Exercises
"Mathematical Modeling"

Sheet 1

Due: Wednesday 07.05.2025, 14:00.
Letterbox 3.21 in the basement of Ernst-Zermelo-Str.1

Please hand in as pairs of students

Exercise 1: (*Nondimensionalization*) (4 Points)

We want to compute the power P, which is necessary to move a body with known shape (for example a ship) in a liquid (for example water). We assume that the power depends on the length l and the velocity v of the ship, the density ρ and the kinematic viscosity η of the liquid, and the gravitational acceleration g. The dimensions of the data are $[l] = L, [\rho] = \frac{M}{L^3}, [v] = \frac{L}{T}, [\eta] = \frac{L^2}{T}, [P] = \frac{ML^2}{T^3}$, and $[g] = \frac{L}{T^2}$, where L denotes the length, M the mass and T the time. Show that under these assumptions the power P is given by

$$\frac{P}{\rho l^2 v^3} = \Phi(\mathrm{Fr}, \mathrm{Re})$$

with a function $\Phi : \mathbb{R}^2 \to \mathbb{R}$ and the dimensionless quantities

$$Re = \frac{|v|l}{\eta} \text{ (Reynolds number) and } Fr = \frac{|v|}{\sqrt{lg}} \text{(Froude number).}$$

Exercise 2: (*Consistency versus convergence*) (2+2 Points)

For a parameter $\varepsilon \in [0, \varepsilon_0)$ with $\varepsilon_0 > 0$ we consider the family of operators

$$F(\cdot, \varepsilon) : B_1 := C_b^2([0, \infty)) \to B_2 := C_b^0([0, \infty)) \times \mathbb{R}^2,$$

$$F(y, \varepsilon) = (y'' + (1 + \varepsilon)y, y(0), y'(0) - 1).$$

Here $C_b^n([0, \infty))$ denotes the vector space of n-times differentiable functions with bounded derivatives up to order n. The norms of the spaces B_1 and B_2 are given by

$$\|y\|_{B_1} = \sup_{t \in (0,\infty)} \{|y(t)| + |y'(t)| + |y''(t)|\},$$

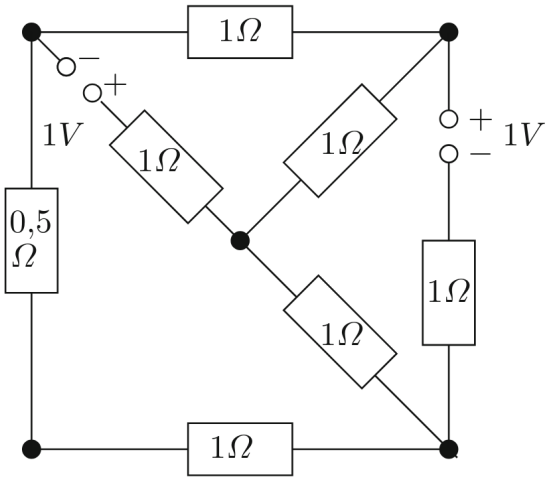
$$\|(f, a, b)\|_{B_2} = \sup_{t \in (0,\infty)} \{|f(t)|\} + |a| + |b|.$$

(a) For the problem $F(y, \varepsilon) = (0, 0, 0)$ compute the exact solution y_ε.

(b) Show: $F(\cdot, \varepsilon)$ is consistent with $F(\cdot, 0)$, but y_ε does not converge to y_0 in B_1 as $\varepsilon \to 0$.

Exercise 3: (4 Points)

Determine the currents and voltages in the following network:



Exercise 4: (2+2 Points)

Construct a network, without electrical devices for the following incidence matrices:

(a) $B = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$

(b) $B = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$