We are looking at

$$-\Delta u = f \quad \text{in } \Omega = (0,1)^2$$
$$u = 0 \quad \text{on } \partial\Omega$$

Discretization

$$-\Delta_h U_{j,m} = f(x_{j,m})$$
$$\shortparallel$$
$$-\frac{1}{h^2}\Big(U_{j+1,m} + U_{j,m+1} - 4U_{j,m} + U_{j-1,m} + U_{j,m-1}\Big)$$

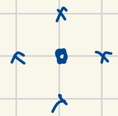
Lemma 2.2.17. (Discrete Maximum Principle)

If $U = (U_{j,m} \,/\, j,m = 0,\dots,J)$ satisfies $-\Delta_h U_{j,m} \le 0$ for all $j,m = 1,\dots,J-1$, then U attains its maximum for $j=0$, $j=J$, $m=0$, or $m=J$.

Pf. From $-\Delta_h U_{j,m} \le 0$, we get

$$U_{j,m} \le \frac{1}{4}\Big(U_{j-1,m} + U_{j+1,m} + U_{j,m-1} + U_{j,m+1}\Big)$$

for $1 \le j, m \le J-1$



If, thus, $U_{j,m}$ is a maximum,

the estimate has to hold with equality,

and the max. is also attained at

all ~~one~~ of the neighbouring points, we can

continue this until we reach a boundary.

$\square$

Lemma 2.2.18 (Discrete boundedness) For all

$(Z_{j,m}, j,m = 0,\dots,J)$ with $Z_{j,m} = 0$

for $j=0$, $j=J$, $m=0$, $m=J$, we have

$$\max_{j,m=0,\dots,J} |Z_{j,m}| \le \frac{1}{2} \sup_{j,m=1,\dots,J-1} |\Delta_h Z_{j,m}|$$

Pf. Write $S = \max\limits_{j,m=1,\dots,J-1} |\Delta_h Z_{j,m}|$, and set

$W_{j,m} = (jh)^2 + (mh)^2$, which,

on the grid points, coincides with

$w(x_1, x_2) = x_1^2 + x_2^2$

Notice $\omega_{j,m} \geq 0 \quad j,m = 0,\dots,N$ and

$$\Delta_h \omega_{j,m} = 4 \quad , j,m = 1,\dots,N-1.$$

Now set

$$V_{j,m} = z_{j,m} + \frac{S}{4}\omega_{j,m}.$$

and get

$$-\Delta_h V_{j,m} = -\Delta_h z_{j,m} - S \leq 0$$

The discrete max. principle implies that $V_{j,m}$ attains its maximum on the boundary, there

$$z_{j,m} = 0 \quad \text{and} \quad 0 \leq \omega_{j,m} \leq 2.$$

Therefore $z_{j,m} = V_{j,m} - \frac{S}{4}\omega_{j,m} \leq \frac{S}{2}$

Similarly, the result

$$-z_{j,m} \leq S/2$$

holds, and we obtain the lemma. $\square$

Prop 2.2.19 (Error estimate)

Let $u \in C^2(\overline{\Omega})$, and $U = (U_{j,m}, j,m = 0 \dots N)$

be sol'ns to the Poisson boundary value problem

$$-\Delta u = f \quad \text{in} \quad (0,1)^2 = \Omega$$
$$u = 0 \quad \text{on} \quad \partial\Omega$$

and its discretization, respectively.

Then we have

$$\sup_{j,m=0,\dots,J} |u(x_{j,m}) - U_{j,m}| \leq$$

$$\frac{h^2}{24}\Big(\overbrace{\|\partial^4_{x_1}u\|_{C^0([0,1]^2)} + \|\partial^4_{x_2}u\|_{C^0([0,1]^2)}}^{M}\Big)$$

Pf. Since $-\Delta u(x_{j,m}) = f(x_{j,m})$ for $j,m = 0,\dots,J$, the error

$$Z_{j,m} = \underset{\overset{\shortparallel}{u_{j,m}}}{u(x_{j,m})} - U_{j,m} \text{ satisfies.}$$

$$-\Delta_h Z_{j,m} = -\Delta_h u_{j,m} + \Delta_h U(x_{j,m})$$

$$= f(x_{j,m}) - f(x_{j,m}) + \Delta u(x_{j,m}) -$$
$$- \Delta_h u(x_{j,m})$$

$$= \partial_{x_1}^2 u(x_{j,m}) - \partial_{x_1}^+ \partial_{x_1}^- u(x_{j,m})$$

$$+ \partial_{x_2}^2 (x_{j,m}) - \partial_{x_2}^+ \partial_{x_2}^- (x_{j,m})$$

From our estimates on the difference quotients, we get

$$|-\Delta_h z_{j,m}| \leq \frac{h^2}{12} M.$$

Together with the discrete boundedness lemma we obtain the result.

$\square$

2.3. Heat equation

We study the equation

$$u_t - \Delta u = 0$$

and its non-homog. pendant

$$u_t - \Delta u = f$$

Subject to appropriate initial and boundary conditions. We take $t > 0$, $x \in \Omega$, $\Omega \subset \mathbb{R}^n$ open. The sought after function is

$$u : \overline{\Omega} \times [0, \infty) \to \mathbb{R}, \quad u = u(x,t),$$

and the Laplacian is taken w.r.t the spatial variables $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. The function $f : \Omega \times [0, \infty) \to \mathbb{R}$ is given.

For a physical interpretation, consider $V \subset \Omega$, then

$$\frac{d}{dt} \int_V u \, dx = - \int_{\partial V} F \cdot \nu \, dS$$

Flux density

(if the quantity does not get produced or destroyed within V).

Again we assume $F = a \cdot \nabla u \quad (a > 0)$, and divergence theorem yields

$$u_t = \operatorname{div} a \nabla u = a \Delta u$$

2.3.1 Fundamental solution.

Consider functions of the form

(*) $$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right) \quad x \in \mathbb{R}^n, t > 0.$$

with constants α, β, and the function $v: \mathbb{R}^n \to \mathbb{R}$ to be found.

Eq. (*) shows up if we look for sol'ns to heat eq. that are invariant under the scaling

$$u(x,t) \mapsto \lambda^\alpha u(\lambda^\beta x, \lambda t)$$

for any $\lambda > 0$, $x \in \mathbb{R}^n$, $t > 0$. Setting $\lambda = \frac{1}{t}$ yields (*) for $v(y) = u(y, 1)$.

Inserting (*) into heat equation, we get

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot \nabla v(y) +$$

$$t^{-\alpha+2\beta}\Delta v(y) = 0$$

for $y = t^{-\beta}x$. Try $\beta = \frac{1}{2}$.

Then, our equation reduces to

$$\alpha v + \frac{1}{2} y \cdot \nabla v + \Delta v = 0.$$

Assume v is radial, i.e. $v(y) = w(|y|)$.
for some $w : \mathbb{R} \to \mathbb{R}$. This yields

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0, \text{ where}$$

$r = |y|$, $' = \frac{d}{dr}$. Setting $\alpha = \frac{n}{2}$,

we get

$$(r^{n-1} w')' + \frac{1}{2}(r^n w)' = 0, \text{ i.e.}$$

$$r^{n-1} w' + \frac{1}{2} r^n w = a$$

Assuming $\lim_{r\to\infty} w, w' = 0$, we

conclude that $a = 0$, so

$w' = -\frac{1}{2} r w$. This has sol'us

$$w = b \cdot e^{-\frac{r^2}{4}}$$

for some constant b. Plugging back in our choices for α, β, we get then

$$\frac{b}{t^{n/2}} e^{\frac{-|x|^2}{4t}}$$

solves the heat equation. This can easily be checked for $x \in \mathbb{R}^n, t > 0$.

Definition 2.3.1. The function

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x|^2}{4t}} & x \in \mathbb{R}^n, t > 0 \\ 0 & x \in \mathbb{R}^n, t < 0 \end{cases}$$

is called the fundamental solution for the heat equation.

Note that we have a singularity at the origin.

Lemma 2.3.2. For $t > 0$, we have

$$\int_{\mathbb{R}^n} \Phi(x,t)\, dx = 1$$

Pf. exercise.

We now use the fundamental sol'n to solve the initial value (or Cauchy) problem

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty)$$
$$u = g \quad \text{on } \mathbb{R}^n \times \{t=0\}$$

Note that $(x,t) \mapsto \Phi(x-y,t)$ solves heat eq. for any given $y \in \mathbb{R}^n$, $t>0$. Therefore, the convolution

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy$$
$$= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy$$

$x \in \mathbb{R}^n, t>0$

should also be a sol'n. to heat eq.

Thm. 2.3.3. Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and define u by

(*) $$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y)\, g(y)\, dy,$$

then

i) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$,

ii) $u_t(x,t) - \Delta u(x,t) = 0$ for $x \in \mathbb{R}^n$, $t > 0$

iii) $$\lim_{\substack{(x,t)\to(x^0,0) \\ x\in\mathbb{R}^n,\, t>0}} u(x,t) = g(x^0)$$ for any $x^0 \in \mathbb{R}^n$.

Pf. 1. Since $\frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}}$ is infinitely differentiable with uniformly bounded derivatives of all orders on $\mathbb{R}^n \times [\delta, \infty)$ for each $\delta > 0$, we see that

$u \in C^\infty(\mathbb{R}^n \times (0,\infty))$. (Exercise...)

Also,

$$u_t(x,t) - \Delta u(x,t) = \int_{\mathbb{R}^n} [(\Phi_t - \Delta_x \Phi)(x-y,t)] \cdot g(y)\, dy$$

$$= 0 \quad (\text{for } x \in \mathbb{R}^n, t > 0).$$

as Φ solves the heat eq.

2. Fix $x^0 \in \mathbb{R}^n$, $\varepsilon > 0$. Choose $\delta > 0$ s.t.

$|g(y) - g(x^0)| < \varepsilon$, if $|y - x^0| < \delta$,

for $y \in \mathbb{R}^n$.