Exercises
Mathematical Modeling

Sheet 4

Due: Wednesday 25.06.2025, 14:00.
Letterbox 3.21 in the basement of Ernst-Zermelo-Str.1

Please hand in as pairs of students

Exercise 13: (4 Points)

For an increment function Φ and $z_k \in \mathbb{R}$, let $z : [t_k, t_{k+1}] \to \mathbb{R}$ be the solution of the initial value problem $\dot{z}(t) = f(t, z(t))$, $\quad z(t_k) = z_k$, and define $z_{k+1} = z_k + \tau\,\Phi(t_k, z_k, z_{k+1}, \tau)$. Define the consistency terms C and $\tilde{C}$ by

$$C(t_k, z_k, \tau) = \frac{z(t_{k+1}) - z_k}{\tau} - \Phi(t_k, z_k, z_{k+1}, \tau),$$

$$\tilde{C}(t_k, z_k, \tau) = \frac{z(t_{k+1}) - z_k}{\tau} - \Phi(t_k, z_k, z(t_{k+1}), \tau).$$

Assume that the increment function Φ is uniformly Lipschitz continuous in its third argument with Lipschitz constant L. Show that for $\tau \le \frac{1}{2L}$, the equivalence

$$c^{-1}\,\tilde{C}(t_k, z_k, \tau) \le C(t_k, z_k, \tau) \le c\,\tilde{C}(t_k, z_k, \tau)$$

holds. Give the constant c, which depends only on L, explicitly.

Exercise 14: (2+2 Points)

Let f be a Lipschitz continuous function.

(i) Show that the implicit Euler method

$$y_{k+1} = y_k + \tau f(t_{k+1}, y_{k+1})$$

is consistent of order $p = 1$, i.e. there exists a constant $c \ge 0$ such that for small enough τ,

$$|C(t_k, z_k, \tau)| \le c\tau.$$

(ii) Let $f \in C^2([0, T] \times \mathbb{R})$. Show that the method

$$y_{k+1} = y_k + \tau f(t_k, y_k) + \frac{\tau^2}{2}\left(\partial_t f(t_k, y_k) + \partial_y f(t_k, y_k) f(t_k, y_k)\right)$$

is consistent of order $p = 2$, meaning there exists some $c \ge 0$ such that for small enough τ,

$$|C(t_k, z_k, \tau)| \le c\tau^2.$$

Exercise 15: (4 Points)

Determine constants $a, b, c, d \in \mathbb{R}$ such that the explicit one-step method defined by the increment function

$$\Phi(t_k, y_k, \tau) = a\,f(t_k, y_k) + b\,f\left(t_k + c\tau,\ y_k + \tau d\,f(t_k, y_k)\right)$$

has consistency order $p = 2$.
Hint: Justify and use the approximation

$$f(t + c\tau,\ y + d\tau f(t, y)) = f(t, y) + \partial_t f(t, y)\,c\tau + \partial_y f(t, y)\,d\tau f(t, y) + \mathcal{O}(\tau^2),$$

and differentiate the differential equation.

Exercise 16: (4 Points)

Use the implicit function theorem to ensure the existence of a unique solution y_{k+1} of the equation

$$y_{k+1} = y_k + \tau\,\Phi(t_k, y_k, y_{k+1}, \tau)$$

under suitable assumptions on the function Φ and the step size τ.