\|f\|_{L^\infty} := \|f\|_\infty := \inf \{ C > 0 \mid |f(x)| \leq C \text{ a.e. on } \Omega \}.$$

Remark. For $\Omega \subset \mathbb{R}^n$ we often write $L^p(\Omega)$ for the L^p-space on Ω with the Lebesgue measure. We sometimes implicitly consider a function in $L^p(\Omega)$ to be a function of $L^p(\mathbb{R}^n)$ by continuation by zero outside Ω. If no ambiguity should occur, we sometimes leave out the measure or set altogether.

Definition 2.3. We set for $1 \leq p \leq \infty$

$$L^p_{loc}(\Omega) := \{ f \colon \Omega \to \mathbb{K} | f \in L^p(K) \text{ for any } K \subset \Omega \text{ compact} \}$$

and we write $f_n \to f$ in L^p_{loc} if f_n converges to f in L^p on any compact subset of Ω.

Notation. Let $1 \leq p \leq \infty$. We denote by p' the conjugate exponent:

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 2.4 (Hölders's inequality)**.** *Assume $f \in L^p, g \in L^{p'}, 1 \leq p \leq \infty$. Then $f \cdot g \in L^1$ with*

$$\int |f \cdot g| \leq \|f\|_{L^p} \cdot \|g\|_{L^{p'}} .$$

Remark. 1. A useful consequence of Hölders's inequality is the following:

Take $f_1, f_2, ..., f_k$ funktions: $f_i \in L^{p_i}$:

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_k} \leq n$$

Then $f = f_1 \cdot f_2 \cdots f_k \in L^p$ and

$$\|f\|_{L^p} \leq \|f_1\|_{L^{p_1}} \cdots \|f_k\|_{L^{p_k}} .$$

In particular, if $f \in L^p \cap L^q, 1 \leq p \leq q \leq \infty$, then

$$f \in L^r \quad \forall p \leq r \leq q$$

and we have the following "interpolation inequality"

$$\|f\|_{L^r} \leq \|f\|_{L^p}^{\alpha} \|f\|_{L^q}^{1-\alpha} ,$$

where

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{p}, \quad 0 \leq \alpha \leq 1.$$

2. We also note that for $1 \leq p < \infty$ we have $((|a| + |b|)^p \leq 2^p(|a|^p + |b|^p)$.

Proof of Theorem 2.4. See exercises. □

Theorem 2.5. *L^p is a vector space and $\|\cdot\|_{L^p}$ is a norm for any $1 \leq p \leq \infty$.*

Proof. The only difficult thing for to check $p < \infty$ is (again) triangle inequality, the checking of which is an exercise. For $p = \infty$ the triangle inequality is easy, but make sure that zero norm implies the function vanishes a.e. □

Theorem 2.6 (Fischer-Riesz)**.** *L^p is a Banach space for any $1 \leq p \leq \infty$.*

Proof. $p = \infty$. Take a Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in L^∞. For all $k \geq 1$ there exists N_k :

$$\|f_m - f_n\|_{L^\infty} < \frac{1}{k} \quad \forall n, m \geq N_k.$$

Thus there exist null sets E_k :

$$|f_m(x) - f_n(x)| \leq \frac{1}{k} \quad \forall x \in \Omega \backslash E_k, n, m \geq N_k. \tag{$*$}$$

Take

$$E = \bigcup_{k\in\mathbb{N}} E_k.$$

We have $\mu(E) = 0$. We see that for all $x \in \Omega\backslash E$, $f_n(x)$ is Cauchy in $\mathbb{K}$. Thus

$$f_n(x) \to f(x) \quad \forall x \in \Omega\backslash E.$$

Taking the limit $n, m \to \infty$ in $(*)$ yields

$$|f(x) - f_n(x)| \leq \frac{1}{k} \quad \forall x \in \Omega\backslash E, n \geq N_k.$$

$$\Rightarrow \quad f_n \xrightarrow{L^\infty} f.$$

$1 \leq p < \infty$. Let (f_n) be a Cauchy sequence in L^p. It is enough to show convergence on a subsequence (why?), so wlog. we may assume

$$\|f_{k+1} - f_k\|_{L^p} \leq \frac{1}{2^k} \quad \forall k \geq 1.$$

Let

$$g_n(x) := \sum_{k=1}^{n} |f_{k+1}(x) - f_k(x)|.$$

We have

$$\|g_n\|_{L^p} \leq 1.$$

g is also increasing, so by the monotone convergence theorem (Beppo-Levi), $(g_n(x))_{n\in\mathbb{N}}$ has a finite limit $g(x)$ a.e. on Ω such that $g \in L^p$. We have for $m \geq n \geq 2$:

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f_{m-1}(x)| + ... + |f_{n+1}(x) - f_n(x)| \leq g(x) - g_{n-1}(x).$$

Therefore, we have that $f_n(x)$ is a Cauchy sequence a.e. on Ω. It follows that $f_n(x)$ converges to a limit $f(x)$. So a.e. on Ω :

$$|f(x) - f_n(x)| \leq g(x) \quad \forall n \geq 2.$$

Thus, $f \in L^p$ and by dominated convergence,

$$\|f_n - f\|_{L^p} \to 0,$$

since $|f_n(x) - f(x)|^p \to 0$ a.e. and $|f_n - f|^p \leq g^p \in L^1$. □

Theorem 2.7. *Let f_n be a sequence in L^p and let $f \in L^p$:*

$$\|f_n - f\|_{L^p} \xrightarrow[n\to\infty]{} 0.$$

Then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ and $h \in L^p$:

1. *$f_{n_k} \to f$ a.e. on Ω.*
2. *$|f_{n_k}| \leq h$ a.e. on Ω.*

Example. Take

$$f_n(x) := \begin{cases} 1, & x \in I_n \\ 0, & \text{otherwise} \end{cases},$$

where $(I_n)_{n\in\mathbb{N}}$ is a sequence of intervals that repeatedly move through all of $[0,1]$ (i.e. every point of $[0,1]$ is in at least one of the intervals of each "round") while steadily getting smaller.

This sequence converges to zero in $L^p, p < \infty$, but it converges nowhere pointwise.

Proof of Theorem 2.7. $p = \infty$ is obvious. Take $1 \leq p < \infty$. Since $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, we consider a relabelled subsequence $(f_k)_{k\in\mathbb{N}}$:

$$f_k \xrightarrow[k\to\infty]{\text{a.e.}} \bar{f} \in L^p.$$

From the previous proof, we know

$$|\bar{f} - f_k| \leq g \in L^p \quad \text{a.e. on } \Omega.$$

By dominated convergence, we know that

$$f_k \xrightarrow{L^p} \bar{f}$$

and thus

$$f = \bar{f} \quad \text{a.e.}$$

In addition, we have

$$|f_k(x)| \leq |\bar{f}(x)| + g(x),$$

which implies 2. □

2.3 Density of smooth functions and separability

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^n$ open. The space $C_c(\Omega)$ is dense in*

$$L^p(\Omega) := L^p(\Omega, \mu) \quad \forall 1 \leq p < \infty, \quad (\mu \text{ Lebesgue measure}).$$

Notation. We define the complex sign function

$$\operatorname{sgn} : \mathbb{C} \to B_1(0), \quad \operatorname{sgn} z := \begin{cases} \frac{z}{|z|}, & z \neq 0 \\ 0, & z = 0 \end{cases},$$

the truncation operator

$$T_n : \mathbb{C} \to \mathbb{C}, \quad T_n z := \operatorname{sgn} z \cdot \min\{|z|, n\},$$

and for a set $E \subset \Omega$, the characteristic function

$$\chi_E : \Omega \to \mathbb{R}, \quad \chi_E(x) := \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}.$$

Proof of Theorem 2.8.
Claim. Given $f \in L^p(\Omega), \varepsilon > 0$, there exists $g \in L^\infty(\Omega)$ and a compact set $K \subset \Omega$ such that

$$g(x) = 0 \quad \forall x \in \Omega \backslash K, \quad \text{and} \quad \|g - f\|_{L^p} < \varepsilon.$$

Proof. Take $(K_n)_{n\in\mathbb{N}}$ an increasing sequence of compact subsets of Ω such that $\bigcup_{n\in\mathbb{N}} K_n = \Omega \setminus E$ for some null set E (this is possible since Lebesgue measurable sets can be approximated from the inside by compact sets).

$$\chi_n := \chi_{K_n}, \quad f_n := \chi_n T_n f.$$

We have

$$f_n \xrightarrow{\text{a.e.}} f.$$

By dominated convergence, we see

$$\|f_n - f\|_{L^p} \xrightarrow[n\to\infty]{} 0.$$

So it suffices to take $g = f_n$ for n large enough. □

By the density of $C_c(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$), $\forall \delta > 0 : \exists g_1 \in C_c(\Omega)$ such that

$$\|g - g_1\|_{L^1} < \delta.$$

To see this if $\Omega \neq \mathbb{R}^n$, first consider the function g as a function on all of $\mathbb{R}^n$ (by continuation by zero outside Ω), then approximate with a functio $\tilde{g}_1$ by density of C_c. Then, since Ω is open and g vanishes outside a compact subset K_n of Ω, we have $\operatorname{dist}(K_n, \partial\Omega) = c > 0$. We can thus find K compact and U open with, $K_n \subset U \subset K \subset \Omega$ and have space to cut off $\tilde{g}_1$ in a continuous fashion in between those two sets such that $\operatorname{supp}(g_1) \subset K$ by taking $g_1(x) = \phi(x)\tilde{g}_1(x)$ where $\phi = 1$ on K_n, $\phi = 0$ on $\Omega \setminus K$, and ϕ continuous. Note that this cutting off can not increase the L^1-distance to g as g vanishes outside K_n anyhow.

We can furthermore assume

$$\|g_1\|_{L^\infty} \leq \|g\|_{L^\infty}$$

(otherwise, replace g_1 with $T_{\|g\|_{L^\infty}} g_1$ and note that the distance again only decreases) and get

$$\begin{aligned} \|g - g_1\|_{L^p} &\leq \|g - g_1\|_{L^1}^{1/p} \cdot \|g - g_1\|_{L^\infty}^{1-1/p} \qquad \text{(the useful interpolation inequality)} \\ &< \delta^{1/p} \left(2\|g\|_{L^\infty}\right)^{1-1/p}, \end{aligned}$$

so by picking δ small enough, the theorem is proven. □

Corollary 2.9. *$L^p(\Omega, \mu)$, $\Omega \subset \mathbb{R}^n$ measurable (not necessarily open), μ Lebesgue, is separable for $1 \leq p < \infty$.*

Proof. First, consider the functions defined on all of $\mathbb{R}^n$ by continuation by zero. Then approximate by a function in $C_c(\mathbb{R}^n)$. Functions in $C_c(\mathbb{R}^n)$ are uniformly continuous, so the can be approximated in L^p by finite step functions. The finite step functions taking (complex) rational values on appropriate sets form a countable dense subset of the finite step functions. If necessary, cut off the step functions outside of Ω by multiplication with χ_Ω. ☐

Remark. Separability also holds for a more general measure space Ω, if that measure space is separable (i.e. its σ-algebra is countably generated).

Remark. $L^\infty((0,1))$ is not separable. To see this, consider $f_\alpha = \chi_{(\alpha,1)}$ for $\alpha \in (0,1)$ and note that $||f_\alpha - f_\beta||_{L^\infty} = 1$ for $\alpha \neq \beta$ and that there are more than countably many real numbers $\alpha \in (0,1)$. Therefore, $\mathcal{B} = \{B_{1/2}(f_\alpha) | \alpha \in (0,1)\}$ is an uncountable and non-intersecting collection of 1/2-balls in $L^\infty((0,1))$. Thus there can not exist a countable, dense subset. Similarly, $L^1(\Omega)$ is not separable for any open set $\Omega \subset \mathbb{R}^n$.

Notation. In the following we write $V \subset\subset \Omega$ for V open and $V \subset K \subset \Omega$ for some compact set $K \subset \Omega$.

Definition 2.10.

1. We define $\eta \in C^\infty(\mathbb{R}^n)$ by
$$\eta(x) = \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$
where $C > 0$ is such that $\int_{\mathbb{R}}^n \eta = 1$.
2. For $\epsilon > 0$ set
$$\eta_\epsilon = \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right).$$

We call η *standard mollifier.*

Remark. Note that $\int_{\mathbb{R}^n} \eta_\epsilon = 1$ and $\operatorname{supp}(\eta_\epsilon) = B_\epsilon(0)$.

Definition 2.11. For $f \in L^1_{loc}(\Omega)$, we define its mollification
$$f^\epsilon(x) = \int_{\mathbb{R}^n} \eta_\epsilon(x-y) f(y) dy = \int_{B_\epsilon(0)} \eta_\epsilon(y) f(x-y) dy$$
for $x \in \Omega$. Note that we implicitly continued f by zero outside of Ω for the integration.

Theorem 2.12 (Properties of mollifiers)**.** *Consider $\Omega \subset \mathbb{R}^n$ open, $f \in L^1_{loc}(\Omega)$. We have*

1. *$f^\epsilon \in C^\infty(\Omega)$*
2. *If $f \in C(\Omega)$, then $f^\epsilon \to f$ uniformly on compact subsets of Ω.*
3. *For $1 \leq p < \infty$, $f \in L^p_{loc}(\Omega)$, $f^\epsilon \to f$ in $L^p_{loc}(\Omega)$.*

Proof.

1. Let $x \in \Omega, i \in \{1, ..., n\}, h > 0$, so that $x + he_i \in \Omega$.
We have
$$\begin{aligned} \frac{f^\epsilon(x+he_i) - f^\epsilon(x)}{h} &= \frac{1}{\epsilon^n} \int_{\mathbb{R}^n} \frac{1}{h}\left(\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right) f(y) dy \\ &= \frac{1}{\epsilon^n} \int_V \frac{1}{h}\left(\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right) f(y) dy \end{aligned}$$
for some $V \subset\subset \mathbb{R}^n$.
We have
$$\frac{1}{h}\left(\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right) \to \frac{1}{\epsilon} \frac{\partial}{\partial x_i} \eta\left(\frac{x-y}{\epsilon}\right)$$
uniformly on V, $\frac{\partial f^\epsilon}{\partial x_i}$ therefore exists and equals
$$\int_\Omega \frac{\partial \eta_\epsilon}{\partial x_i}(x-y) f(y) dy$$
The same argument holds for any other derivative.

2. Let $K \subset V \subset\subset \Omega$ be compact and $\epsilon > 0$. By uniform continuity of f on V, there exists $\delta > 0$ with
$$|f(x-y) - f(x)| < \epsilon \quad \forall x \in K, y \in B_\delta(0).$$
For $x \in K$ and δ so small that $\bigcup_{z \in K} B_\delta(z) \subset V$ we now obtain
$$|f^\delta(x) - f(x)| \leq \epsilon \int \eta_\epsilon = \epsilon.$$

3. Let $1 \leq p < \infty$, $f \in L^p_{loc}(\Omega)$ and $V \subset\subset W \subset\subset \Omega$. Claim: For $\epsilon > 0$ sufficiently small we have
$$\|f^\epsilon\|_{L^p(V)} \leq \|f\|_{L^p(W)}$$
Let $x \in V$. We have
$$|f^\epsilon(x)| = \left|\int_{B_\epsilon(x)} \eta_\epsilon(x-y) f(y) dy\right| \leq \int B_\epsilon(x) \eta_\epsilon^{1-1/p}(x-y) \eta_\epsilon^{1/p}(x-y)|f(y)|dy$$
$$\leq \underbrace{\left|\int_{B_\epsilon(x)} \eta_\epsilon(x-y)dy\right|^{1-1/p}}_{=1} \left|\int_{B_\epsilon(x)} \eta_\epsilon^{1/p}(x-y)|f(y)|^p dy\right|^{1/p}$$
$$\Rightarrow \int_V |f^\epsilon(x)|^p dx = \int_V \left(\int_{B_\epsilon(x)} \eta_\epsilon^{1/p}(x-y)|f(y)|^p dy\right) dx$$
$$\leq \int_W |f(y)|^p \left(\int_{B_\epsilon(y)} \eta_\epsilon(x-y)dx\right) dy = \int_W |f(x)|^p dx$$
if $B_\epsilon(x) \subset W, \forall x \in V$.
Now finally choose again $V \subset\subset W \subset\subset \Omega$, $\delta > 0$ and $g \in C_c(W)$ (by Theorem 2.8), so that
$$\|f - g\|_{L^p(W)} < \delta$$
Then
$$\|f^\epsilon - f\|_{L^p(V)} \leq \|f^\epsilon - g^\epsilon\|_{L^p(V)} + \|g^\epsilon - g\|_{L^p(V)} + \|g - f\|_{L^p(V)}$$
$$\leq 2\|f - g\|_{L^p(W)} + \|g^\epsilon - g\|_{L^p(V)} \leq 2\delta + \|g^\epsilon - g\|_{L^p(V)}$$
$$\|g^\epsilon - g\|_{L^p(V)} \to 0, (\epsilon \to 0),$$
since $g^\epsilon \to g$ uniformly on V.
$$|g^\epsilon(x) - g(x)| \leq C \int_{B_\epsilon(x)} |g(x) - g(y)|dy \to 0$$
And the the claim follows.

□

Remark. By first multiplying with χ_n as in the proof of Theorem 2.8, we also get density of $C_c^\infty(\Omega)$ in $L^p_{loc}(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open. Remember that we use the Fréchet-Metric on L^p_{loc}, i.e., $d(f,g) = \sum_{j \in \mathbb{N}} 2^{-j} \frac{||g|_{K_j}||_{L^p}}{1+||g|_{K_j}||_{L^p}}$ for compact $K_j \subset \Omega$ such that $\Omega = \bigcup_{j \in \mathbb{N}} K_j$ modulo a null set. For bounded Ω this is equivalent to the usual distance from $||\cdot||_{L^p(\Omega)}$ (try it!).

3 Continuous linear maps

We take $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ to be normed spaces and consider a linear map $A : X \to Y$.

Definition 3.1. A is called *bounded*, if

$$\sup_{\|x\|_X \leq 1} \|Ax\|_Y < \infty.$$

Theorem 3.2. *The following are equivalent:*

1. *A is continuous in $0 \in X$.*
2. *A is continuous.*
3. *A is Lipschitz continuous (i.e. $\exists L > 0 : \forall x, y \in X : \|Ax - Ay\| \leq L \|x - y\|$).*
4. *A is bounded.*

Proof. 4. $\Rightarrow$ 3. Let $x_1 + x_2 \in X$.

$$\begin{aligned} \|Ax_1 - Ax_2\| &= \|A(x_1 - x_2)\| \\ &= \left\| A \frac{x_1 - x_2}{\|x_1 - x_2\|} \right\| \cdot \|x_1 - x_2\| \\ &\leq \underbrace{\sup_{\|z\| \leq 1} \|Az\|}_{=:L \leq \infty} \|x_1 - x_2\|. \end{aligned}$$

3. $\Rightarrow$ 2. Trivial.

2. $\Rightarrow$ 1. Trivial.

1. $\Rightarrow$ 4. By contradiction: Assume A is not bounded, that is

$$\exists (x_k)_{k \in \mathbb{N}} \text{ in } X : \|x_k\| = 1, \|Ax_k\| \to \infty.$$

We can construct

$$z_k := \|Ax_k\|^{-1} x_k \to 0.$$

However, we have $\|Az_k\| = 1 \ \forall k$, contradicting sequential continuity of A at 0. $\square$

Corollary 3.3. *If $\dim X < \infty$, then all linear operators $A : X \to Y$ are continuous.*

Example. Take

$$\begin{aligned} (X, \|\cdot\|_X) &:= (C([0,1]), \|\cdot\|_{L^1}), \\ (Y, \|\cdot\|_Y) &:= (C([0,1]), \|\cdot\|_\infty). \end{aligned}$$

Then

$$\mathrm{id} : X \to Y, \quad f \mapsto f$$

is not continuous. To see this, consider $f_k(t) = t^k$.

Definition 3.4.

$$\begin{aligned} \mathscr{L}(X, Y) &:= \{A : X \to Y \mid A \text{ is linear and continuous}\}, \\ \|A\|_{\mathscr{L}(X,Y)} &:= \sup_{\|x\|_X \leq 1} \|Ax\|_Y. \end{aligned}$$

If $Y = \mathbb{K}$, then we call $X' := \mathscr{L}(X, \mathbb{K})$, the *space of (continuous linear) functionals* or the *(topological) dual space* of X.

Proposition 3.5. $\left(\mathscr{L}(X,Y), \|\cdot\|_{\mathscr{L}(X,Y)}\right)$ *is a normed space. For all* $A \in \mathscr{L}(X,Y), x \in X$ *:*

$$\|Ax\|_Y \le \|A\|_{\mathscr{L}(X,Y)} \|x\|_X .$$

Proof. Exercise. □

Theorem 3.6. *Let* X, Y, Z *be normed spaces,* $A \in \mathscr{L}(X,Y), B \in \mathscr{L}(Y,Z)$. *Then* $BA \in \mathscr{L}(X,Z)$ *with*

$$\|BA\|_{\mathscr{L}(X,Z)} \le \|B\|_{\mathscr{L}(Y,Z)} \|A\|_{\mathscr{L}(X,Y)} .$$

Furthermore, the mapping

$$(A,B) \mapsto BA$$

is continuous.

Proof. Let $\|x\| = 1$. Then we have

$$\begin{aligned}\|BAx\|_Z &\le \|B\|_{\mathscr{L}(Y,Z)} \cdot \|Ax\|_Y \\ &\le \|B\|_{\mathscr{L}(Y,Z)} \cdot \|A\|_{\mathscr{L}(X,Y)} \cdot \underbrace{\|x\|_X}_{=1} .\end{aligned}$$

Continuity of $(A,B) \mapsto BA$ follows from

$$\begin{aligned}\|B_1A_1 - B_2A_2\|_{\mathscr{L}(X,Z)} &= \|B_1(A_1 - A_2) + (B_1 - B_2)A_2\|_{\mathscr{L}(X,Z)} \\ &\le \|B_1\|_{\mathscr{L}(Y,Z)} \|A_1 - A_2\|_{\mathscr{L}(X,Y)} + \|A_2\|_{\mathscr{L}(X,Y)} \|B_1 - B_2\|_{\mathscr{L}(Y,Z)} \\ &\xrightarrow[\|A_1-A_2\|_{\mathscr{L}(X,Y)}, \|B_1-B_2\|_{\mathscr{L}(Y,Z)} \to 0]{} 0.\end{aligned}$$

□

Theorem 3.7. *If* Y *is a Banach space, then so is* $\mathscr{L}(X,Y)$.

Remark. Note that we do not require completeness of X.

Proof of Theorem 3.7. Let $(A_k)_{k\in\mathbb{N}}$ be a Cauchy sequence. It follows that

$$\sup_{\|x\|_X \le 1} \|A_kx - A_lx\|_Y \xrightarrow[k,l\to\infty]{} 0$$

and for any fixed $x \in X\setminus\{0\}$, we have

$$\frac{1}{\|x\|_X} \|A_kx - A_lx\|_Y \xrightarrow[k,l\to\infty]{} 0.$$

That means, $(A_kx)_{k\in\mathbb{N}}$ is a Cauchy sequence in Y. Thus there exists a limit

$$Y \ni y(x) := \lim_{k\to\infty} A_kx.$$

Claim (1). $y(x)$ is linear in x.

Proof (1). Take $\alpha \in \mathbb{K}, x_1, x_2 \in X$. We have

$$\begin{aligned}y(\alpha x_1 + x_2) &= \lim_{k\to\infty} A_k(\alpha x_1 + x_2) \\ &= \lim_{k\to\infty} (\alpha A_kx_1 + A_kx_2) \\ &= \alpha \lim_{k\to\infty} A_kx_1 + \lim_{k\to\infty} A_kx_2 \\ &= \alpha y(x_1) + y(x_2).\end{aligned}$$

So y is a linear map and we write

$$y = Ax.$$

□

Claim (2). A is continuous.

Proof (2).

$$
\begin{aligned}
\|Ax\|_Y &= \left\|\lim_{k\to\infty} A_k x\right\|_Y \\
&= \lim_{k\to\infty} \|A_k x\|_Y \\
&\le \limsup_{k\to\infty} \|A_k\|_{\mathscr{L}(X,Y)} \|x\|_X \\
&\le L \|x\|_X ,
\end{aligned}
$$

since Cauchy sequences are bounded. □

Claim (3). $\|A_k - A\|_{\mathscr{L}(X,Y)} \xrightarrow[k\to\infty]{} 0$.

Proof (3).

$$
\begin{aligned}
\|Ax - A_k x\|_Y &= \left\|\lim_{l\to\infty} A_l x - A_k x\right\|_Y \\
&\le \limsup_{l\to\infty} \|A_l - A_k\|_{\mathscr{L}(X,Y)} \|x\|_X .
\end{aligned}
$$

If we take the sup over $\|x\| \le 1$, we get

$$
\|A_l - A_k\|_{\mathscr{L}(X,Y)} \le \limsup_{l\to\infty} \|A_l - A_k\| \to 0,
$$

since $(A_k)_{k\in\mathbb{N}}$ is a Cauchy sequence. □

□

Remark. For an interesting application see the exercise with the exponential and Neumann series and the connection to Voterra's integral equation.

4 Hahn-Banach theorem and some consequences

Question. *Is the dual space rich enough?*

Example. X vector space, take $x, y \in X, x \neq y$. Does there exist some $f \in X' : f(x) \neq f(y)$.

4.1 Analytic version of the theorem

Theorem 4.1 (Hahn-Banach)**.** *Let X be a vector space over the real numbers and take $p : X \to \mathbb{R}$ with the following properties:*

1. $p(\lambda x) = \lambda p(x) \quad \forall \lambda > 0, x \in X$ *(positive homogeneity)*
2. $p(x+y) \le p(x) + p(y) \quad \forall x, y \in X$ *(sublinearity)*

Furthermore, let $G \subset X$ be a linear subspace and let $g : G \to \mathbb{R}$ be linear with

$$
g(x) \le p(x) \quad \forall x \in G.
$$

Then there exists a linear map

$$
f : X \to \mathbb{R}
$$

such that

1. $f|_G = g$ *(that is $f(x) = g(x) \ \ \forall x \in G$).*
2. $f(x) \le p(x) \quad \forall x \in X.$

We say that f is a continuation (extension) *of g.*

Remark. On a normed vector space, the norm satisfies the requirements on p, whence the extension is a bounded linear functional.

Example. Series with increasing difficulty:

1. $X = \mathbb{R}^2 = \operatorname{span}\{(1,0),(0,1)\}$.
$$\begin{aligned} \tilde{x} \in G &:= \operatorname{span}\{(1,0)\}, \\ g(\tilde{x}) &= \tilde{x}, \\ p(x) &= \|x\|_2 = \left(x_1^2 + x_2^2\right)^{1/2}. \end{aligned}$$
Clearly, we have an extension f of g, namely
$$f(x) = x_1.$$
2. $X = L^2((0,1))$ Take
$$\begin{aligned} \phi \in X &\text{ such that } \|\phi\|_2 = 1, \\ G &:= \operatorname{span}\{\phi\} := \{\tilde{x} \in X \mid \tilde{x} = \lambda\phi, \lambda \in \mathbb{R}\}, \\ g(\tilde{x}) &:= \lambda \quad \forall \tilde{x} \in G, \\ p(x) &:= \|x\|_2. \end{aligned}$$
We can take
$$f(x) = \int_0^1 x(t)\,\phi(t)\,dt = (x,\phi)_{L^2}$$
(bound follows from Cauchy-Schwarz).
3. $X := L^1((0,1))$
$$\|x\|_1 = \int_0^1 |x(t)|\,dt.$$
Take $\phi \in X$, $\phi(t) = \frac{1}{2\sqrt{t}}$. We thus have $\|\phi\|_1 = 1$.
$$\begin{aligned} G &:= \operatorname{span}\{\phi\} = \{\tilde{x} \in X \mid \tilde{x} = \lambda\phi, \lambda \in \mathbb{R}\}, \\ g(\tilde{x}) &:= \lambda \quad \forall \tilde{x} \in G, \\ p(x) &:= \|\cdot\|_1. \end{aligned}$$
Note that
$$f(x) = \int_0^1 x(t)\,\phi(t)\,dt$$
does *not* work. However, the Hahn-Banach theorem guarantees the existence of such a functional
$$f : X \to \mathbb{R}, \quad \|f\| \le 1, \quad f\,|_G = g.$$
4. $X = L^\infty((0,1))$ Take the linear subspace of L^∞
$$\begin{aligned} G &:= \left(C((0,1)), \|\cdot\|_\infty\right), \\ g(\tilde{x}) &:= \tilde{x}\left(\frac{1}{2}\right), \\ p(x) &:= \|x\|_{L^\infty}. \end{aligned}$$
Existence of a continuation is guaranteed, even though a general function in L^∞ does not admit a "value at $1/2$".

Idea of the proof. We consider the set P of all continuations h of g on $D(h)$ (domain of h, subspace of X) and order the elements of P by set inclusion with respect to their domain $D(h)$, that is
$$h \le h' \quad :\Leftrightarrow \quad D(h') \supset D(h) \text{ and } h(x) = h'(x) \;\; \forall x \in D(h).$$
"All" that is left to do then is to find the "largest" extension.

Definition 4.2. 1. A set P with a relation $\leq$ is called *partially ordered*, if $\forall a, b, c \in P$:

(a) $a \leq a$ (Reflexivity)

(b) $(a \leq b$ and $b \leq c) \Rightarrow a \leq c$ (Transitivity)

(c) $(a \leq b$ and $b \leq a) \Rightarrow a = b$ (Antisymmetry)

2. A set Q with a relation $\leq$ is called *totally ordered*, if it is partially ordered and we further have
$$a \leq b \ \vee \ b \leq a \quad \forall a, b \in Q.$$

3. Let $R \subset P$, P partially ordered. An element $c \in P$ is called an *upper bound* of R, if
$$a \leq c \quad \forall a \in R.$$

4. An element $m \in P$ is called *maximal*, if for all $a \in P$, we have
$$m \leq a \quad \Rightarrow \quad m = a.$$

Remark.
- The canonical example for a partial order is set inclusion.
- If for $a, b \in P$ we neither have $a \leq b$ nor $b \leq a$, then a and b are called *not comparable.*

Zorn's Lemma. *Let $(P, \leq)$ be not empty, partially ordered and assume that any totally ordered subset Q of P admits an upper bound in P. Then P admits a maximal element.*

Remark. Zorn's Lemma is equivalent to the *axiom of choice:*

Axiom of choice. *Take A to be a set of non-empty sets ,then there exists a choice function F on A, such that*
$$F(X) \in X \quad \forall X \in A.$$

Proof of equivalence. Not here. □

Proof (Theorem 4.1). Consider
$$P := \{h : D(h) \to \mathbb{R} \mid G \subset D(h),\ h \text{ is a continuation of } g \text{ onto } D(h), h(x) \leq p(x) \quad \forall x \in D(h)\}$$
with the relation
$$h_1 \leq h_2 \quad :\Leftrightarrow \quad \begin{cases} D(h_1) \subset D(h_2) \\ h_1(x) = h_2(x) \quad \forall x \in D(h_1) \end{cases}.$$

Claim (1). P is not empty.

Proof (1). $g \in P$. □

Claim (2). "$\leq$" is a partial order on P.

Proof (2).
1. $a \leq a$ is clear by definition.
2. $a \leq b \wedge b \leq c \Rightarrow D(a) \subset D(b) \subset D(c)$ and $c\,|_{D(a) \subset D(b)} = a$.
3. $a \leq b \wedge b \leq a \Rightarrow D(a) = D(b) \wedge a\,|_{D(b)} = b \Rightarrow a = b$. □

Claim (3). Every totally ordered subset of P admits an upper bound in P.

Proof (3). Let
$$Q := \{h_j\}_{j \in I},$$
$$D(h) := \bigcup_{j \in I} D(h_j),$$
$$h(x) := h_j(x) \quad \forall x \in D(h_j).$$

Claim (3a). h is well defined.

Proof (3a). $h_1, h_2 \in Q$, $x \in D(h_1) \cap D(h_2)$. Since Q is totally ordered, we have either

$$h_1 \leq h_2 \text{ or } h_2 \leq h_1.$$

W.l.o.g. let's assume the former. $\Rightarrow$

$$x \in D(h_1) \Rightarrow x \in D(h_2). \qquad \square$$

Claim (3b). $h \in P$.

Proof (3b).

- $D(h)$ is a subspace of X, because

  $$x, y \in D(h) \Rightarrow \exists j \in I : x, y \in D(h_j).$$

  Therefore, we have

  $$(\alpha x + y) \in D(h) \quad \forall \alpha \in \mathbb{R}.$$

  Linearity of h follows in the same way.

- $h(x) \leq p(x)$ on $D(h)$ is also clear, since

  $$\exists j \in I : x \in D(h_j) \Rightarrow h(x) = h_j(x) \leq p(x).$$

- $h(x) = g(x)$ on G is again clear, since for any $j \in I$, we have $G \subset D(h_j)$ and $h_j = g$ on G.

- $h_j \leq h$ for all $h_j \in Q$ follows from the definition of h. $\square$

With that, all requirements of Zorn's Lemma are fulfilled and we can deduce existence of a maximal element $f \in P$. $\square$

Claim (4). f is the sought after continuation of g.

Proof (4). Since $f \in P$, the only thing that remains to show is that $D(f) = X$. We argue by contradiction.

Let $x_0 \in X$ such that $x_0 \notin D(f)$. Take

$$D\left(\tilde{f}\right) := D(f) + \operatorname{span}(\{x_0\}).$$

We have

$$\tilde{x} = x + t x_0 \in D\left(\tilde{f}\right) \quad \forall t \in \mathbb{R}.$$

Set

$$\tilde{f}(\tilde{x}) := f(x) + \alpha t$$

for a suitable $\alpha \in \mathbb{R}$ such that $\tilde{f} \in P$.

Claim (4a). Such an α exists.

Proof (4a). We only have to show that $\tilde{f} \leq p$ which is nothing but

$$\begin{aligned} & f(x) + \alpha t \leq p(x + t x_0) \quad \forall x \in D(f), t \in \mathbb{R} \\ \Leftrightarrow \quad & f\left(\frac{x}{t}\right) + \alpha \leq p\left(\frac{x}{t} + x_0\right) && (t > 0) \\ \Leftrightarrow \quad & f\left(\frac{x}{t}\right) - \alpha \leq p\left(\frac{x}{t} - x_0\right) && (t < 0) \end{aligned}$$

by positive homogeneity. This is equivalent to showing that

$$\forall x \in D(f) : \quad \begin{cases} f(x) + \alpha \leq p(x + x_0) \\ f(x) - \alpha \leq p(x - x_0) \end{cases} \tag{$*$}$$

due to homogeneity of p. However, (using sublinearity of p) we have that $\forall x, y \in D(f)$:

$$\begin{aligned} f(x) + f(y) &\leq p(x+y) \\ &\leq p(x+x_0) + p(y-x_0) \\ f(y) - p(y-x_0) &\leq p(x+x_0) - f(x). \end{aligned}$$

We can thus choose α in between the two sides of the inequality, which satisfies $(*)$. □

Therefore, we have $\tilde{f} \in P$. Note that $f \leq \tilde{f} \neq f$, in contradiction to maximality of f. □

This completes the proof. □

Corollary 4.3. *Let $(X, \|\cdot\|)$ be a normed $\mathbb{K}$-vector space ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) and let G be a subspace of X. Let*

$$\begin{aligned} g &: G \to \mathbb{K} \ \textit{linear}, \\ \|g\|_{G'} &:= \sup_{\substack{x \in G \\ \|x\|_X \leq 1}} |g(x)|. \end{aligned}$$

Then there exists a continuation $f \in X'$ of g such that

$$\|f\|_{X'} = \|g\|_{G'}.$$

Proof. Exercise. □

Corollary 4.4. *Let X be a $\mathbb{K}$-vector space. Then $\forall x_0 \in X : \exists f_0 \in X'$:*

$$\|f_0\|_{X'} = \|x_0\|_X \quad \textit{and} \quad f_0(x_0) = \|x_0\|^2.$$

Proof. Take

$$\begin{aligned} G &:= \mathbb{K} \cdot x_0 := \operatorname{span}(\{x_0\}), \\ g(tx_0) &:= t\|x_0\|^2. \end{aligned}$$

Then

$$\|g\|_{G'} = \|x_0\|.$$

With Corollary 4.3 the existence of f_0 follows. □

Corollary 4.5. *Let X be a $\mathbb{K}$-vector space, $x, y \in X, x \neq y$. Then there exists $f \in X'$ such that*

$$f(x) \neq f(y).$$

Proof. Corollary 4.4 with $x_0 = x - y$. □

Corollary 4.6. *Let X be a normed $\mathbb{K}$-vector space. Then for all $x \in X$ we have*

$$\|x\| = \sup_{\substack{f \in X' \\ \|f\|_{X'} \leq 1}} |f(x)| = \max_{\substack{f \in X' \\ \|f\|_{X'} \leq 1}} |f(x)|.$$

Proof. Let $x \neq 0$. We immediately have

$$\sup_{\substack{f \in X' \\ \|f\|_{X'} \leq 1}} |f(x)| \leq 1 \cdot \|x\|$$

Using Corollary 4.4 there exists $f_0 \in X'$ such that

$$\begin{aligned} \|f_0\|_{X'} &= \|x\|_X, \\ f_0(x) &= \|x\|_X^2. \end{aligned}$$

Now let

$$f_1 := \frac{1}{\|x\|_X} f_0,$$

and we have $\|f_1\|_{X'} = 1$ and $f_1(x) = \|x\|_X$. □

4.2 Separation of convex sets

In the following, let X be a normed $\mathbb{R}$-vector space (complex at the end of the section).

Definition 4.7. An *affine hyperplane* is a set of the form

$$H := \{x \in X \mid f(x) = \alpha\},$$

where $0 \not\equiv f : X \to \mathbb{R}$ is a linear (not necessarily continuous) map.

H is called *hyperplane of the equation*

$$[f = \alpha].$$

Proposition 4.8. *The hyperplane of the equation $[f = \alpha]$ is closed iff f is continuous.*

Proof. Exercise. □

Definition 4.9. Consider $A, B \subset X$. The hyperplane H of the equation $[f = \alpha]$ *separates* A from B if we have

$$f(x) \leq \alpha \leq f(y) \quad \forall x \in A, y \in B.$$

H *strictly separates* A from B if there exists $\varepsilon > 0$:

$$f(x) + \varepsilon \leq \alpha \leq f(y) - \varepsilon \quad \forall x \in A, y \in B.$$

Theorem 4.10 (First separation theorem)**.** *Consider $A, B \subset X$ convex, not empty and disjoint and let A be open. Then there exists a closed hyperplane that separates A from B.*

Lemma 4.11 (Minkowsky Functional)**.** *Let $C \subset X$ be convex and open with $0 \in C$. For all $x \in X$, we define*

$$p(x) := \inf\left\{\alpha > 0 \mid \alpha^{-1} \cdot x \in C\right\}.$$

Then p is positively homogeneous of degree 1 and sublinear with respect to vector addition. Furthermore, we have

1. $\exists M > 0 : \quad 0 \leq p(x) \leq M\|x\| \quad \forall x \in X.$
2. $C = \{x \in X : p(x) < 1\}.$

p is called Minkowsky functional *or* gauge *of C.*

Proof. Homogeneity is clear by definition.

1. Let $r > 0 : B_r(0) \subset C$. $\Rightarrow$

   $$p(x) \leq \frac{1}{r}\|x\|.$$

   Property 1. follows.

2. Let $x \in C$, thus $\exists \varepsilon > 0 : (1+\varepsilon)x \in C$. With that we have

   $$p(x) \leq \frac{1}{1+\varepsilon} < 1.$$

   Let $p(x) < 1$. Then $\exists 0 < \alpha < 1$:

   $$\alpha^{-1}x \in C.$$

   We thus have

   $$x = \alpha\underbrace{\left(\alpha^{-1}x\right)}_{\in C} + (1-\alpha)\underbrace{0}_{\in C} \in C. \qquad \text{(convex combination)}$$

   So 2. follows.

Sublinearity: Let $x, y \in X, \varepsilon > 0$. With homogeneity and 1., we have

$$\frac{x}{p(x)+\varepsilon} \in C, \quad \frac{y}{p(y)+\varepsilon} \in C.$$

Due to convexity, we have

$$\frac{tx}{p(x)+\varepsilon} + \frac{(1-t)y}{p(y)+\varepsilon} \in C \quad \forall t \in [0,1].$$

Take

$$t := \frac{p(x)+\varepsilon}{p(x)+p(y)+2\varepsilon}$$

and we get

$$\frac{x+y}{p(x)+p(y)+2\varepsilon} \in C.$$

By 2.

$$p\left(\frac{x+y}{p(x)+p(y)+2\varepsilon}\right) < 1.$$

So (by homogeneity) we have

$$p(x+y) \le p(x)+p(y)+2\varepsilon \quad \forall \varepsilon > 0.$$

The claim follows by arbitrary choice of ε ($\varepsilon \to 0$). □

Lemma 4.12 (Separation of a point and a convex set). *Take $C \subset X$ open, non-empty, and convex, take $x_0 \in X \setminus C$. Then there exists $f \in X'$:*

$$f(x) \le f(x_0) \quad \forall x \in C.$$

In particular, the hyperplane $[f = f(x_0)]$ separates the set C from $\{x_0\}$.

Proof. W.l.o.g. we can assume $0 \in C$. Consider $p(x)$, the Minkowsky functional of C. Take

$$G := \mathbb{R}x_0 \quad \text{and} \quad g(tx_0) = t.$$

Then, we have

$$g(x) \le p(x) \quad \forall x \in G.$$

By Hahn-Banach theorem 4.1, there exists $f \in X'$ with the required properties:

$$f(x_0) = 1 \quad \text{and} \quad f(x) < 1 \ \forall x \in C.$$ □

Proof (Theorem 4.10). Take

$$C := \bigcup_{y \in B} (A - y) = \bigcup_{y \in B} \bigcup_{x \in A} \{x - y\}.$$

By definition, C is open and $0 \notin C$ (since by assumption $A \cap B = \emptyset$).

By Lemma 4.12, there exists $f \in X'$:

$$f(z) \le 0 \quad \forall z \in C.$$

Therefore,

$$f(x) \le f(y) \quad \forall x \in A, y \in B,$$

since

$$z \in C \quad \Rightarrow \quad z = x - y, \ x \in A, y \in B,$$

and thus

$$0 \ge f(z) = f(x) - f(y).$$

With $\alpha \in \mathbb{R}$ such that

$$\sup_{x \in A} f(x) \le \alpha \le \inf_{y \in B} f(y),$$

the theorem follows. □

Theorem 4.13 (Second separation theorem)**.** *Take $A, B \subset X$ convex, non-empty and disjoint. Let A be closed and B compact. Then there exists a closed hyperplane that strictly separates A from B.*

Proof. Fix $\varepsilon > 0$. Take

$$A_\varepsilon := A + B_\varepsilon(0) := \bigcup_{x \in A} \bigcup_{y \in B_\varepsilon(0)} \{x + y\},$$
$$B_\varepsilon := B + B_\varepsilon(0).$$

With that, $A_\varepsilon, B_\varepsilon$ are open, convex, and non-empty.
Claim. For sufficiently small ε, we have $A_\varepsilon \cap B_\varepsilon = \emptyset$.

Proof. Assume that for all $\varepsilon : A_\varepsilon \cap B_\varepsilon \neq \emptyset$. Then there exists $(\varepsilon_n)_{n \in \mathbb{N}}$:

$$\varepsilon_n \to 0, \quad \varepsilon_n > 0,$$

with the property that there exist $(x_n)_{n \in \mathbb{N}} \in A^{\mathbb{N}}, (y_n)_{n \in \mathbb{N}} \in B^{\mathbb{N}}$, such that

$$\|x_n - y_n\| < 2\varepsilon_n.$$

Using compactness of B, there exists a subsequence

$$y_{n_k} \to y \in B.$$

But

$$\forall \varepsilon > 0 : \exists x \in A : \quad y \in B_\varepsilon(x).$$

Thus, we have $y \in \overline{A} = A$. This is a contradiction to $A \cap B = \emptyset$. □

Therefore, by Theorem 4.10 we have a closed hyperplane $[f = \alpha]$ separating A_ε from B_ε. Thus we have

$$f(x + \varepsilon z) \leq \alpha \leq f(y + \varepsilon z) \quad \forall x \in A, y \in B, z \in B_1(0).$$

But that gives us

$$f(x) + \varepsilon \|f\| \leq \alpha \leq f(y) - \varepsilon \|f\|,$$

and the claim follows. □

Remark.
- The most common application of this theorem is to take $B = \{x_0\}$.
- Let $A, B \subset X$ non-empty, disjoint and convex. Without assumptions, it is only generally possible to separate A from B in the finite-dimensional case.
- With the following Lemma, one can extend the above theorems to the complex setting. By taking a *real* closed hyperplane $[\operatorname{Re} f = \alpha], \alpha \in \mathbb{R}$.

Lemma 4.14. *Take X to be a Banach space over $\mathbb{C}$, $A \subset X$ convex, non-empty and open. Take $x_0 \in X \backslash A$. Then there exists $f \in X'$:*

$$\operatorname{Re} f(x) \leq \operatorname{Re} f(x_0) \quad \forall x \in A.$$

Corollary 4.15. *Let $F \subset X$ be a subspace, such that $\overline{F} \neq X$. Then there exists $0 \not\equiv f \in X'$:*

$$f(x) = 0 \quad \forall x \in F.$$

Remark. This is very useful for proving the denseness of subspaces:

$$[(\forall x \in F : f(x) = 0) \ \Rightarrow \ f \equiv 0] \quad \Rightarrow \quad \overline{F} = X.$$

Proof. Take $x_0 \in X$ such that $x_0 \notin \overline{F}$. By Theorem 4.13 with $A = \overline{F}, B = \{x_0\}$, there exists $0 \not\equiv f \in X'$, such that the hyperplane

$$[(\mathrm{Re})\, f = \alpha]$$

separates $\overline{F}$ and $\{x_0\}$ strictly. We have

$$(\mathrm{Re})\, f(x) < \alpha < \mathrm{Re}\, f(x_0) \quad \forall x \in F.$$

Since F is a subspace of X, it follows

$$\begin{aligned} \lambda \,\mathrm{Re}\, f(x) < \alpha &\quad \forall \lambda \in \mathbb{R}, x \in F \\ \Rightarrow \quad \mathrm{Re}\, f(x) = 0 &\quad \forall x \in F \\ \Rightarrow f(x) = 0 &\quad \forall x \in F \end{aligned}$$

See exercise 10. □

5 Baire category argument

Lemma 5.1 (Baire)**.** *Let X be a complete metric space and take a sequence $(F_n)_{n\in\mathbb{N}}$ of closed subsets of X. If*

$$F_n^\circ = \emptyset \quad \forall n \in \mathbb{N},$$

then

$$\left(\bigcup_{n\in\mathbb{N}} F_n\right)^\circ = \emptyset.$$

Remark.
- A set F is called *nowhere dense*, if $\left(\overline{F}\right)^\circ = \emptyset$.
- A set M is called *meagre* (or *set of category 1*), if $\exists\, (F_n)_{n\in\mathbb{N}}, F_n$ nowhere dense and
$$\bigcup_{n\in\mathbb{N}} F_n = M.$$
- Non-meagre sets are called *fat* (or *of category 2*).
- In particular, every complete metric space is fat.

Remark. The most common application of Baire's Lemma is the following: Take $(A_n)_{n\in\mathbb{N}}$ a sequence of sets in a complete metric space X. If

$$\bigcup_{n\in\mathbb{N}} A_n = X \quad \Rightarrow \quad \exists n \in \mathbb{N} : \left(\overline{A_n}\right)^\circ \neq \emptyset.$$

Proof. Let $O_n := F_n^c$. Then O_n is open and dense in X. Required to prove is that

$$G := \bigcap_{n\in\mathbb{N}} O_n$$

is still dense in X. Take ω a non-empty open set in X. We will show that

$$\omega \cap G \neq \emptyset.$$

Take $x_0 \in \omega, r_0 > 0$ such that

$$\overline{B_{r_0(x_0)}} \subset \omega.$$

Now pick $x_1 \in B_{r_0}(x_0) \cap O_1$ and $r_1 > 0$ such that

$$\overline{B_{r_1}(x_1)} \subset B_{r_0}(x_0) \cap O_1, \quad 0 < r_1 < \frac{r_0}{2}$$

(This is possible, since O_1 is open and dense.) In the same manner we can inductively construct $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}, (r_n)_{n\in\mathbb{N}} \in \mathbb{R}_{>0}^{\mathbb{N}}$ such that $\forall n \in \mathbb{N}:$

$$\overline{B_{r_{n+1}}(x_{n+1})} \subset B_{r_n}(x_n) \cap O_{n+1}, \quad 0 < r_{n+1} < \frac{r_n}{2}.$$

Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and we have

$$x_n \to x \in \overline{X},$$

and

$$x_{n+p} \in B_{r_n}(x_n) \quad \forall p > 0, n \in \mathbb{N}.$$

But then for the limit, we get

$$x \in \overline{B_{r_n}(x_n)} \quad \forall n \in \mathbb{N}.$$

Thus $x \in \omega \cap G$. □

5.1 Banach-Steinhaus theorem

Theorem 5.2 (Banach-Steinhaus)**.** *Let X, Y be normed $\mathbb{K}$-vector spaces, X a Banach space. Let $(T_i)_{i\in I}$ a family of linear bounded maps from X to Y (not necessarily countably many). Assume that*

$$\sup_{i\in I} \|T_i x\| < \infty \quad \forall x \in X.$$

Then we have

$$\sup_{i\in I} \|T_i\| < \infty.$$

Remark. Theorem 5.2 is also called *uniform boundedness principle*, since from the pointwise statement, it follows that

$$\|T_i x\| \leq C\,\|x\| \quad \forall i \in I, x \in X.$$

Proof. Let

$$X_n := \{x \in X : \forall i \in I : \|T_i x\| \leq n\}.$$

With that, we have X_n closed and by assumption of pointwise boundedness, we have

$$\bigcup_{n\in\mathbb{N}} X_n = X.$$

By Lemma 5.1, there exists $n_0 \in \mathbb{N}:$

$$X_{n_0}^{\circ} \neq \emptyset.$$

Now pick $x_0 \in X_{n_0}, r > 0:$

$$B_r(x_0) \subset X_{n_0}.$$

But then we have

$$\|T_i(x_0 + rz)\| \leq n_0 \quad \forall i \in I, z \in B_1(0).$$

It follows

$$r \cdot \|T_i z\| \leq n_0 + \|T_i x_0\| \quad \forall i \in I, z \in B_1(0),$$
$$\|T_i\| \leq \frac{n_0 + \|T_i x_0\|}{r} \quad \forall i \in I,$$

and the theorem is proven. □

Corollary 5.3. *Let X be a Banach space, Y a normed space, $(T_n)_{n\in\mathbb{N}}$ a sequence of bounded linear maps $T_n : X \to Y$:*

$$T_n x \xrightarrow[n\to\infty]{} Tx \quad \forall x \in X.$$

Then we have

1. $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$.

2. $T \in \mathscr{L}(X,Y)$.

3. $\|T\| \le \liminf_{n\to\infty} \|T_n\|$.

Proof. Exercise. □

Corollary 5.4. *Let Z be a Banach space, $B \subset Z$. Assume $\forall f \in Z'$ we have*

$$f(B) := \bigcup_{b\in B} \{f(b)\}$$

is bounded in $\mathbb{K}$. Then B is bounded set.

Proof. We will use Theorem 5.2with

$$X = Z', Y = \mathbb{K}, I = B.$$

For any $b \in B$, we define

$$T_b(f) = f(b) \quad \forall f \in Z'.$$

Thus by assumption we have

$$\sup_{b\in B} (T_b(f)) < \infty \quad \forall f \in Z'$$

Banach-Steinhaus 5.2 yields $C > 0$:

$$|f(b)| \le C\|f\| \quad \forall f \in Z', b \in B.$$

With Corollary 4.4, it follows that

$$\|b\| \le C \quad \forall b \in B.$$

□

Corollary 5.5. *Let Z be a Banach space, $B' \subset Z'$ and the sets*

$$\bigcup_{f\in B'} f(x) \text{ be bounded (in } \mathbb{K}) \quad \forall x \in Z.$$

Then B' is bounded (in the operator norm $\|\cdot\|_{\mathscr{L}(Z,\mathbb{K})}$).

Proof. Use Theorem 5.2 with $X = Z, Y = \mathbb{K}, I = B'$. The conclusion is made as in 5.4. □

5.2 Open mapping and closed graph theorems

Theorem 5.6 (Open mapping theorem)**.** *Let X,Y be Banach spaces, $T : X \to Y$ bounded and linear and* surjective*. Then there exists $c > 0$:*

$$T(\underbrace{B_1(0)}_{in\ X}) \supset \underbrace{B_c(0)}_{in\ Y}.$$

Remark. It follows that the image of any open set in X is an open set in Y (under surjective maps). Such a map is called an *open map.*

Proof. 1. We first show that for $T : X \to Y$ surjective, linear, we have $\exists c > 0$:

$$\overline{T(B_1(0))} \supset B_{2c}(0). \qquad (*)$$

Let

$$Y_n := \overline{T(B_n(0))}.$$

By surjectivity of T, we have

$$Y = \bigcup_{n\in\mathbb{N}} Y_n.$$

Due to Baire's Lemma, we have some $n_0 \in \mathbb{N}$:

$$Y_{n_0}^{\circ} \neq \emptyset.$$

But we have

$$B_{n_0}(0) = \left\{ x \in X | \frac{1}{n_0} x \in B_1(0) \right\}.$$

Using linearity of T, we have

$$Y_{n_0} = \overline{T(B_{n_0}(0))} = \left\{ y \in Y | \frac{1}{n_0} y \in \overline{T(B_1(0))} \right\}.$$

Therefore, we also have

$$\left(\overline{T(B_1(0))} \right)^{\circ} \neq \emptyset.$$

Now pick $c > 0, y_0 \in Y$:

$$B_{4c}(y_0) \subset \overline{T(B_1(0))}.$$

By linearity of T, we have that not only $y_0 \in \overline{T(B_1(0))}$, but also $-y_0 \in \overline{T(B_1(0))}$. Thus

$$\begin{aligned} B_{4c}(0) &\subset B_{4c}(y_0) + B_{4c}(-y_0) && \left(A + B := \bigcup_{(x,y) \in A \times B} \{x + y\}\right) \\ &\subset \overline{T(B_1(0))} + \overline{T(B_1(0))}. \end{aligned}$$

Linearity and convexity hold

$$\overline{T(B_1(0))} + \overline{T(B_1(0))} = \overline{T(B_2(0))}.$$

The claim follows by additional rescaling by a factor $\frac{1}{2}$.

2. *Claim.* Assume T is a continuous linear operator from X to Y satisfying $(*)$. Then we have

$$T(B_1(0)) \supset B_c(0)$$

for the $c > 0$ from 1.

Proof. Choose any $y \in Y$ such that $\|y\| < c$. We need to find $x \in X$:

$$\|x\| < 1 \quad \text{and} \quad Tx = y.$$

By $(*)$ we know that $\forall \varepsilon > 0 \exists z \in X$:

$$\|z\| < \frac{1}{2} \quad \text{and} \quad \|y - Tz\| < \varepsilon.$$

Choosing $\varepsilon = \frac{c}{2}$, we find $z_1 \in X$:

$$\|z_1\| < \frac{1}{2} \quad \text{and} \quad \|y - Tz_1\| < \frac{c}{2}.$$

By the same construction applied to $y - Tz_1$ instead of y and $\varepsilon = \frac{c}{4}$, we find $z_2 \in X$:

$$\|z_2\| < \frac{1}{4} \quad \text{and} \quad \|\underbrace{\underbrace{(y - Tz_1)}_{\in B_{c/2}(0)} - Tz_2}_{\in B_{c/4}(0)}\| < \frac{c}{4}$$

This way, we can construct a sequence $(z_n)_n$:

$$\|z_n\| < \frac{1}{2^n} \quad \text{and} \quad \left\| y - T\left(\sum_{k=1}^{n} z_k \right) \right\| < \frac{c}{2^n} \quad \forall n \in \mathbb{N}.$$

It follows that

$$x_n := \sum_{k=1}^{n} z_k$$

is a Cauchy sequence. Let

$$x = \lim_{n\to\infty} x_n.$$

Clearly $\|x\| < 1$ and

$$\|y - Tx\| = \left\| y - T\left(\lim_{n\to\infty} x_n\right)\right\| = \lim_{n\to\infty} \|y - Tx_n\| = 0. \qquad \text{(by continuity of } T\text{)}$$

Thus

$$y = Tx.$$

□

□

Remark. Both completeness of Y *and* completeness of X are necessary. Counterexamples are in the exercises.

Corollary 5.7. *Let X, Y be Banach spaces, $T : X \to Y$ linear, bounded and* bijective. *Then T^{-1} is continuous.*

Proof. By Theorem 5.6, we have

$$\|T(x)\| < c \Rightarrow \|x\| < 1.$$

By homogeneity of the norm, we have

$$\|x\| < \frac{1}{c}\|T(x)\| \quad \forall x \in X.$$

So T^{-1} is bounded, thus continuous. □

Remark. Let X be a vector space and $\|\cdot\|_1, \|\cdot\|_2$ two norms on X, such that X is complete w.r.t *both* norms. Assume that

$$\|x\|_2 \le c\|x\|_1 \quad \forall x \in X.$$

By Corollary 5.7 with $T = \mathrm{id}_X$, the norms are already equivalent.

Theorem 5.8 (Closed graph theorem)**.** *Let X, Y be Banach spaces, $T : X \to Y$ linear. Assume*

$$G(T) := \{(x, T(x)) \in X \times Y\} \qquad \text{(graph of } T\text{)}$$

to be closed in the product norm. Then, T is continuous.

Remark. We also have T continuous $\Rightarrow G(T)$ closed (proven by sequence criterion).

Proof. Consider the two norms

$$\|x\|_1 := \|x\|_X + \|T(x)\|_Y,$$
$$\|x\|_2 := \|x\|_X.$$

Since we assumed that $G(T)$ is closed, $G(T)$, as a closed linear subspace of a Banach space, is a Banach space, so $(X, \|\cdot\|_1)$ is a Banach space as well. Obviously

$$\|x\|_2 \le \|x\|_1 \quad \forall x \in X.$$

By the remark to Corollary 5.7, $\exists C > 0$:

$$\|T(x)\|_Y \le C \cdot \|x\|_X \quad \forall x \in X.$$

Thus, T is bounded. □

Remark. Theorem 5.8 sometimes simplifies the proof of continuity for a linear operator $T : X \to Y$: Instead of showing

$$x_k \to x \quad \Rightarrow \quad T(x_k) \to y \text{ and } y = T(x),$$

we only need to prove

$$(x_k \to x \text{ and } T(x_k) \to y) \quad \Rightarrow \quad y = T(x).$$

6 The weak topology

So in ∞-dimensional spaces, bounded sequences do not necessarily admit a converging subsequence. What now?

Theorem 6.1. *Let X be a separable Banach space and let $(f_k)_{k\in\mathbb{N}}$ be a sequence in X' with*

$$\|f_k\|_{X'} \leq 1 \quad \forall k \in \mathbb{N}.$$

Then there exists $f \in X'$ and a subsequence $\left(f_{k_j}\right)_{j\in\mathbb{N}}$:

$$\underbrace{\left(f - f'_{k_j}\right)}_{\in X'}(x) \xrightarrow[j\to\infty]{} 0 \quad \forall x \in X.$$

Example. For $X = \ell^p$, we have $X' = \ell^{p'}$ See Exercise 31. Therefore, immediately any bounded sequence in l^p admits a weak($*$)ly convergent subsequence.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence with dense span in X. We have $(f_k,(x_n))_{k\in\mathbb{N}}$ bounded $\mathbb{K}$, for all $n \in \mathbb{N}$. Using a diagonal sequence argument, we have $(f_{k_j})_{j\in\mathbb{N}}$, such that for all $n \in \mathbb{N}$ we have

$$\lim_{j\to\infty} f_{k_j}(x_n)$$

exists in $\mathbb{K}$.

But thus we have for any $y \in Y = \operatorname{span}(\{x_n\}_{n\in\mathbb{N}})$ existence of the limit

$$f(y) = \lim_{j\to\infty} f_{k_j}(y),$$

for which linearity follows immediately. Since $|f(y)| \leq \|y\|$ we have uniform continuity of f on Y. Thus we can find a unique continuation of f onto all of X' (since Y dense in X).

We also call this continuation f, noting that it is linear. It follows that $\|f\| \leq 1$ und

$$\forall x \in X, y \in Y : |(f - f_{k_y})[x]| \leq |(f - f_{k_j})[x - y]| + |(f - f_{k_j})[y]| \leq 2\|x - y\| + \underbrace{|(f - f_{k_j})[y]|}_{\to 0\,(j\to\infty)}$$

Since Y is dense in X, we can choose $\|x - y\|$ arbitrarily small and the theorem follows. $\square$

Question. *Can we generalise this?*

6.1 The weak topology $\sigma(X, X')$

Let X be a set, $f_i : X \to Y_i$ maps, Y_i topological spaces for $i \in I$.

Goal. Find the coarsest topology $\mathcal{T}$ on X such that all functions f_i are continuous.

Remark. There is such a topology, since with the discrete topology on X all functions are continuous. However, this is not necessarily the coarsest.

Let now $\left(w_i^j\right)_{j\in J_i}$ be the open sets in Y_i. In order for all f_i to be continuous, we must have

$$f_i^{-1}\left(w_i^j\right) \in \mathcal{T} \quad \forall i \in I, j \in J_i.$$

So we need

$$\left\{f_i^{-1}\left(w_i^j\right) \mid i \in I, j \in J_i\right\} \subset \mathcal{T}.$$

Proposition 6.2. *Let S be a family of subsets of X containing $\emptyset$ and X. Let Φ be the set of subsets of X which can be written as finite intersections of sets in S, i.e.*

$$\Phi = \left\{\varphi \subset X \mid \exists k \in \mathbb{N}, (s_j)_{j=1}^k, s_j \in S, \varphi = \bigcap_{j=1}^k s_j\right\}.$$

Furthermore, let Ψ be the set of all arbitrary unions of sets in Φ, i.e.

$$\Psi = \left\{ \psi \in X \mid \exists (\phi_\lambda)_{\lambda \in \Lambda}, \psi = \bigcup_{\lambda \in \Lambda} \phi_\lambda, \phi_\lambda \in \Phi \right\}.$$

Then Ψ is the coarsest topology on X that contains all sets in S.

Proof. We have to show that

1. Ψ is a topology on X, that is, it is stable under finite intersections and arbitrary unions and contains $\emptyset, X$.

2. Ψ contains all sets in S.

3. $\Psi' \subset \Psi, \Psi'$ is a topology on X, $S \subset \Psi'$, then $\Psi' = \Psi$.

1. was Proposition 1.8 (the only thing to show is stability under finite intersections, which follows as noted below), 2. and 3. are fairly obvious.

Take $\psi_1, \psi_2 \in \Psi$. We have

$$(A_\lambda)_{\lambda \in \Lambda}, (B_\kappa)_{\kappa \in \mathcal{K}}, \quad A_\lambda, B_\kappa \in \Phi \ \forall \lambda, \kappa$$

such that

$$\psi_1 = \bigcup_{\lambda \in \Lambda} A_\lambda, \quad \psi_2 = \bigcup_{\kappa \in \mathscr{K}} B_\kappa.$$

But then we have

$$\begin{aligned} \psi_1 \cap \psi_2 &= \left(\bigcup_\lambda A_\lambda \right) \cap \left(\bigcup_\kappa B_\kappa \right) \\ &= \bigcup_\lambda \left(A_\lambda \cap \bigcup_\kappa B_\kappa \right) \\ &= \bigcup_\lambda \bigcup_\kappa (A_\lambda \cap B_\kappa) \in \Psi. \end{aligned}$$

□

Corollary 6.3. *Taking*

$$S = \left\{ f_i^{-1}\left(w_i^j\right) \mid i \in I, j \in J_i \right\}$$

and Ψ as in Proposition 6.2, we get the required coarsest topology that makes all f_i continuous.

Proposition 6.4. *Let $(X, \mathcal{T})$ be a topological space, such that $\mathcal{T}$ is the coarsest topology such that $f_i : X \to Y_i$ is continuous $\forall i$ (f_i, Y_i as above). Then we have*

$$x_k \xrightarrow{\mathcal{T}} x \quad \Leftrightarrow \quad f_i(x_k) \to f_i(x) \ \ \forall i \in I.$$

Proof. "⇒" True due to continuity of the $f_i, i \in I$.

"⇐" Let $U \in \mathcal{T} : x \in U$. We have

$$U = \bigcup_{\lambda \in \Lambda} \bigcap_{\kappa=1}^{K(\lambda)} f_{i(\kappa,\lambda)}^{-1}(V_{\kappa,\lambda}), \quad V_{\kappa,\lambda} \subset Y_{i(\kappa,\lambda)} \text{ open.}$$

There exists $\lambda_0 \in \Lambda$:

$$x \in \bigcap_{\kappa=1}^{K(\lambda_0)} f_{i(\kappa,\lambda_0)}^{-1}(V_{\kappa,\lambda_0}).$$

By assumption, we have

$$f_{i(\kappa,\lambda_0)}(x_k) \to f_{i(\kappa,\lambda_0)}(x) \quad \forall \kappa \in \{1, ..., K(\lambda_0)\}$$

$\Rightarrow \exists N \in \mathbb{N}:$

$$f_{i(\kappa,\lambda_0)}(x_n) \in V_{\kappa,\lambda_0} \quad \forall n \geq N, \kappa \in \{1, ..., K(\lambda_0)\}.$$

$\Rightarrow$

$$x_n \in \bigcap_{\kappa=1}^{K(\lambda_0)} f^{-1}_{i(\kappa,\lambda_0)}(V_{\kappa,\lambda_0}) \subset U \quad \forall n \geq N.$$

$\Rightarrow$

$$x_n \xrightarrow{\mathcal{T}} x.$$

□

Proposition 6.5. *Let Z be a topological space, $\psi : Z \to (X, \mathcal{T}), \mathcal{T}$ as above. Then ψ is continuous, iff $f_i \circ \psi$ is continuous for all $i \in I$.*

Proof. "$\Rightarrow$" clear.

"$\Leftarrow$" Let $U \in \mathcal{T}$. Then

$$U = \bigcup_{\text{arb.}} \bigcap_{\text{fin.}} f_i^{-1}(V_{i,k}), \quad V_{i,k} \in Y_i.$$

Thus we have

$$\begin{aligned} \psi^{-1}(U) &= \bigcup_{\text{a}} \bigcap_{\text{f}} \psi^{-1}\left(f_i^{-1}(V_{i,k})\right) \\ &= \bigcup_{\text{a}} \bigcap_{\text{f}} (f_i \circ \psi)^{-1}(V_{i,k}) \end{aligned}$$

open in Z. □

Definition 6.6. Let X be a topological space.

1. $N \subset X$ is called a *neighbourhood* of $x_0 \in X$, if

$$\exists U \in \mathcal{T}: \quad x_0 \in U \subset N.$$

2. A family W of open sets in X is called a *neighbourhood basis* of x_0, if every $N \in W$ is a neighbourhood of x_0 and we have: M neighbourhood of $x_0 \Rightarrow$

$$\exists N \in W: \quad N \subset M.$$

Definition 6.7. Let X be a Banach space. The *weak topology* $\sigma(X, X')$ is the coarsest topology that renders all maps in the dual space X' continuous as maps from X to $\mathbb{K}$ with its usual topology.

Remark. We immediately see that if U is open w.r.t. $\sigma(X, X')$, then U is open w.r.t. the topology induced by $\|\cdot\|_X$.

Proposition 6.8. *The space $(X, \sigma(X, X'))$ is a Hausdorff space.*

Proof. Let x_1, x_2 be in $X, x_1 \neq x_2$. By the Hahn-Banach theorem 4.1, there exists $f \in X'$:

$$f(x_1) \neq f(x_2).$$

So we have U_1, U_2 open in $\mathbb{K}$:

$$U_1 \cap U_2 = \emptyset \quad \text{and} \quad f(x_1) \in U_1, f(x_2) \in U_2.$$

As preimages of disjoint sets are disjoint, we have

$$f^{-1}(U_1) \cap f^{-1}(U_2) = \emptyset.$$

□

Proposition 6.9. *Let $x_0 \in X$. A neighbourhood basis w.r.t. $\sigma(X, X')$ of x_0 can be constructed of sets of the form*

$$V = \{x \in X \mid |f_i(x - x_0)| < \varepsilon \ \forall i \in I\},$$

where I is a finite index set and

$$f_i \in X' \quad \forall i \in I, \quad \varepsilon > 0.$$

Proof. It's clear that all sets of the form of V are open and contain x_0. Let N be a neighbourhood of x_0 w.r.t. $\sigma(X, X')$. Let $U \subset N, x_0 \in U, U \in \sigma(X, X')$. We have

$$U = \bigcup_{\lambda \in \Lambda} \bigcap_{\kappa=1}^{\kappa(\lambda)} f_{\kappa,\lambda}^{-1}(V_{\kappa,\lambda}), \quad f_{\kappa,\lambda} \in X', V_{\kappa,\lambda} \subset \mathbb{K} \text{ open.}$$

So there exists $\lambda_0 \in \Lambda$:

$$x_0 \in \bigcap_{\kappa=1}^{\kappa(\lambda_0)} f_{\kappa,\lambda_0}^{-1}(V_{\kappa,\lambda_0}).$$

Let now

$$y_{\kappa,\lambda_0} = f_{\kappa,\lambda_0}(x_0).$$

We have $\varepsilon > 0$:

$$B_\varepsilon(y_{\kappa,\lambda_0}) \subset V_{\kappa,\lambda_0}.$$

Now consider

$$V := \bigcap_{\kappa=1}^{\kappa(\lambda_0)} f_{\kappa,\lambda_0}^{-1}(B_\varepsilon(y_{\kappa,\lambda_0})) = \{x \in X \mid |f_i(x - x_0)| < \varepsilon\}.$$

Thus V is of the required form and by construction we have $V \subset U$. □

Notation. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. By

$$x_n \rightharpoonup x$$

we denote convergence of x_n to x w.r.t. the weak topology. If there is the danger of ambiguity, we write

$$x_n \xrightharpoonup{\text{weakly}} x$$

for convergence w.r.t. $\sigma(X, X')$ and

$$x_n \xrightarrow{\text{strongly}} x$$

for convergence w.r.t. $\|\cdot\|_X$. We call the topology induced by $\|\cdot\|_X$ the strong topology.

Proposition 6.10. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Banach space X and $x \in X$. We have*

1. $x_n \rightharpoonup x \quad \Leftrightarrow \quad f(x_n) \to f(x) \quad \forall f \in X'.$
2. $x_n \xrightarrow{\text{strongly}} x \quad \Rightarrow \quad x_n \xrightharpoonup{\text{weakly}} x.$
3. $x_n \rightharpoonup x \quad \Rightarrow \quad \exists C > 0 : \quad \|x_n\|_X < C \ \ \forall n \in \mathbb{N}$ *and* $\|x\| \le \liminf_{n\to\infty} \|x_n\|.$
4. $x_n \rightharpoonup x, f_n \xrightarrow{X'} f \quad \Rightarrow \quad f_n(x_n) \to f(x).$

Proof. 1. Proposition 6.4

2. Observe
$$|f(x_n) - f(x)| \le \|f\|_{X'} \|x_n - x\|,$$
then use "if" from 1. to get the result.

3. By Corollary 5.4, we only need to check that
$$\{f(x_n)\}_{n\in\mathbb{N}} \text{ is bounded } \forall f.$$
This is true by assumption of convergence of $f(x_n)$. For the second statement note
$$|f(x_n)| \le \|f\|_{X'} \|x_n\|_X \quad \text{and} \quad f(x_n) \to f(x) \quad \forall f \in X'.$$
Thus we have
$$|f(x)| \le \|f\|_{X'} \liminf_{n\to\infty} \|x_n\|.$$

4. By the triangle inequality we have

$$\begin{aligned}|f_n(x_n) - f(x)| &\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\ &\leq \|f_n - f\| \|x_n\| + |f(x_n - x)|.\end{aligned}$$

The first term vanishes due to 3. and strong convergence of f_n, the second term does so by assumption. ☐

Theorem 6.11. *In finite dimensional Banach spaces the strong and weak topologies agree.*

Proof. Exercise. ☐

Remark. However, in ∞-dimensional Banach spaces, they *never* agree.

Example. $S := \{x \in X \mid \|x\| = 1\}$, X ∞-dimensional Banach space. We will show

$$\overline{S}^{\sigma(X,X')} \supset \overline{B_1(0)}^{\|\cdot\|_X} =: \overline{B_X} =: \overline{B}.$$

Let $x_0 \in \overline{B}$, U neighbourhood of x_0 w.r.t. to $\sigma(X, X')$, then $U \cap S \neq \emptyset$:

By Proposition 6.9, we can assume

$$V = \{x \in X \mid |f_i(x - x_0)| < \varepsilon \ \ \forall i \in \{1, ..., n\}\}, \quad f_i \in X', n \in \mathbb{N}, \varepsilon > 0.$$

Now let $0 \neq y_0 \in X$:

$$f_i(y_0) = 0 \quad \forall i \in \{1, ..., n\}.$$

Such a y_0 exists: Consider

$$\phi : X \to \mathbb{K}^n, \quad \phi(x) = (f_i(x))_{i \in \{1,...,n\}}.$$

If there were no such y_0, ϕ would only vanish on the origin and thus be injective, so X could be at most n-dimensional.

Let now

$$g(t) := \|x_0 + ty_0\|,$$

which is continuous and

$$g(0) = \|x_0\| \leq 1 \quad \text{and} \quad g(t) \xrightarrow[t \to \infty]{} \infty.$$

Therefore, there exists $t_0 \in \mathbb{R} : g(t_0) = 1$ and thus

$$\|x_0 + t_0 y_0\|_X = 1 \quad \rightsquigarrow \quad x_0 + t_0 y_0 \in S,$$

and

$$x_0 + t_0 y_0 \in V.$$

Example. $B_1(0)$ is not open w.r.t. $\sigma(X, X')$, if X is ∞-dimensional. Indeed, we even have

$$(B_1(0))^{\circ\sigma(X,X')} = \emptyset.$$

Proof works exactly as above.

Remark.

- The weak topology on an ∞-dimensional space is never metrisable (i.e. not induced by any metric).
- Two metrics on X that induce the same converging sequences also induce the same topology. However this is not necessarily true if only the topologies induce the same converging sequences.

Example. $X = l^1$. We have

$$x_n \xrightarrow{\|\cdot\|_{l^1}} x \quad \Leftrightarrow \quad x_n \xrightarrow{\sigma(l^1, (l^1)')} x.$$

Luckily, such examples are "rare".

Theorem 6.12. *Let $C \subset X$ convex. Then C is strongly closed, if and only if it is weakly closed.*

Remark. Together with the example above, this shows that

$$\overline{S}^{\sigma(X,X')} = \overline{B_1(0)}^{\|\cdot\|_X}.$$

Proof. "$\Leftarrow$" We always have

$$U \text{ weakly open} \quad \Rightarrow \quad U \text{ strongly open.}$$

Thus, if C is weakly closed, it is strongly closed.

"$\Rightarrow$" Let C be strongly closed and convex, $x_0 \notin C$. We need to show $\exists U \in \sigma(X, X')$:

$$x_0 \in U \quad \text{and} \quad U \cap C = \emptyset.$$

By the second separation theorem 4.13, there exists $f \in X', \alpha \in \mathbb{R}$:

$$\operatorname{Re} f(x_0) < \alpha < \operatorname{Re} f(x) \quad \forall x \in C.$$

Let now

$$U := \{x \in X \mid \operatorname{Re} f(x) < \alpha\}.$$

Then $U \in \sigma(X, X')$:

$$x_0 \in U \quad \text{and} \quad U \cap C = \emptyset.$$

□

Lemma 6.13 (Mazur)**.** *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X,*

$$x_n \xrightarrow{\text{weakly}} x.$$

Then there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of convex combinations of $(x_j)_{j=1}^n$, i.e.

$$y_n = \sum_{j=1}^n \lambda_{n,j} x_j, \quad \sum_{j=1}^n \lambda_{n,j} = 1, \quad \lambda_{n,j} \geq 0 \ \forall n, j,$$

such that

$$y_n \xrightarrow{\text{strongly}} x.$$

Proof. Exercise. □

Remark. In particular, we have

$$x \in \operatorname{conv}\left(\{x_n\}_{n\in\mathbb{N}}\right). \qquad \text{(convex hull)}$$

Theorem 6.14. *Let X, Y be Banach spaces, $T : X \to Y$ linear. Then T is continuous w.r.t. the strong topology on both spaces, if and only if it is continuous w.r.t. the weak topology on both spaces.*

Proof. "$\Rightarrow$" By Proposition 6.5, it suffices to show $\forall f \in Y'$:

$$F : x \mapsto f(Tx)$$

is continuous w.r.t. $\sigma(X, X')$ as a map to $\mathbb{K}$. This, however, is clear, since

$$f(Tx) \in X'.$$

"$\Leftarrow$" The graph $G(T)$ of T is closed in $X \times Y$ w.r.t. $\sigma(X, X') \otimes \sigma(Y, Y')$, since, by assumption, T is continuous. By Theorem 6.12, $G(T)$ is also strongly closed and the closed graph theorem 5.8 yields the result. □

Remark. In general, this does *not* hold for non-linear functions.

6.2 The weak* topology $\sigma(X', X)$

Let X be a Banach space, X' the dual space of X endowed with its usual norm

$$\|f\|_{X'} = \sup_{\substack{x\in X\\ \|x\|\leq 1}} |f(x)|.$$

Let further X'' be the bidual of X, i.e. the dual space of X' endowed with the norm

$$\|\xi\|_{X''} = \sup_{\substack{f\in X'\\ \|f\|\leq 1}} |\xi(f)|.$$

Definition 6.15. The *canonical injection* $J : X \to X''$ is given by

$$x \mapsto J(x), \quad J(x)(f) := f(x).$$

Remark. 1. For fixed $x \in X$, the map $f \mapsto f(x)$ is continuous as a map from X' to $\mathbb{K}$ and also linear, so $J(x)$ is indeed a continuous linear form on X'.

2. J is isometric, since

$$\begin{aligned}\|J(x)\|_{X''} &= \sup_{\|f\|\leq 1} |J(x)(f)| \\ &= \sup_{\|f\|\leq 1} |f(x)| \\ &= \|x\|_X \qquad \text{(Hahn-Banach 4.1)}\end{aligned}$$

This also implies injectivity.

3. There are examples where J is not surjective (see exercise). One can, however, always identify the subspace $J(X)$ of X'' with X.

On X' we have already defined two topologies

1. The strong topology, induced by $\|\cdot\|_{X'}$,
2. The weak topology $\sigma(X', X'')$.

We now introduce a third topology.

Definition 6.16. The *weak* topology* $\sigma(X', X)$ on X' is the coarsest topology on X' that renders all maps of the form

$$\varphi_X : X' \to \mathbb{K}, \quad f \mapsto \varphi_X(f) = (Jx)(f) = f(x)$$

continuous for all $x \in X$.

Remark. 1. If $\dim X < \infty$, weak, weak* and strong topologies all agree.

2. If $J(X)$ is a strict subspace of X'' (i.e. $J(X) \neq X''$), then the weak topology $\sigma(X', X'')$ is strictly finer than the weak* topology $\sigma(X', X)$.

Example. Let $\xi \in X'' \backslash J(X)$. Then

$$H := \{f \in X' \mid \xi(f) = 0\}$$

is closed w.r.t. $\sigma(X', X'')$, but not w.r.t. $\sigma(X', X)$.

Note. convex, strongly closed $\Rightarrow$ weakly closed, however convex, strongly closed $\not\Rightarrow$ weakly* closed.

Proposition 6.17. *The space $(X', \sigma(X', X))$ is Hausdorff.*

Proof. Let $f_1, f_2 \in X', f_1 \not\equiv f_2$. $\Rightarrow \exists x \in X$:

$$f_1(x) \neq f_2(x).$$

Without loss of generality, for some $\alpha \in \mathbb{R}$, we have

$$\operatorname{Re} f_1(x) < \alpha < \operatorname{Re} f_2(x).$$

Now let

$$U_1 := \{f \in X' \mid \operatorname{Re} f(x) < \alpha\},$$
$$U_2 := \{f \in X' \mid \operatorname{Re} f(x) > \alpha\}.$$

$f_1 \in U_1, f_2 \in U_2$ U_1, U_2 open, $U_1 \cap U_2 = \emptyset$. □

Proposition 6.18. *Let $f_0 \in X'$. A neighbourhood basis w.r.t. the weak* topology is given by sets of the form*

$$V = \{f \in X' \mid |(f - f_0)(x_j)| \leq \varepsilon \ \ \forall j \in J\}, \quad |J| < \infty, x_j \in X \forall j, \varepsilon > 0.$$

Proof. Same as the proof of Proposition 6.9. □

Notation. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in X'. We write

$$f_n \overset{*}{\rightharpoonup} f,$$

if f_n converges to $f \in X'$ w.r.t. the weak* topology.

Proposition 6.19. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in X'. We have*

1. $f_n \overset{*}{\rightharpoonup} f \quad \Leftrightarrow \quad f_n(x) \to f(x) \ \ \forall x \in X.$
2. $f_n \xrightarrow{\|\cdot\|_{X'}} f \quad \Rightarrow \quad f_n \overset{*}{\rightharpoonup} f.$
3. $f_n \rightharpoonup f \quad \Rightarrow \quad f_n \overset{*}{\rightharpoonup} f.$
4. $f_n \overset{*}{\rightharpoonup} f \quad \Rightarrow \quad (\|f_n\|)_{n\in\mathbb{N}}$ *is bounded in* $\mathbb{R}$ *and* $\|f\| \leq \liminf_{n\to\infty} \|f_n\|.$
5. $f_n \overset{*}{\rightharpoonup} f, x_n \xrightarrow{strongly} x \quad \Rightarrow \quad f_n(x_n) \to f(x).$

Proof. As in Proposition 6.10. 3. follows from $J(X)$ being a subspace of X'' ($\sigma(X', X)$ is certainly not finer than $\sigma(X', X'')$). □

Remark. A counterexample to $f_n \overset{*}{\rightharpoonup} f, x_n \rightharpoonup x \Rightarrow f_n(x_n) \to f(x)$ can be found in the exercises (in l^2, where $(l^2)' = l^2 = (l^2)''$).

Now, we look at the fundamental reason, why we deal with weak* topologies.

Theorem 6.20 (Banach-Alaoglu)**.** *The set*

$$\{f \in X' \mid \|f\|_{X'} \leq 1\} = \overline{B_1(0)} \subset X'$$

is compact w.r.t. the weak topology* $\sigma(X', X)$.

Remark. The proof uses Tychonov's theorem, that says, that the product space of compact topological spaces (even uncountably many) is compact w.r.t. the product space topology.

Definition 6.21 (Product space topology)**.** Let $(Y_i)_{i\in I}$ be a family of topological spaces and let

$$Y := \prod_{i\in I} Y_i := \left\{(y_i)_{i\in I} \mid y_i \in Y_i \ \ \forall i \in I\right\}.$$

The *product topology* on Y is the coarsest topology that renders all maps of the form

$$\varphi_i : Y \to Y_i, \quad \varphi_i(y) = \varphi_i\left((y_i)_{i\in I}\right) = y_i \qquad \text{(projections)}$$

continuous.

Remark. For finite products ($|I| < \infty$), this topology agrees with the box topology. This is not the case in ∞-dimensional spaces. In particular, the following theorem is untrue for the box topology.

Theorem 6.22 (Tychonov)**.** *Let*

$$X = \prod_{i \in I} X_i, \quad X_i \text{ compact } \forall i \in I.$$

Then X is compact w.r.t. the product topology.

Remark. We will prove Tychonov's theorem by means of Alexandrov's subbase theorem.

Theorem 6.23 (Alexandrov)**.** *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{B}$ a subbasis of $\mathcal{T}$. If every cover of X by sets in $\mathcal{B}$ admits a finite subcover, X is compact.*

Proof. We argue by contradiction.

- Assume X is *not* compact, however every cover by sets in $\mathcal{B}$ admits a finite subcover.
- Let P be the set of all open covers of X that do *not* admit a finite subcover.
- We endow P with the partial order of set inclusion.
  - P is not empty.
  - P is partially ordered.

Let C be a chain in P (i.e. a totally ordered subset of P). We have an upper bound of C by taking

$$S := \bigcup_{V_j \in C} V_j = \bigcup_{V_j \in C} \{U \mid U \in V_j\}.$$

Claim. $S \in P$.

Proof. Assume $U_1, ..., U_n$ is an open subcover of X by sets in S. We have

$$\forall i \in \{1, ..., n\} \exists V_{j_i} \in C : \quad U_i \in V_{j_i}.$$

However, $\{V_{j_i}\}_{i=1...n} \subset C$ is totally ordered, thus there is a maximal element V_{j_0}. We have

$$U_i \in V_{j_0} \quad \forall i = 1...n.$$

But then $\{U_i\}_{i=1...n}$ is a finite subcover by sets in V_{i_0}. This contradicts $V_{i_0} \in P$. □

By Zorn's Lemma, we thus have a maximal element in P, i.e. $\exists M \in P$:

$$A \in P, M \subset A \quad \Rightarrow \quad A = M.$$

This maximal element has the following properties

1. $U \notin M$ open $\Rightarrow M \cup \{U\} \notin P \Rightarrow M \cup \{U\}$ admits a finite subcover that must contain U. Thus there exist $\{U_j\}_{j=1}^n$ in M :
$$X = \bigcup_{j=1}^n U_j \cup U.$$
2. For U open
$$X = \bigcup_{j=1}^n U_j \cup U \quad \Rightarrow \quad U \notin M.$$
3. If $U_1, ..., U_n \notin M$, then
$$\bigcap_{j=1}^n U_j \notin M.$$

Proof. $\forall j \in \{1, ..., n\} \exists V_{j,k} \in M, k = 1...l_j$:

$$U_j \cup \bigcup_{k=1}^{l_j} V_{j,k} = X. \qquad \text{(by 1.)}$$

Let $x \in X$. Then either $x \in V_{j,k}$ for some j, k or

$$x \in U_j \quad \forall j \in \{1, ..., n\}.$$

With that, we have

$$X = \left(\bigcap_{j=1}^{n} U_j \right) \cup \underbrace{\left(\bigcup_{j=1}^{n} \bigcup_{k=1}^{l_j} V_{j,k} \right)}_{\text{finite}}$$

and by 2.

$$\bigcap_{j=1}^{n} U_j \notin M.$$

□

4. $U \notin M, U \subset W$ open $\Rightarrow W \notin M$. (follows from 1. and 2.)

We will now apply the subbasis.
Claim. $\mathcal{B} \cap M$ is a cover of X.

Proof. Let $x \in X$. We want to show:

$$\exists A \in \mathcal{B} \cap M : \quad x \in A.$$

Clearly, there exists $U \in M : x \in U$. U is open, so we can write it as

$$U = \bigcup_{\text{arb}} \bigcap_{\text{fin}} B_j, \quad B_j \in \mathcal{B}.$$

Thus there exists $\{B_j\}_{j=1}^n$ in $\mathcal{B}$:

$$x \in \bigcap_{j=1}^{n} B_j \subset U.$$

We now necessarily have a $j_0 : B_{j_0} \in M$ (otherwise, by 3. $\bigcap_j B_j \notin M \overset{4.}{\Rightarrow} U \notin M$). But then,

$$x \in B_{j_0} \in \mathcal{B} \cap M.$$

□

By assumption, $\mathcal{B} \cap M \subset \mathcal{B}$ contains a finite subcover of X. This subcover is also a finite subcover to M, contradicting $M \in P$. Hence P is empty and the proof is complete. □

Proof (Tychonov 6.22). Let $X_i, i \in I$ be compact topological spaces,

$$X = \prod_{i \in I} X_i.$$

We need to prove that X is compact w.r.t. the product topology.

Let thus Γ be an open cover of X. By Theorem 6.23 we may assume Γ only consists of sets of an arbitrary subbasis of the product topology. Such a subbasis is given by

$$\mathcal{S} := \left\{ U \subset X : U = \varphi_i^{-1}(U_i), U_i \subset X_i \text{ open}, i \in I \right\}. \qquad (\varphi_i \text{ projection on } X_i)$$

Let S_i be the subset of $\mathcal{S}$ generated by a particular φ_i :

$$S_i := \left\{ U \subset X \mid U = \varphi_i^{-1}(V), V \subset X_i \text{ open} \right\}, \quad i \in I.$$

We have

$$\mathcal{S} = \bigcup_{i \in I} S_i.$$

Claim. There exists $i_0 \in I : \Gamma \cap S_{i_0}$ is a cover of X.

Proof. Assume the contrary, namely

$$\forall i \in I : \exists y_i \in X : \quad y_i \notin A \;\; \forall A \in \Gamma \cap S_i.$$

Take

$$x_i := \varphi_i (y_i) \notin \varphi_i (A) \quad \forall A \in \Gamma \cap S_i. \qquad \text{(note that } \varphi_i^{-1}\varphi_i (A) = A\text{)}$$

Let now $x := (x_i)_{i\in I}$. We have $x \in X$, but

$$x \notin A \quad \forall A \in \Gamma \cap \bigcup_{i\in I} S_i = \Gamma \cap \mathcal{S} = \Gamma,$$

in contradiction to Γ being a cover of X. □

We now have

$$\Gamma \cap S_{i_0} = \left\{\varphi_{i_0}^{-1} (V_j) \subset X \mid j \in J\right\}$$

for some sets

$$V_j \subset X_{i_0} \text{ open} \quad \forall j \in J.$$

Since $\Gamma \cap S_{i_0}$ covers X, we have

$$\bigcup_{j\in J} V_j = X_{i_0}.$$

X_{i_0} is compact, thus we can select a finite Subset

$$\{V_k\}_{k=1}^{n} \subset \{V_j\}_{j\in J} : \quad \bigcup_{k=1}^{n} V_k = X_{i_0}.$$

Therefore,

$$\{\varphi_{i_0} (V_k) \subset X \mid k \in \{1, ..., n\}\}$$

is a finite subcover of X to Γ. □

Proof (Banach-Alaoglu 6.20)*.* Let X be a Banach space. Consider the product space

$$Y = \mathbb{K}^X = \left\{\omega = (\omega_x)_{x\in X} \mid \omega_x \in \mathbb{K}\right\}$$

with the product topology and X' with the weak* topology $\sigma (X', X)$. Consider further the map

$$\Phi : X' \to Y, \quad f \mapsto (f (x))_{x\in X}.$$

Claim. Φ is a homeomorphism from X' to $\Phi (X')$ (bijective, continuous, with continuous inverse).

Proof.

1. Continuity of Φ is follows from continuity of

$$f \mapsto (\Phi (f))_x = f (x)$$

for any $x \in X$ by Proposition 6.5.

2. Injectivity: Let $f_1, f_2 \in X', f_1 \neq f_2$.

$$\begin{aligned} &\Rightarrow \exists x \in X : & f_1 (x) &\neq f_2 (x) \\ &\Rightarrow & (\Phi (f_1))_x &\neq (\Phi (f_2))_x \\ &\Rightarrow & \Phi (f_1) &\neq \Phi (f_2). \end{aligned}$$

3. Continuity of the inverse: Again, by 6.5 it suffices to show that

$$\omega \mapsto \Phi^{-1} (\omega) (x)$$

is continuous for any $x \in X$. But

$$\omega \mapsto \Phi^{-1} (\omega) (x) = \omega_x$$

is continuous by definition of the product topology. □

We have
$$\Phi\left(\overline{B_1(0)}\right) = \mathscr{K}, \quad \left(\overline{B_1(0)} \subset X'\right),$$
where
$$\mathscr{K} := \{\omega \in Y \mid |\omega_x| \leq \|x\|, \omega_{x+y} = \omega_x + \omega_y, \omega_{\lambda x} = \lambda\omega_x \ \forall x, y \in X, \lambda \in \mathbb{K}\}.$$
With that, we have $\mathscr{K} = \mathscr{K}_1 \cap \mathscr{K}_2$, where
$$\mathscr{K}_1 := \prod_{x\in X} [-\|x\|, \|x\|], \qquad \text{(compact by Theorem 6.22)}$$
$$\mathscr{K}_2 := \left(\bigcap_{x,y\in X} A_{x,y}\right) \cap \left(\bigcap_{x\in X,\lambda\in\mathbb{K}} B_{x,\lambda}\right),$$
$$A_{x,y} := \{\omega \in Y \mid \underbrace{\omega_x + \omega_y - \omega_{x+y}}_{\text{continuous function in } \omega} = 0\}, \quad x, y \in X,$$
$$B_{x,\lambda} := \{\omega \in Y \mid \underbrace{\omega_{\lambda x} - \lambda\omega_x}_{\text{continuous in } \omega} = 0\}, \quad x \in X, \lambda \in \mathbb{K}.$$

All $A_{x,y}, B_{x,\lambda}$ are closed as continuous preimages of $\{0\}$, and thus $\mathscr{K}_2$ is closed as an intersection of closed sets. Therefore, $\mathscr{K}$ is compact as an intersection of a compact and a closed set and finally,
$$X' \supset \overline{B_1(0)} = \Phi^{-1}(\mathscr{K})$$
is compact as a continuous image of a compact set. □

6.3 Reflexive spaces and separable spaces

For this section denote by $\overline{B_X}$ the strongly closed unit ball in a Banach space X.

Definition 6.24. Let X be a Banach space and let J be the canonical injection in the bidual space X''
$$J : X \to X'', \quad x \mapsto J(x) : X' \to \mathbb{K}, \ \ f \mapsto f(x).$$
X is called reflexive, if
$$J(X) = X''.$$
In reflexive spaces, we can thus identify X with X''.

Remark. It is necessary to use the canonical injection, since there are Banach spaces X, that are not reflexive, however do admit a surjective isometry $X \to X''$.

Lemma 6.25 (Helly)**.** *Let X be a Banach space, $f_1, ..., f_n \in X'$, $\alpha_1, ..., \alpha_n \in \mathbb{K}$. Then the following are equivalent:*

1. $\forall \varepsilon > 0 : \exists x_\varepsilon \in X, \|x_\varepsilon\| \leq 1 :$
$$|f_i(x_\varepsilon) - \alpha_i| < \varepsilon.$$
2. $\left|\sum_{i=1}^n \beta_i\alpha_i\right| \leq \left\|\sum_{i=1}^n \beta_i f_i\right\|_{X'} \quad \forall \beta_1, ..., \beta_n \in \mathbb{K}.$

Proof. Exercise. □

Lemma 6.26 (Goldstine)**.** *Let X be a Banach space. Then $J\left(\overline{B_X}\right)$ is dense in $\overline{B_{X''}}$ w.r.t. $\sigma(X'', X')$ (weak* topology on X'').*

Proof. Let $\xi \in \overline{B_{X''}}$ and V be a neighbourhood of ξ w.r.t. $\sigma(X'', X')$. We have to show that
$$J\left(\overline{B_X}\right) \cap V \neq \emptyset.$$
By construction of the neighbourhood basis in $\sigma(X'', X')$ we can assume that V is of the type
$$V := \{\eta \in X'' \mid |(\eta - \xi)(f_i)| < \varepsilon \ \forall f_1, ..., f_n \in X'\}, \quad \varepsilon > 0.$$

We are thus looking for some $x \in \overline{B_X}$:

$$|f_i(x) - \xi(f_i)| < \varepsilon \quad \forall i \in \{1, ..., n\}.$$

Let

$$\alpha_i := \xi(f_i) \in \mathbb{K}.$$

Since $\|\xi\| \leq 1$, we have $\forall \beta_1, ..., \beta_n \in \mathbb{K}$:

$$\left|\sum_{i=1}^{n} \beta_i \alpha_i\right| = \left|\xi\left(\sum_{i=1}^{n} \beta_i f_i\right)\right| \leq \left\|\sum_{i=1}^{n} \beta_i f_i\right\|_{X'}.$$

By Lemma 6.25 there is $x_\varepsilon \in \overline{B_X}$:

$$|f_i(x_\varepsilon) - \alpha_i| < \varepsilon \quad \forall i \in \{1, ..., n\}.$$

So

$$J(x_\varepsilon) \in J\left(\overline{B_X}\right) \cap V.$$

□

Theorem 6.27 (Kakutani)**.** *Let X be a Banach space. X is reflexive, if and only if $\overline{B_X}$ is compact w.r.t. the weak topology $\sigma(X, X')$.*

Proof. "⇒" By reflexivity

$$J\left(\overline{B_X}\right) = \overline{B_{X''}} = \{\xi \in X'' \mid \|\xi\|_{X''} \leq 1\}.$$

Thus $J\left(\overline{B_X}\right)$ is compact w.r.t. $\sigma(X'', X')$ (by Banach-Alaoglu 6.20).

Hence it is enough to show that

$$J^{-1} : (X'', \sigma(X'', X')) \to (X, \sigma(X, X'))$$

is continuous. However, we have

$$f \circ J^{-1} : (X'', \sigma(X'', X')) \to \mathbb{K}, \quad f\left(J^{-1}(\xi)\right) = \xi(f)$$

is continuous $\forall f \in X'$. Thus J^{-1} is continuous and

$$J^{-1}\left(\overline{B_{X''}}\right) = \overline{B_X}$$

is compact.

"⇐" Let $\overline{B_X}$ be compact w.r.t. $\sigma(X, X'')$. By Theorem 6.14 (strong-strong continuity ⇔ weak-weak continuity), J is continuous as a map

$$(X, \sigma(X, X')) \to (X'', \sigma(X'', X''')).$$

Since $\sigma(X'', X''')$ is finer than $\sigma(X'', X')$, it is also continuous as a map

$$(X, \sigma(X, X')) \to (X'', \sigma(X'', X')).$$

(There are "no more" elements in X' than in X'''.)

Therefore $J\left(\overline{B_X}\right)$ is compact w.r.t. $\sigma(X'', X')$ and dense by Lemma 6.26. Compact sets are closed in Hausdorff spaces, so we have

$$\overline{B_{X''}} = \overline{J\left(\overline{B_X}\right)} = J\left(\overline{B_X}\right).$$

By linearity of J, we further have

$$J(X) = X''.$$

□

Remark. 1. $J\left(\overline{B_X}\right)$ is always strongly closed in X''.
In general, however, $J\left(\overline{B_X}\right)$ is not strongly dense in $\overline{B_{X''}}$.

2. Finite dimensional vector spaces are reflexive.

Lemma 6.28. *Let X be a reflexive Banach space, $M \subset X$ a closed subspace. Then M is reflexive w.r.t. its induced norm.*

Proof. On M, we have defined two weak topologies:

1. The topology $\sigma(M, M')$.
2. The subspace topology of $\sigma(X, X')$.

By restriction or extension of continuous linear functionals, those two topologies agree.
However, we have $\overline{B_M}$ is compact w.r.t. $\sigma(M, M')$ as a weakly closed subset of a compact set. □

Corollary 6.29. *Let X be a Banach space. X is reflexive, if and only if X' is reflexive.*

Proof. "⇒" By Banach-Alaoglu 6.20, $\overline{B_{X'}}$ is compact w.r.t. $\sigma(X', X)$, however, by reflexivity of X :

$$\sigma(X', X) = \sigma(X', X'').$$

Thus X' is reflexive by Theorem 6.27.

"⇐" The same argument as above holds X'' is reflexive. By 6.28, $J(X)$ is reflexive as a closed subspace of X''. f, however, is certainly a surjective isometry

$$X \to J(X).$$

Therefore, X is reflexive, since

$$(T : X \to Y \text{ surjective Isometry}) \quad \Rightarrow \quad (X \text{ reflexive } \Leftrightarrow Y \text{ reflexive}).$$ □

Corollary 6.30. *Let X be a reflexive Banach space, $K \subset X$ strongly closed, convex and bounded. Then K is compact w.r.t. $\sigma(X, X')$.*

Proof. $\exists m > 0 : K \subset m \cdot \overline{B_X}$ and K is weakly closed. Therefore, K is compact as a closed subset of a compact set. □

Question. *What are we going to use this for? We will prove*:
Let ϕ be a convex and strongly lower semicontinuous function, i.e.

$$x_i \to x \quad \Rightarrow \quad \liminf \phi(x_i) \geq \phi(x).$$

Then ϕ admits a minimum on any convex bounded set.

Definition 6.31. Let X be a topological space, $A \subset X$. A function

$$\phi : A \to (-\infty, \infty]$$

is called *lower semicontinuous*, if

$$\{y \in X \mid \phi(y) > \alpha\} \subset X \text{ open} \quad \forall \alpha \in \mathbb{R}.$$

Note. On metric spaces, this is the same as sequential lower semicontinuity:

$$(x_i \to x \text{ in } X) \quad \Rightarrow \quad \liminf_{j \to \infty} \phi(x_j) \geq \phi(x).$$

Theorem 6.32. *Let X be a reflexive Banach space, $A \subset X$ convex, closed, not empty, and let*

$$\phi : A \to (-\infty, \infty]$$

be a strongly lower semicontinuous function that is convex and $\not\equiv \infty$.

In case A is not bounded, assume in addition that

$$\lim_{\|x\| \to \infty, x \in A} \phi(x) = \infty.$$

Then ϕ attains its minimum on A, i.e. $\exists x_0 \in A$:

$$\phi(x_0) \leq \phi(x) \quad \forall x \in A.$$

Proof. Let

$$\tilde{A} := \{x \in A \mid \phi(x) \leq \lambda_0\}, \quad \lambda_0 = \phi(a) < \infty, a \in A.$$

$\tilde{A}$ is thus bounded by growth, convex by convexity and closed by strong lower semicontinuity of ϕ.

Claim. ϕ attains its minimum on $\tilde{A}$.

Note. If we prove this claim, we are done, since

$$\phi(x) > \phi(x_0) \quad \forall x \in \tilde{A}.$$

Proof. Take a sequence $(x_k)_{k \in \mathbb{N}}$ in $\tilde{A}$:

$$\phi(x_k) \leq \inf_{x \in \tilde{A}} \phi(x) + \frac{1}{k}.$$

By compactness of $\tilde{A}$ (by Corollary 6.30), this sequence admits an accumulation point $x_0 \in \tilde{A}$.

We continue to argue by contradiction. Assume thus $\exists c > 0$:

$$\phi(x_0) \geq \inf_{x \in \tilde{A}} \phi(x) + c.$$

The set

$$U := \left\{ y \in \tilde{A} \mid \phi(y) > \inf_{x \in \tilde{A}} \phi(x) + \frac{c}{2} \right\}$$

is weakly open, since its complement is convex and strongly closed, thus weakly closed.

We have $x_0 \in U$, thus U is an open neighbourhood of x_0. However, only finitely many elements of the sequence can be in U. □

□

Example. u in a Sobolev space W is a minimizer of

$$\phi(a) = \int_D |\nabla u|^2 - \int_D f \cdot u, \quad D \subset W \text{ bounded}$$

$\Leftrightarrow -\Delta u = f$.

Now we turn our attention to separable spaces, which will enable us to extract converging subsequences, not just find accumulation points.

Lemma 6.33. *Subsets of separable metric spaces are separable.*

Proof. Take $(x_n)_{n \in \mathbb{N}}$ dense in X (X separable metric space). Take $\emptyset \neq Y \subset X, 0 < r_n \to 0$. We can then pick

$$y_{m,n} \in B_{r_n}(x_m) \cap Y.$$

This set is countable and dense in Y. □

Theorem 6.34. *Let X be a Banach space. We have*

$$X' \text{ separable} \quad \Rightarrow \quad X \text{ separable}.$$

Remark. The converse does not necessarily hold. e.g. L^1 (next week).

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a dense sequence in X'. Since

$$\|f_n\|_{X'} = \sup_{\|x\|_X\le 1, x\in X} |f_n(x)|,$$

there exist $x_n \in X$:

$$\|x_n\|_X = 1, \quad f_n(x_n) \ge \frac{1}{2}\|f_n\| \quad \forall n.$$

Let L_0 be the set of all rational, finite linear combinations of elements in $\{x_n\}_{n\in\mathbb{N}}$. This set is countable and clearly dense in

$$L := \operatorname{span}\left(\{x_n\}_{n\in\mathbb{N}}\right)$$

the set of real finite linear combinations of the x_n.

Claim. L is dense in X.

Proof. Let $f \in X'$:

$$f(x) = 0, \quad \forall x \in L.$$

Therefore, we have $\forall \varepsilon > 0 : \exists \{f_n\}_n$ dense, countable subset:

$$\|f - f_n\| < \varepsilon$$

and we have

$$\frac{1}{2}\|f_n\| \le f_n(x_n) = \underbrace{(f_n - f)}_{\le\varepsilon}(x_n) + \underbrace{f(x_n)}_{=0}.$$
$$\Rightarrow \|f\| < 3\varepsilon \quad \forall \varepsilon > 0.$$
$$\Rightarrow \|f\| = 0.$$

We have thus shown that $\forall f \in X', f(x) = 0 \ \forall x \in L$:

$$f \equiv 0.$$

The statement of Corollary 4.15 says exactly that L must be dense in X. □

In the complex case, as usual we consider the characterization of complex linear functionals as

$$f(x) = g(x) - ig(ix)$$

with $g : X \to \mathbb{R}$ linear. □

Corollary 6.35. *Let X be a reflexive, separable Banach space. Then X' is separable.*

Proof. X separable, reflexive. $\Rightarrow X''$ separable $\overset{6.34}{\Longrightarrow} X'$ separable. □

Theorem 6.36. *Let X be a Banach space. X is separable, if and only if the weak* topology on $\overline{B_{X'}}$ is metrisable (that is, there exists a metric $d : \overline{B_{X'}} \times \overline{B_{X'}} \to \mathbb{R}$ that induces the weak* topology).*

Remark. The weak* topology is *never* metrisable on the whole space if X is ∞-dimensional.

Proof. "$\Rightarrow$" Let $(x_n)_{n\in\mathbb{N}}$ be dense in $\overline{B_X}$. For $f, g \in \overline{B_{X'}}$, set

$$d(f,g) := \sum_{n=1}^{\infty} \frac{1}{2^n} |(f-g)(x_n)|.$$

Claim (1). d is a metric.

Proof.
- Positive definiteness follows from the density of x_n.

- Triangle inequality and symmetry follow from respective properties of $|\cdot|$ as seen in exercises. □

In the following, we will prove that if

$$V \text{ open neighbourhood of } x_0 \text{w.r.t. } \sigma(X, X'),$$

then there exists

$$U \subset V \text{ open neighbourhood of } x_0 \text{w.r.t. } d(\cdot,\cdot)$$

and vice versa.

Question. *Why is that enough?*

Take a set O open w.r.t. $\sigma(X', X)$ then $\forall f \in O$ there is an open neighbourhood V of f w.r.t. $\sigma(X', X)$ in O. The claim provides an open neighbourhood $U \subset V \subset O$ of f w.r.t. $d(\cdot,\cdot)$, so

$$O = O^{\circ_d} \quad (\textit{interior of } O \textit{w.r.t. } d)$$

and vice versa.

Claim (2). Take $f_0 \in \overline{B_{X'}}$, U a neighbourhood of f_0 w.r.t. $\sigma(X', X)$. Then $\exists r > 0$:

$$U := \left\{ f \in \overline{B_{X'}} \mid d(f_0, f) < r \right\} \subset V.$$

Proof. We can, as usual, take V of the form

$$V = \left\{ f \in \overline{B_{X'}} \mid |(f - f_0)(y_i)| < \varepsilon \ \ \forall i \in \{1, ..., k\} \right\}, \quad y_i \in X, \varepsilon > 0.$$

Without loss of generality, we can assume

$$\|y_i\| = 1 \quad \forall i.$$

Now for all $i \in \{1, ..., k\}$ we pick $n_i \in \mathbb{N}$:

$$\|y_i - x_{n_i}\| < \frac{\varepsilon}{4}.$$

We also pick $r > 0$:

$$2^{n_i} \cdot r < \frac{\varepsilon}{2} \quad \forall i \in \{1, ..., k\}.$$

For $f : d(f, f_0) < r$, we have

$$\frac{1}{2^{n_i}} |(f - f_0)(x_{n_i})| < r \quad \forall i \in \{1, ..., k\}.$$

But then we have

$$\begin{aligned} |(f - f_0)(y_i)| &= |\underbrace{(f - f_0)}_{\|\cdot\| \le 2} \underbrace{(y_i - x_{n_i})}_{\|\cdot\| < \frac{\varepsilon}{4}} + \underbrace{(f - f_0)(x_{n_i})}_{\|\cdot\| \le 2^{n_i} r < \frac{\varepsilon}{2}}| \\ &< 2 \cdot \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

Claim (3). Take $f_0 \in \overline{B_{X'}}, r > 0$. Then there exists a neighbourhood V of f_0 w.r.t. $\sigma(X', X)$:

$$V \subset U := \left\{ f \in \overline{B_{X'}} \mid d(f, f_0) < r \right\}.$$

Proof. We have

$$\begin{aligned} d(f,f_0) &= \sum_{j=1}^{\infty} \frac{1}{2^j} \left|(f-f_0)(x_j)\right| \\ &= \sum_{j=1}^{k} \frac{1}{2^j} \left|(f-f_0)(x_j)\right| + \underbrace{\sum_{j=k+1}^{\infty} \frac{1}{2^j} \big|\underbrace{(f-f_0)}_{\|\cdot\|\le 2}\underbrace{(x_j)}_{\|\cdot\|\le 1}\big|}_{\le \sum_{j=k+1}^{\infty} \frac{2}{2^j} = \frac{1}{2^{k-1}}}. \end{aligned}$$

Now take

$$V := \left\{ f \in \overline{B_{X'}} \mid \left|(f-f_0)(x_i)\right| < \frac{r}{2} \ \ \forall i \in \{1,...,k\} \right\},$$

where we pick k :

$$\frac{1}{2^{k-1}} < \frac{r}{2}.$$

Then, for all $f \in V$:

$$d(f,f_0) \le \underbrace{\sum_{j=1}^{k} \frac{1}{2^j} \underbrace{\left|(f-f_0)(x_j)\right|}_{<\frac{r}{2}}}_{<\frac{r}{2}} + \underbrace{\frac{1}{2^{k-1}}}_{<\frac{r}{2}} < r.$$

Therefore $V \subset U$. □

This proves, that in a separable Banach space the weak* topology on $\overline{B_{X'}}$ is metrisable.

"$\Leftarrow$" Take

$$U_n := \left\{ f \in \overline{B_{X'}} \mid d(f,0) < \frac{1}{n} \right\}, \quad n \in \mathbb{N}.$$

By assumption, there exists a neighbourhood V_n of the origin w.r.t. $\sigma(X',X)$:

$$V_n \subset U_n.$$

We can take

$$V_n := \left\{ f \in \overline{B_{X'}} \mid |f(x)| < \varepsilon_n \ \ \forall x \in \Phi_n \right\},$$

where $\Phi_n \subset \overline{B_X}$ finite. Thus, the Set

$$D := \bigcup_{n\in\mathbb{N}} \Phi_n$$

is countable.

Claim. D is dense in $\overline{B_X}$.

Proof. Assume that

$$f(x) = 0 \quad \forall x \in D.$$

Then

$$\begin{aligned} & f \in V_n \quad \forall n \\ \Rightarrow\ & f \in \bigcap_{n\in\mathbb{N}} V_n \\ \Rightarrow\ & f \in \bigcap_{n\in\mathbb{N}} U_n \\ \Rightarrow\ & d(f,0) = 0 \\ \Rightarrow\ & f \equiv 0. \end{aligned}$$

By corollary 4.15, D is dense in $\overline{B_X}$, since $\operatorname{span} D$ is dense in X. □

□

Theorem 6.37. *Let X be a Banach space. Then X' is separable, if and only if $\overline{B_X}$ is metrisable w.r.t. $\sigma(X, X')$.*

Proof. Exactly as in the proof of 6.36:

$$X' \text{ separable} \quad \Rightarrow \quad \left(\overline{B_X}, \sigma(X, X')\right) \text{ metrisable.}$$

□

Remark. The other direction holds as well, but the proof is much harder.

Corollary 6.38. *Let X be a separable Banach space, $(f_n)_{n\in\mathbb{N}}$ a bounded sequence in X'. Then there exists a weakly* convergent subsequence of $(f_n)_{n\in\mathbb{N}}$.*

Proof. Since $K \cdot \overline{B_{X'}}$ is compact and metric, there exists a convergent subsequence of

$$\{f_n\}_{n\in\mathbb{N}} \subset K \cdot \overline{B_{X'}}.$$

□

Remark. See the exercises for an example where X is not separable and a bounded sequence in X' does not admit a convergent subsequence.

Theorem 6.39. *Let X be a reflexive Banach space and let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X. Then there exists a weakly converging subsequence.*

Proof. Let

$$M_0 := \operatorname{span}\{x_n\}_{n\in\mathbb{N}} \quad \text{and} \quad M := \overline{M_0}.$$

M is separable and reflexive as a closed subspace of a reflexive space. $\Rightarrow M''$ is separable $\Rightarrow M'$ separable. Therefore

$$\left(\overline{B_M}, \sigma(M, M')\right)$$

is metrisable and weakly compact as the closed unit ball in a reflexive space. Thus, there exists a subsequence of $(x_n)_{n\in\mathbb{N}}$ that converges w.r.t. $\sigma(M, M')$. $\forall f \in X' : \exists \tilde{f} \in M'$:

$$f|_M = \tilde{f}.$$

Therefore, the subsequence also converges w.r.t. $\sigma(X, X')$. □

We even have the converse statement:

Theorem 6.40 (Eberlein-Shmulian)**.** *Let X be a Banach space, such that every sequence admits a weakly converging subsequence. Then X is reflexive.*

Proof. See Rudin. □

Definition 6.41. A Banach space is called *uniformly convex*, if $\forall \varepsilon > 0 : \exists \delta > 0$:

$$\forall x, y \in X, \|x\| \le 1, \|y\| \le 1, \|x - y\| > \varepsilon : \quad \frac{\|x + y\|}{2} < 1 - \delta.$$

Example. $X = \left(\mathbb{R}^2, \|\cdot\|_p\right)$,

$$\|x\|_p = \begin{cases} \left(|x_1|^p + |x_2|^p\right)^{1/p}, & 1 \le p < \infty \\ \max\{|x_1|, |x_2|\}, & p = \infty \end{cases}, \quad x = (x_1, x_2) \in X.$$

The unit balls X are uniformly convex $\forall 1 < p < \infty$.

Theorem 6.42 (Milman-Pettis)**.** *Every uniformly convex Banach space is reflexive.*

Proof. Let $\xi \in X''$. W.l.o.g. assume $\|\xi\|_{X''} = 1$. We show

$$\xi \in J\left(\overline{B_X}\right).$$

Since $J\left(\overline{B_X}\right)$ strongly closed, it is enough to show, that $\forall \varepsilon > 0 : \exists x \in \overline{B_X}$:

$$\|\xi - J(x)\|_{X''} < \varepsilon.$$

Let now $\varepsilon > 0$ and fix δ to be the constant from uniform convexity. Let $f \in X'$:

$$|\xi(f)| > 1 - \frac{\delta}{2}$$

(this f exists since

$$\|\xi\|_{X''} = \sup_{\substack{\|f\| \le 1 \\ f \in X'}} |\xi(f)| = 1.)$$

Let

$$V := \left\{ \eta \in X'' \mid |(\eta - \xi)(f)| < \frac{\delta}{2} \right\}$$

be a neighbourhood of ξ w.r.t. $\sigma(X'', X')$. Since $J\left(\overline{B_X}\right)$ is weakly* dense in $\overline{B_{X''}}$ (by Goldstine), we have

$$V \cap J\left(\overline{B_X}\right) \neq \emptyset.$$

Now let $x \in \overline{B_X}$:

$$J(x) \in V.$$

Claim. $\|\xi - J(x)\|_{X''} < \varepsilon$.

Proof. Assume for a contradiction that

$$\xi \in \left(\overline{B_\varepsilon(J(x))}\right)^c =: W.$$

With that, W is an open neighbourhood of ξ w.r.t. $\sigma(X'', X')$, since $\varepsilon\overline{B_{X''}}$ is weakly* compact and thus closed. Therefore, we have

$$(V \cap W) \cap J\left(\overline{B_X}\right) \neq \emptyset.$$

Now we pick $\hat{x} \in \overline{B_X}$:

$$J(\hat{x}) \in V \cap W.$$

We have

$$|f(x) - \xi(f)| < \frac{\delta}{2}, \quad |f(\hat{x}) - \xi(f)| < \frac{\delta}{2}$$

and thus

$$\begin{aligned} 2|\xi(f)| &\le |f(x + \hat{x})| + \delta \\ &\le \|\hat{x} + x\| + \delta. \end{aligned}$$

Since $|\xi(f)| > 1 - \frac{\delta}{2}$, we have

$$2|\xi(f)| > 2 - \delta,$$

thus

$$\|x + \hat{x}\| + \delta > 2 - \delta$$

and therefore

$$\left\|\frac{x + \hat{x}}{2}\right\| > 1 - \delta.$$

Due to uniform convexity we now have

$$\|x - \hat{x}\| \le \varepsilon.$$

However, $\|x - \hat{x}\| > \varepsilon$, since $J(\hat{x}) \in W$. This is a contradiction. □

□

Theorem 6.43. *Let X be a uniformly convex Banach space, $(x_n)_{n\in\mathbb{N}}$ a sequence in X :*

$$x_n \rightharpoonup x \quad \text{and} \quad \limsup_{n\to\infty} \|x_n\| \leq \|x\| .$$

Then we have

$$x_n \to x.$$

Proof. Let $\|x\| \neq 0$ (otherwise the statement is trivial). Take

$$\lambda_n := \max\{\|x_n\|, \|x\|\},$$

so

$$\lambda_n \to \|x\| .$$

Now let

$$y_n := \lambda_n^{-1} x_n, \quad y := \|x\|^{-1} x.$$

We have

$$\|y\| \leq \liminf_{n\to\infty} \left\| \frac{y_n + y}{2} \right\| .$$

However, $\|y\| = 1, \|y_n\| \leq 1$, so we have

$$\left\| \frac{y_n + y}{2} \right\| \to 1.$$

Using uniform convexity, it follows that

$$\|y - y_n\| \to 0.$$

□

7 Lebesgue-Spaces Part II

7.1 The Dual of L^p

7.1.1 Case $1 < p < \infty$

Theorem 7.1. *L^p is reflexive for $1 < p < \infty$.*

Proof.

1. $2 \leq p < \infty$. Then L^p is a strictly convex space.
   (This is a consequence of Carlson's first inequality and will be proven as an exercise.)
   Thus by the theorem of Milman and Pettis 6.42, L^p is reflexive for $2 \leq p < \infty$.

2. $1 < p < 2$. First define for $1 < p < \infty$:

$$T : L^p \to \left(L^{p'}\right)', \quad u \mapsto Tu : L^{p'} \to \mathbb{K}, \; f \mapsto \int u \cdot f.$$

This map is well defined, since Tu is a linear functional on $L^{p'}$ and continuous by Hölder's inequality 2.4.

Claim.

$$\|Tu\|_{\left(L^{p'}\right)'} = \|u\|_{L^p} \quad \forall u \in L^p.$$

Proof. By Hölder's inequality, we have

$$|(Tu)(f)| \leq \|u\|_{L^p} \|f\|_{L^{p'}} \quad \forall f \in L^{p'}.$$

Therefore,

$$\|Tu\|_{\left(L^{p'}\right)'} \leq \|u\|_{L^p} .$$

For the other direction, set

$$f_0(x) := \begin{cases} |u(x)|^{p-2}\,\overline{u(x)}, & u(x) \neq 0 \\ 0, & \text{else} \end{cases}.$$

A short calculation shows

$$\|f_0\|_{L^{p'}} = \|u\|_{L^p}^{p-1} < \infty \quad \Rightarrow \quad f_0 \in L^{p'},$$
$$(Tu)(f_0) = \int |u|^{p-2} \cdot \underbrace{\overline{u} \cdot u}_{=|u|^2} = \|u\|_{L^p}^p .$$

Therefore,

$$\|Tu\|_{(L^{p'})'} \geq \frac{|(Tu)(f_0)|}{\|f_0\|_{L^{p'}}} = \|u\|_{L^p} .$$

This proves the claim. □

We have thus shown that

$$T : L^p \to \left(L^{p'}\right)'$$

is an isometry. Therefore, $T(L^p)$ is a closed subset of $\left(L^{p'}\right)'$ for all $1 < p < \infty$.

For $1 < p \leq 2$, we have $p' \geq 2$ and thus by 1. $L^{p'}$ is reflexive.

By Corollary 6.29, $\left(L^{p'}\right)'$ is reflexive and by Lemma 6.28, $T(L^p)$ is reflexive.

Since

$$T : L^p \to T(L^p)$$

is a surjective isometry, L^p is also reflexive.

$$\begin{array}{ccc} X'' & \underset{\tilde{T}}{\leftrightarrows} & Y'' \\ {\scriptstyle J}\uparrow\downarrow & & {\scriptstyle J}\uparrow\downarrow {\scriptstyle TJ^{-1}\tilde{T}^{-1}} \\ X & \underset{T}{\leftrightarrows} & Y \end{array}$$

□

Notation. For the dual pairing, $f \in (L^p)'$, $g \in L^p$, we often write

$$f(g) = \langle f, g \rangle .$$

This is very common on L^p or Sobolev spaces and somewhat common in general.

Theorem 7.2 (Riesz representation theorem). *Let $1 < p < \infty$ and $\phi \in (L^p)'$. Then there exists a* unique *function $u \in L^{p'}$:*

$$\langle \phi, f \rangle = \phi(f) = \int u \cdot f \quad \forall f \in L^p.$$

Furthermore,

$$\|u\|_{L^{p'}} = \|\phi\|_{(L^p)'} .$$

Remark. The elements of the abstract space $(L^p)'$ can be uniquely identified with a concrete function in $L^{p'}$. We systematically make this identification

$$(L^p)' = L^{p'}.$$

Proof. We consider the operator $T : L^{p'} \to (L^p)'$ defined by

$$\langle Tu, f \rangle = \int u \cdot f \quad \forall u \in L^{p'}, f \in L^p.$$

The argument in the proof of Theorem 7.1 shows that

$$\|Tu\|_{(L^p)'} = \|u\|_{L^{p'}} .$$

It remains to show that T is surjective. Take

$$E = T\left(L^{p'}\right).$$

Since E is a closed subspace of $(L^p)'$, we only have to show that E is dense. Consider $h \in (L^p)''$ such that

$$\begin{aligned} &\langle h, Tu\rangle = 0 \quad \forall u \in L^p \\ \Leftrightarrow \quad &\langle h, v\rangle = 0 \quad \forall v \in E. \end{aligned} \tag{$*$}$$

Claim. $h \equiv 0$.

Proof. Since L^p is reflexive, we can simply take $h \in L^p$ and then $(*)$ implies that

$$\int u \cdot h = 0 \quad \forall u \in L^{p'}.$$

By the choice

$$u = |h|^{p-2}\,\overline{h},$$

we see $h = 0$ and the claim is proven. □

By Corollary 4.15, for a subspace E :

$$(\langle h, v\rangle = 0 \quad \forall v \in E \quad \Rightarrow \quad h = 0) \quad \Rightarrow \quad E \text{ is dense.}$$

So the Theorem is proven. □

7.1.2 The space $L^1(\mu)$

For σ-additive μ.

Theorem 7.3 (Riesz representation theorem)**.** *Take $\phi \in \left(L^1\right)'$. Then there exists a unique function $u \in L^\infty$:*

$$\phi(f) = \int u \cdot f \quad \forall f \in L^1.$$

Moreover,

$$\|u\|_{L^\infty} = \|\phi\|_{(L^1)'}.$$

Remark. This allows us to identify $\left(L^1\right)'$ with L^∞.

Proof. Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of measurable sets in Ω such that:

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n \quad \text{and} \quad \mu(\Omega_n) < \infty \quad \forall n.$$

Set $\chi_n := \chi_{\Omega_n}$.
Claim (1)*.* If such a u exists, it is unique.

Proof (1)*.* Suppose, we have $u_1, u_2 \in L^\infty$:

$$\int u_1 \cdot f = \int_2 u_2 \cdot f \quad \forall f \in L^1.$$

Then

$$\int \underbrace{(u_1 - u_2)}_{=:\tilde{u}} f = 0 \quad \forall f \in L^1.$$

Choosing

$$f = \chi_n \cdot \operatorname{sgn} u,$$

we see $\forall n$:

$$\int_{\Omega_n} |\tilde{u}| = 0 \quad \Rightarrow \quad \tilde{u} = 0 \text{ on } \Omega_n.$$

$\Rightarrow \tilde{u} = 0$ on Ω. □

Now for existence: Consider a function $\theta \in L^2(\Omega)$:

$$\theta(x) \geq \varepsilon_n > 0 \quad \forall x \in \Omega_n.$$

(It is clear, that such a function exists, take eg.

$$\begin{aligned} \theta &:= \alpha_1 \text{ on } \Omega_1, \\ \theta &:= \alpha_n \text{ on } \Omega_n \setminus \Omega_{n-1} \quad \forall n > 1 \end{aligned}$$

and pick α_n such that θ remains square integrable.) The map

$$L^2 \to \mathbb{K}, \quad f \mapsto \phi(\theta \cdot f)$$

is a continuous linear functional on L^2, since by Hölder's inequality, $\theta \cdot f \in L^1$:

$$\|\theta \cdot f\|_{L^1} \leq \|\theta\|_{L^2} \|f\|_{L^2}$$

and ϕ is a continuous linear functional on L^1. By Theorem 7.2, there exists a function $v \in L^2$:

$$\phi(\theta \cdot f) = \int v \cdot f \quad \forall f \in L^2(\Omega). \tag{7.1}$$

Set

$$u(x) := \frac{v(x)}{\theta(x)}. \qquad (\theta > 0 \text{on } \Omega)$$

u is measurable and

$$u \cdot \chi_n \in L^1.$$

Claim (2). u is the sought after function.

Proof (2). Note

$$\phi(\chi_n g) = \int u \chi_n g \quad \forall g \in L^\infty(\Omega), n. \tag{7.2}$$

This follows from picking

$$f = \frac{\chi_n g}{\theta}$$

in (7.1). Further note $f \in L^2$, since it is bounded on Ω_n on zero otherwise.

Claim (2a).

$$\|u\|_{L^\infty} \leq \|\phi\|_{(L^1)'}. \tag{7.3}$$

Proof (2a). Fix any constant $C > \|\phi\|_{(L^1)'}$ and set

$$A := \{x \in \Omega \mid |u(x)| > C\}.$$

Further choose

$$g = \chi_A \cdot \operatorname{sgn} u$$

in (7.2) and get

$$\int_{A \cap \Omega_n} |u| \leq \|\phi\|_{(L^1)'} \cdot \mu(A \cap \Omega_n).$$

Therefore, we have

$$C \cdot \mu(A \cap \Omega_n) \leq \|\phi\|_{(L^1)'} \cdot \mu(A \cap \Omega_n).$$

Since $C > \|\phi\|_{(L^1)'}$, this can only hold, if

$$\mu(A \cap \Omega_n) = 0.$$

By arbitrary choice of n,

$$\mu(A) = \sum_{n \in \mathbb{N}} \mu(A \cap \Omega_n) = 0. \qquad \square$$

Claim (2b). We have

$$\phi(h) = \int u \cdot h \quad \forall h \in L^1(\Omega). \tag{7.4}$$

Proof (2b). Choose

$$g_n := T_n h \in L^\infty$$

and apply (7.2). Note

$$\chi_n T_n h \xrightarrow{L^1(\Omega)} h$$

and take limits on both sides of (7.2) to prove Claim (2b). □

Claim (2c). We have

$$\|u\|_{L^\infty} = \|\phi\|_{(L^1)'}.$$

Proof (2c). By (7.4), we immediately have

$$|\phi(h)| \le \|u\|_{L^\infty} \cdot \|h\|_{L^1}.$$

Therefore,

$$\|\phi\|_{(L^1)'} \le \|u\|_{L^\infty}.$$

Together with (7.3), we get Claim (2c). □

Those claims prove Claim (2). □

The theorem follows from the claims. □

Remark.

1. $L^1(\mathbb{R}^n)$ is not reflexive. Let $\varepsilon_n \to 0$ and take

$$f_n = \chi_{B_{\varepsilon_n}(0)} \cdot \frac{1}{\mu(B_{\varepsilon_n}(0))}.$$

We have

$$\|f_n\|_{L^1} = 1.$$

If $L^1(\mathbb{R}^n)$ were reflexive, we could extract a weakly converging subsequence with limit f, i.e.

$$\int f_n \cdot u \to \int f \cdot u \quad \forall u \in L^\infty.$$

Since

$$\int f_n \cdot u \to 0 \quad \forall u \in L^\infty, 0 \notin \operatorname{supp}(u),$$

in particular for

$$u = \chi_{\mathbb{R}^n \setminus B_{\varepsilon_k}(0)} \quad \forall k \in \mathbb{N},$$

we have,

$$f = 0 \quad \text{a.e.}$$

However

$$\int f_n \cdot 1 = 1 \to 1 \neq \int 0 \cdot 1.$$

2. Indeed, $L^1(\mu)$ is never reflexive, unless μ consists of finitely many atoms (and $L^1(\mu)$ is thus finite dimensional).

7.1.3 Study of L^∞

We already know by 7.3, that $L^\infty = \left(L^1\right)'$ is the dual of a separable space. Therefore, we have

1. $\overline{B_{L^\infty}}$ is weakly* compact. (Banach-Alaoglu)
2. If $\Omega \subset \mathbb{R}^n$ measurable, $(f_n)_{n\in\mathbb{N}}$ bounded in L^∞, by Corollary 6.38 and 2.9, there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ and $f \in L^\infty$:
$$f_{n_k} \xrightarrow[k\to\infty]{*} f.$$
3. However, L^∞ is *not* reflexive (otherwise, L^1 would be reflexive by Corollary 6.29) as long as μ does not consist of finitely many atoms.

   Hence, the dual space of L^∞ is *not* equal to L^1, i.e. there are continuous linear functionals on L^∞ that can not be expressed as an integral with an L^1 function.

   Example. Let
$$\phi_0 : \mathscr{C}_c\left(\mathbb{R}^n\right) \to \mathbb{R}, \quad \phi_0\left(f\right) := f\left(0\right).$$
   ϕ_0 is a continuous linear functional on $\mathscr{C}_c\left(\mathbb{R}^n\right)$. $\mathscr{C}_c\left(\mathbb{R}^n\right)$ is a linear subspace of $L^\infty\left(\mathbb{R}^n\right)$. By Hahn-Banach 4.1, we can extend it to L^∞ and have
$$\phi\left(f\right) = f\left(0\right) \quad \forall f \in \mathscr{C}_c\left(\mathbb{R}^n\right).$$
   Claim. There is no $u \in L^1\left(\mathbb{R}^n\right)$:
$$\phi\left(f\right) = \int u \cdot f \quad \forall f \in L^\infty.$$
   Proof. By contradiction: Note
$$\int uf = 0 \quad \forall f \in \mathscr{C}_c\left(\mathbb{R}^n\right), f\left(0\right) = 0.$$
   This implies
$$u = 0 \quad \text{a.e. on } \mathbb{R}^n.$$
   Thus
$$\phi\left(f\right) = 0 \quad \forall f \in L^\infty\left(\mathbb{R}^n\right),$$
   which is a contradiction. □

Example. Take $\Omega = (0,1), \mu =$ Lebesgue measure. Then $L^\infty(\mu)$ is not separable.

Proof. Take
$$u_t := \chi_{(0,t)}, \quad t \in (0,1).$$
We have
$$t \neq t' \quad \Rightarrow \quad \left\|u_t - u_{t'}\right\|_{L^\infty} \geq 1.$$
Assume now that there were a countable dense set $(v_j)_{j\in\mathbb{N}}$ in L^∞. Let
$$O_t := B_{1/2}\left(u_t\right), \quad t \in (0,1).$$
$\Rightarrow \forall t \in (0,1) : \exists v_j \in O_t$. This produces a map
$$t \mapsto j\left(t\right).$$
This map is injective: Assume $j(t) = j(t')$. $\Rightarrow$
$$v_{j(t)} = v_{j(t')} \in O_t \cap O_{t'}.$$
However, $O_t \cap O_{t'} = \emptyset$, unless $t = t'$. So $(0,1)$ has to be countable, which is a contradiction. □

Conclusion. For $\Omega \subset \mathbb{R}^n$ open, μ Lebesgue measure, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$:

	Reflexivity	Separability	Dual
L^p	✓	✓	$L^{p'}$
L^1	×	✓	L^∞
L^∞	×	×	$\supsetneq L^1$

.

7.2 Weak convergence in $L^p(\mu)$

Corollary 7.4. *1. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $L^p, 1 \le p < \infty$. Then*

$$f_k \xrightarrow{L^p} f \quad \Leftrightarrow \quad \int_\Omega (f_k - f)\cdot g \to 0 \quad \forall g \in L^{p'}.$$

2. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in L^∞. Then

$$f_k \xrightarrow{*\ in\ L^\infty} 0 \quad \Leftrightarrow \quad \int_\Omega (f_k - f)\, g \to 0 \quad \forall g \in L^1.$$

Proof. The theorem follows immediately from Riesz representation 7.2/7.3 and characterisation of weak* convergence in Proposition 6.19. □

Corollary 7.5. *Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $L^p(\mu)$:*

$$\sup_{k\in\mathbb{N}} \|f_k\|_{L^p} < \infty.$$

1. $1 < p < \infty$. There exists a subsequence $\left(f_{k_j}\right)_{j\in\mathbb{N}}$:

$$f_{k_j} \xrightarrow[j\to\infty]{L^p} f.$$

2. $p = \infty$. There exists a subsequence $\left(f_{k_j}\right)_{j\in\mathbb{N}}$:

$$f_{k_j} \xrightarrow[j\to\infty]{L^\infty\ *} f.$$

Proof. Follows from Corollaries to metrisability of weak(*) convergence. □

Example. A typical weakly convergent sequence looks like this:

$$\tilde{f}(x) := \begin{cases} -1, & x \in (0, 1/2) \\ 1, & x \in (1/2, 1) \end{cases} \quad \text{periodically extended to } \mathbb{R}.$$

We have

$$f_k := \tilde{f}(k\cdot) \begin{cases} \xrightarrow{L^p((0,1))} 0, & 1 \le p < \infty \\ \xrightarrow{L^\infty((0,1))\ *} 0 \end{cases}.$$

Idea of the proof:

1. Show $\int_0^1 f_k \cdot \varphi \to 0 \quad \forall \varphi \in \mathscr{C}_c(0,1)$.
2. Use density of $\mathscr{C}_c(0,1)$ in $L^p(0,1)$ (be careful with L^∞).

8 Hilbert spaces

In the following, let H denote a Hilbert space. Thus, we have

1. Cauchy-Schwarz inequality:

$$|(x,y)_H| \le \|x\|_H \|y\|_H \quad \forall x, y \in H.$$

2. Parallelogram identity

$$\frac{1}{2}\|x+y\|_H^2 + \frac{1}{2}\|x-y\|_H^2 = \|x\|_H^2 + \|y\|_H^2 \quad \forall x, y \in H.$$

Example.
- $L^2(\Omega)$ with

$$(x,y)_{L^2} = \int_\Omega xy^*.$$

- $H^1(\Omega)$ with

$$(x,y)_{H^1} = \int_\Omega xy^* + \int_\Omega DxDy^*.$$

8.1 Projection onto convex sets

Proposition 8.1. *H is uniformly convex, thus reflexive.*

Proof. Follows from the Parallelogram identity: Take $\varepsilon > 0, x, y \in H : \|x\| < 1, \|y\| < 1, \|x - y\| > \varepsilon \Rightarrow$

$$\frac{1}{4}\|x+y\|^2 < 1 - \frac{\varepsilon^2}{4}.$$

Thus

$$\left\|\frac{x+y}{2}\right\| < 1 - \delta, \quad \delta := 1 - \left(1 - \frac{\varepsilon^2}{4}\right)^{1/2} > 0.$$

□

Theorem 8.2 (Projection)**.** *Take $K \subset H$ closed, not empty, convex. Further take $f \in H$. Then there exists a unique $u \in K$:*

$$\|u - f\| = \min_{v \in K} \|v - f\|$$

and u is characterised as the only $u \in K$:

$$\operatorname{Re}(f - u, v - u) \le 0 \quad \forall v \in K.$$

Notation. We write

$$u = P_K f.$$

Proof.

1. *Existence*:

$$\varphi : K \to \mathbb{R}, \quad v \mapsto \|f - v\|$$

is strongly continuous, convex and

$$\varphi(v) \xrightarrow[\|v\| \to \infty]{} \infty.$$

By Theorem 6.32, there exists a minimiser u of φ.

2. *Characterisation*: Exercise.
3. *Uniqueness*: Exercise as well.

□

Proposition 8.3. *For $K \subset H$ not empty, closed, convex, $P_K : H \to H$ is continuous, in particular*

$$\|P_k f_1 - P_k f_2\| \le \|f_1 - f_2\| \quad \forall f_1, f_2 \in H.$$

Proof. Take $u_j := P_K f_j, \quad j \in [2]$. By Theorem 8.2, we have

$$\begin{aligned} \operatorname{Re}(f_1 - u_1, v - u_1) &\le 0 \quad \forall v \in K, \\ \operatorname{Re}(f_2 - u_2, v - u_2) &\le 0 \quad \forall v \in K. \end{aligned}$$

Take $v = u_2$ in the first and $v = u_1$ in the second inequality.

$$\begin{aligned} \operatorname{Re}(f_1 - u_1, u_2 - u_1) &\le 0 \\ \operatorname{Re}(f_2 - u_2, u_1 - u_2) &\le 0 \\ \Rightarrow \quad \operatorname{Re}(f_1 - f_2 + u_2 - u_1, u_2 - u_1) &\le 0 \\ \operatorname{Re}(f_1 - f_2, u_2 - u_1) + \|u_2 - u_1\|^2 &\le 0 \\ \|u_2 - u_1\|^2 &\le -\operatorname{Re}(f_1 - f_2, u_2 - u_1) \\ &\le \|f_2 - f_1\| \cdot \|u_2 - u_1\| \qquad \text{(CS inequality)} \end{aligned}$$

□

Corollary 8.4. *Let $M \subset H$ be a closed subspace. Then*

$$u = P_M f$$

is characterised by $u \in M$:

$$(f - u, v) = 0 \quad \forall v \in M.$$

Proof. Exercise. □

8.2 Dual spaces of Hilbert spaces and the theorem of Lax and Milgram

Theorem 8.5 (Riesz-Fréchet)**.** *Take $\varphi \in H'$. Then there exists a unique $f \in H$:*

$$\varphi(u) = (f, u)_H \quad \forall u \in H.$$

Proof. Take

$$T : H \to H', \quad Tf : H \to \mathbb{R}, \ (Tf)(u) := (f, u)_H .$$

We certainly have

$$\|Tf\|_{H'} = \|f\|_H . \qquad \text{(CS inequality and } u = f \text{as a test function)}$$

Thus T is a linear isometry and $T(H)$ is a closed subspace of H'.

It remains to show density of $T(H)$ in H'. Let $\tilde{h} \in H''$:

$$\tilde{h}(\varphi) = 0 \quad \forall \varphi \in T(H).$$

By reflexivity, we can represent $\tilde{h}$ by $h \in H$:

$$\varphi(h) = \tilde{h}(\varphi) = 0 \quad \forall \varphi \in T(H).$$

Thus

$$\begin{aligned} (Tf)(h) &= 0 \quad \forall f \in H \\ (f, h)_H &= 0 \quad \forall f \in H \\ h &= 0. \end{aligned}$$

It follows immediately that $\tilde{h} = 0$ and thus $T(H)$ is dense in H'. By closedness,

$$T(H) = H'$$

and T is a linear, isometric bijection. □

Definition 8.6. A sesquilinear map

$$a : H \times H \to \mathbb{K}$$

is called

- *continuous,* if $\exists C > 0$:
$$|a(u, v)| \leq C \|u\| \|v\| \quad \forall u, v \in H.$$
- *coercive*, if $\exists \alpha > 0$:
$$|a(v, v)| \geq \alpha \|v\|^2 \quad \forall v \in H.$$

Theorem 8.7 (Lax-Milgram, real version)**.** *Let H be a real Hilbert space, $a : H \times H \to \mathbb{R}$ a continuous, coercive, bilinear form and $f \in H'$. Then there exists a unique $u \in H$:*

$$a(u, v) = f(v) \quad \forall v \in H.$$

Furthermore, if a is symmetric, u is characterised by $u \in H$:

$$\frac{1}{2} a(u, u) - f(u) = \min_{v \in H} \left(\frac{1}{2} a(v, v) - f(v) \right).$$

Proof. Fix $v \in H$. The map

$$u \mapsto a(u, v)$$

is a continuous linear functional an H. By Theorem 8.5, there exists a unique $w \in H$:

$$a(u, v) = (w, v)_H .$$

Doing that, we created a map $A : H \to H$ given by

$$A : u \mapsto Au := w.$$

Claim (1). A is linear and continuous.

1. Linearity follows immediately by bilinearity of $a(\cdot,\cdot)$. We also have

$$\|Au\|_H^2 = (Au, Au)_H = a(u, Au) \le C\|u\|_H \|Au\|_H. \qquad (a\text{continuous})$$

Dividing by $\|Au\|_H$ yields continuity. □

Claim (2). A is injective and $A(H)$ is a closed subspace of H.

2.

$$\begin{aligned} & \alpha\|u\|_H^2 \le a(u,u) = (u, Au)_H \le \|u\|_H \|Au\|_H \\ \Rightarrow\quad & \alpha\|u\|_H \le \|Au\|_H \end{aligned}$$

and injectivity follows. Closedness of $A(H)$ follows by the sequence criterion. □

Claim (3). $A(H) = H$.

Proof. Due to closedness, it suffices to show density. Consider thus $\tilde{h} \in H'$:

$$\tilde{h}(v) = 0 \quad \forall v \in A(H).$$

By Riesz-Fréchet 8.5, equivalently take $h \in H$:

$$(v, h)_H = 0 \quad \forall v \in A(H).$$

We have

$$0 = (Ah, h)_H = a(h, h) \ge \alpha\|h\|_H^2.$$

$\Rightarrow \|h\|_H = 0$ and the claim follows. □

A second application of Riesz-Fréchet 8.5 yields the existence of a unique $w \in H$:

$$f(v) = (v, w)_H \quad \forall v \in H.$$

Now pick $u \in H$:

$$Au = w.$$

With this choice, we have

$$a(u, v) = (Au, v)_H = (w, v)_H = f(v) \quad \forall v \in H.$$

Existence is thus proven.

Uniqueness follows from coercivity:

$$\begin{aligned} & a(u_1, v) = f(v) = a(u_2, v) && \forall v \in H \\ \Rightarrow\quad & a(u_1 - u_2, v) = 0 && \forall v \in H \\ \Rightarrow\quad & a(u_1 - u_2, u_1 - u_2) = 0 \\ \Rightarrow\quad & \|u_1 - u_2\|_H = 0. \end{aligned}$$

Let now a be symmetrical. Then $(H, a(\cdot,\cdot))$ is also a Hilbert space and the norm from the a-scalar product is equivalent to the original norm. Using Riesz-Fréchet 8.5, we can thus represent the continuous linear form f by an element $g \in H$:

$$a(g, v) = f(v) \quad \forall v \in H.$$

By $a(u, v) = f(v)$, it follows that

$$a(u - g, v) = 0 \quad \forall v.$$

This however, is nothing but the projection of g onto the whole space H w.r.t. the new a-scalar product. Therefore, u solves the minimisation problem

$$\min_{w\in H}\sqrt{a\left(g-w,g-w\right)},$$

which is equivalent to minimising

$$w\mapsto a\left(g-w,g-w\right)=a\left(w,w\right)-2a\left(g,w\right)+\underbrace{a\left(g,g\right)}_{\text{constant}},$$

and therefore equivalent to minimzing

$$w\mapsto a\left(w,w\right)-2\left(g,w\right).$$ □

Remark. We see, that in the symmetic case, the proof reduces to a single application of Riesz-Fréchet 8.5.

Theorem 8.8 (Complex version of Lax-Milgram)**.** *Assume $A\in\mathscr{L}\left(H\right),\alpha>0$ satisfy*

$$\left|\left(Au,u\right)\right|\geq\alpha\left\|u\right\|^{2}\quad\forall u\in H.$$

Then A is bijective.

Proof. Follows from exactly as the proof of the claims in the real version. □

8.3 Orthonormal basis in Hilbert spaces

Definition 8.9. Let $\left(E_n\right)_{n\in\mathbb{N}}$ be a sequence of closed subspaces of H. We say that H is the *Hilbert sum* of the E_n and write

$$H=\bigoplus_{n}E_n,$$

if we have

1. The E_n are pairwise orthogonal, i.e.

$$\left(u,v\right)_H=0\quad\forall u\in E_n,v\in E_m,n\neq m.$$

2. $\operatorname{span}\left\{E_n\right\}_{n\in\mathbb{N}}$ is dense in H.

Lemma 8.10. *Let $\left(v_n\right)_{n\in\mathbb{N}}$ be a sequence in H :*

$$\left(v_n,v_m\right)=0\quad\forall n\neq m$$

and assume that

$$\sum_{k=1}^{\infty}\left\|v_k\right\|_H^2<\infty.$$

Define

$$S_n:=\sum_{k=1}^{n}v_n.$$

Then

$$S:=\lim_{n\to\infty}S_n$$

exists and

$$\left\|S\right\|_H^2=\sum_{k=1}^{\infty}\left\|v_k\right\|_H^2.$$

Proof. For $m > n$, we have

$$\|S_m - S_n\|_H^2 = \sum_{k=n+1}^{m} \|v_n\|^2 + \underbrace{\text{mixed terms}}_{=0 \text{ by orthogonality}}.$$

Therefore S_m is a Cauchy sequence and converges to $S \in H$. Furthermore, for all$n \in \mathbb{N}$:

$$\|S_n\|^2 = \sum_{k=1}^{n} \|v_k\|^2$$

and the claim follows in the limit. □

Theorem 8.11 (Bessel-Parseval identity)**.** *Take $H = \bigoplus_n E_n, u \in H$ and consider*

$$u_n := P_{E_n} u.$$

Let

$$S_n := \sum_{k=1}^{n} u_n.$$

Then we have

$$\lim_{n\to\infty} S_n = u, \quad \sum_{k=1}^{\infty} \|u_k\|^2 = \|u\|^2.$$

Proof. Let $u_n = P_{E_n} u$. By the projection theorem 8.2, we have

$$(u - u_n, v)_H = 0 \quad \forall v \in E_n.$$

In particular,

$$(u_n, u)_H = (u - u_n, u_n)_H + (u_n, u_n)_H = \|u_n\|_H^2.$$

Thus,

$$(u, S_n) = \sum_{k=1}^{n} \|u_k\|^2.$$

At the same time, we have

$$\sum_{k=1}^{n} \|u_k\|_H^2 = \|S_n\|_H^2 \qquad \text{(by Lemma 8.10)}$$

and therefore,

$$(u, S_n)_H = \|S_n\|_H^2.$$

Using Cauchy-Schwarz inequality on the left, we get

$$\|u\|_H \|S_n\|_H \geq \|S_n\|_H^2$$

and thus

$$\|S_n\|_H \leq \|u\|_H.$$

This yields

$$\sum_{k=1}^{n} \|u_k\|_H^2 = \|S_n\|_H^2 \leq \|u\|_H^2.$$

Using Lemma 8.10, we get

$$S = \lim_{n\to\infty} S_n.$$

Claim. Let $F := \operatorname{span} E_n$. Then $S = P_{\overline{F}} u$.

Proof. Note that we have

$$u - S_n = (u - u_m) - \sum_{\substack{k \le n \\ k \ne m}} u_k \quad \forall m \le n.$$

It follows

$$(u - S_n, v)_H = 0 \quad \forall v \in E_m, m \le n.$$

Taking $n \to \infty$, we get

$$\begin{aligned} & (u - S, v)_H = 0 \quad \forall v \in E_m, m \in \mathbb{N} \\ \Rightarrow \quad & (u - S, v)_H = 0 \quad \forall v \in \operatorname{span} E_m = F \\ \Rightarrow \quad & (u - S, v)_H = 0 \quad \forall v \in \overline{F} \qquad \text{(continuity.)} \end{aligned}$$

This proves the claim. □

By density of $\operatorname{span} E_n$ in H it follows that

$$S = P_H u = u.$$

The Bessel-Parseval identity follows with

$$\sum_{j=1}^{n} \|u_j\|_H^2 = \|S_n\|_H^2$$

in the limit. □

Definition 8.12. A sequence $(e_n)_{n\in\mathbb{N}}$ in H is called *Hilbert basis* (or ONB) of H, if

1. $\|e_n\|_H = 1, \quad (e_n, e_m)_H = 0 \quad \forall n \ne m.$
2. $\operatorname{span} \{e_n\}_{n\in\mathbb{N}}$ is dense in H.

Corollary 8.13. *Let $(e_n)_{n\in\mathbb{N}}$ be a Hilbert basis of H, $u \in H$. Then*

$$u = \sum_{k=1}^{\infty} (u, e_k)_H \, e_k := \lim_{n\to\infty} \sum_{k=1}^{n} (u, e_k)_H \, e_k$$

and

$$\|u\|_H^2 = \sum_{k=1}^{\infty} |(u, e_k)_H|^2 .$$

Conversely, for $(\alpha_k)_{k\in\mathbb{N}} \in l^2$:

$$\sum_{k=1}^{\infty} \alpha_k e_k =: u, \quad (u, e_n)_H = \alpha_n.$$

Proof.

$$H = \bigoplus_n E_n, \quad E_n := \mathbb{K} e_n.$$

We thus have

$$P_{E_n} u = (u, e_n) \, e_n.$$

The claim follows with Theorem 8.11 and Lemma 8.10. □

Remark. We don't necessarily have absolute convergence. Find an example, such that

$$\sum_{k=1}^{\infty} |(u, e_k)| = \infty.$$

E.g. $H = l^2, u = \left(\frac{1}{k}\right)_{k\in\mathbb{N}}$, e_n as usual.

Theorem 8.14. *Every separable Hilbert space admits a Hilbert basis.*

Proof. Take $(v_n)_{n\in\mathbb{N}}$ dense (countable) in H and

$$F_n = \operatorname{span}\{v_k\}_{k=1}^n .$$

$(F_n)_n$ is a sequence of finite dimensional, thus closed subspaces of H. We have

$$F_k \supset F_j \quad \forall k \geq j \quad \text{and} \quad \overline{\bigcup_{k=1}^{\infty} F_n} = H.$$

We now iteratively pick

$$\begin{aligned} e_1 \in F_1 : & & \|e_1\|_H = 1, & \\ e_2 \in F_j, j > 1 : & \ \dim F_j = 2, & \|e_2\|_H = 1, & \quad (e_1, e_2)_H = 0, \\ e_k \in F_{j_k}, j_k > j_{k-1} : & \dim F_{j_k} = k, & \|e_K\|_H = 1, & \quad (e_i, e_k)_H = 0 \ \forall i < k. \end{aligned}$$

□

Remark. It follows that every separable ∞-dimensional Hilbert space is isomorphic to l^2.

9 Some theory of compact operators

In the following, X, Y are Banach spaces and B_X is the unit ball in X.

9.1 Compact Operators and the adjoint Operator

Definition 9.1. A bounded linear operator $T \in \mathscr{L}(X,Y)$ is called *compact*, if $\overline{T(B_X)}$ is compact in Y (w.r.t. strong topology).

The set of compact operators from X to Y is called $\mathscr{K}(X,Y)$. We write $\mathscr{K}(X) := \mathscr{K}(X,X)$.

Theorem 9.2. *$\mathscr{K}(X,Y)$ is a closed subspace of $\mathscr{L}(X,Y)$ (w.r.t. operator norm*

$$\|A\|_{\mathscr{L}(X,Y)} = \sup_{\substack{x\in X \\ \|x\|_X \leq 1}} \|Ax\|_Y).$$

Proof. Linearity is obvious. To show closedness, consider $(T_n)_{n\in\mathbb{N}}, T \in \mathscr{L}(X,Y)$:

$$T_n \in \mathscr{K}(X,Y) \quad \forall n, \quad \text{and} \quad \|T_n - T\|_{\mathscr{L}(X,Y)} \to 0.$$

Fix $n \in \mathbb{N}$:

$$\|T_n - T\|_{\mathscr{L}(X,Y)} < \frac{\varepsilon}{2}.$$

By compactness of T_n, there exists a finite cover of $T_n(\overline{B_X})$ by $\frac{\varepsilon}{2}$-balls:

$$T_n(\overline{B_X}) \subset \bigcup_{j=1}^{N} B_{\varepsilon/2}(y_j), \quad y_j \in Y.$$

It follows

$$T(\overline{B_X}) \subset \bigcup_{j=1}^{N} B_{\varepsilon}(y_i), \quad y_i \in Y.$$

Thus $T(\overline{B_X})$ is precompact and $\overline{T(\overline{B_X})}$ is compact. □

Definition 9.3. The *range* of an operator $T \in \mathscr{L}(X,Y)$ is

$$R(T) := \{y \in Y \mid \exists x \in X : T(x) = y\}.$$

The *null space* of T is

$$N(T) := \{x \in X \mid T(x) = 0\}.$$

Definition 9.4. An operator $T \in \mathscr{L}(X,Y)$ has *finite range*, if

$$\dim R(T) < \infty.$$

Remark. Every finite range operator is compact.

Corollary 9.5. *The limit of a sequence of finite range operators is compact.*

Proof. The space of compact operators is closed. □

Proposition 9.6. *Let X, Y, Z be Banach spaces, $T \in \mathscr{K}(Y,Z), S \in \mathscr{L}(X,Y)$. Then*

$$T \circ S \in \mathscr{K}(X,Z).$$

Take $U \in \mathscr{L}(Y,Z), V \in \mathscr{K}(X,Y)$, Then

$$U \circ V \in \mathscr{K}(X,Z).$$

Proof. Exercise. □

Definition 9.7. Let $T \in \mathscr{L}(X,Y)$. The adjoint operator $T^* \in \mathscr{L}(Y', X')$ is defined by

$$(T^* f)(x) := f(Tx) \quad \forall x \in X, f \in Y'.$$

Remark. Take $v \in Y'$. We define $\forall u \in X$:

$$g(u) := v(Tu) \quad \Rightarrow \quad |g(u)| \leq C \|u\|_X.$$

Write $T^* v = g$. It follows, that

$$\|T^*\|_{\mathscr{L}(Y',X')} = \|T\|_{\mathscr{L}(X,Y)}.$$

Remark. Hilbertspaces.

Theorem 9.8 (Schauder)**.**

$$T \in \mathscr{K}(X,Y) \quad \Leftrightarrow \quad T^* \in \mathscr{K}(Y', X').$$

Proof. "⇒" Consider $(v_n)_{n\in\mathbb{N}}$ a sequence in $B_{Y'}$. We show that $(T^* v_n)_{n\in\mathbb{N}}$ admits a converging subsequence. Take

$$S := \overline{T\left(\overline{B_X}\right)}.$$

S is a compact metric space. Define

$$\mathcal{H} := \{\varphi_n : S \to \mathbb{K},\ x \mapsto v_n(x)\}_{n\in\mathbb{N}} \subset \mathscr{C}(S).$$

These functions are not only continuous, but uniformly equicontinuous.
By Arzelà-Ascoli, it follows $\exists \varphi : S \to \mathbb{K}, (\varphi_{n_k})_{k\in\mathbb{N}}$:

$$\varphi_{n_k} \xrightarrow{\text{unif. } \mathscr{C}(S)} \varphi.$$

Thus,

$$\begin{aligned}
& \sup_{x\in S} |\varphi_{n_k}(x) - \varphi(x)| \xrightarrow[k\to\infty]{} 0 \\
\Rightarrow \quad & \sup_{u\in B_X} |v_{n_k}(Tu) - \varphi(Tu)| \xrightarrow[k\to\infty]{} 0 \\
\Rightarrow \quad & \sup_{u\in B_X} |v_{n_k}(Tu) - v_{n_l}(Tu)| \xrightarrow[k,l\to\infty]{} 0 \\
\Rightarrow \quad & \|T^* v_{n_k} - T^* v_{n_l}\|_{\mathscr{L}(Y',X')} \xrightarrow[k,l\to\infty]{} 0 \\
\Rightarrow \quad & T^* v_{n_k} \text{ converges in } X'.
\end{aligned}$$

"$\Leftarrow$" $\quad \mathrm{J}(T(\overline{B_X})) = T^{**}(\overline{B_X})$ is precompact in Y'' and $\mathrm{J}(Y)$ is closed in Y''. □

Remark. It follows that $x_n \xrightarrow{X} x, T \in \mathscr{K}(X,Y) \Rightarrow$

$$Tx_n \xrightarrow{Y} Tx.$$

Definition 9.9 (Anihilator)**.** Let X be a Banach space, $M \subset X$ a linear subspace. the anihilator of M is

$$M^\perp := \{f \in X' \mid f(x) = 0 \;\; \forall x \in M\}.$$

Let $N \subset X'$ be a linear subspace. The *anihilator* of N is

$$N^\perp := \{x \in X \mid f(x) = 0 \;\; \forall f \in N\}.$$

Remark.

- This simplifies in Hilbert spaces.
- It is clear that $M^\perp, N^\perp$ are both closed subspaces.
- By definition $N^\perp \subset X$ and not $N^\perp \subset X''$!

Proposition 9.10. *We have*

1. $\left(M^\perp\right)^\perp = \overline{M}$,
2. $\left(N^\perp\right)^\perp \supset \overline{N}$.

Proof. 1. From the definition, it is clear that $\left(M^\perp\right)^\perp \supset M$, since

$$x \in M \quad \Rightarrow \quad f(x) = 0 \;\; \forall f \in M^\perp \quad \Rightarrow \quad x \in \left(M^\perp\right)^\perp.$$

Since $\left(M^\perp\right)^\perp$ is closed, we also have

$$\left(M^\perp\right)^\perp \supset \overline{M}.$$

Assume that $\exists x_0 \in \left(M^\perp\right)^\perp \setminus \overline{M}$.
By the second separation theorem 4.13, there exists $f \in X', \alpha \in \mathbb{R}:$

$$\operatorname{Re} f(x) < \alpha < \operatorname{Re} f(x_0) \quad \forall x \in \overline{M}.$$

By$-M = M$, it follows

$$\begin{aligned} \operatorname{Re} f(x) &= 0 \quad \forall x \in \overline{M}. \\ \operatorname{Re} f(x_0) &> 0. \end{aligned}$$

Thus, by $iM \subset M$,

$$\begin{aligned} & f(x) = 0 \quad \forall x \in M. \\ \Rightarrow \quad & f \in M^\perp \\ \Rightarrow \quad & f(x_0) = 0 \quad ↯. \end{aligned}$$

2. The inclusion follows as in 1. □

Remark. $\overline{N} = \left(N^\perp\right)^\perp$ only holds in reflexive spaces.

Proposition 9.11. *Consider $A \in \mathscr{L}(X)$. We have*

1. $N(A^*) = R(A)^\perp$,
2. $N(A) = R(A^*)^\perp$.

Proof. 1. Take $f \in N(A^*)$, i.e.

$$\begin{aligned} & A^* f(x) = 0 \quad \forall x \in X \\ \Leftrightarrow \quad & f(Ax) = 0 \quad \forall x \in X \\ \Leftrightarrow \quad & f(y) = 0 \quad \forall y \in R(A) \\ \Leftrightarrow \quad & f \in R(A)^{\perp}. \end{aligned}$$

2. Analogous. □

Corollary 9.12.

$$N(A^*) = \overline{R(A)}.$$

Proof. Follows immediately from 9.11 and 9.10. □

We state the following theorem only in Hilbert spaces, for a full version in Banach spaces (which is slightly more technical to prove), see the book by Brezis.

Theorem 9.13 (Fredholm alternative)**.** *Let H be a Hilbert space, $K \in \mathscr{K}(H, H)$ a compact operator. Then*

1. $N(\mathrm{id} - K)$ *is finite dimensional,*
2. $R(\mathrm{id} - K)$ *is closed,*
3. $R(\mathrm{id} - K) = N(\mathrm{id} - K^*)^{\perp}$,
4. $N(\mathrm{id} - K) = \{0\}$ *if and only if* $R(\mathrm{id} - K) = H$,
5. $\dim N(\mathrm{id} - K) = \dim N(\mathrm{id} - K^*)$.

Remark. The theorem concerns the solvability of the equation $u - Tu = f$:

- Either the equation admits a unique solution for all $f \in X$
- or the homogeneous equation $u - Tu = 0$ has n linearly independent solutions.
- $u - Tu = f$ is then solvable exactly if f admits n orthogonal solutions, i.e.

$$f \in N(\mathrm{id} - T^*)^{\perp}.$$

Proof. See Evans, Partial Differential Equations, Appendix D.5. □

For information on applications and Sobolev spaces I recommend Chapter 5 in the book by Evans.