Albert-Ludwigs-Universität Freiburg

Lecture notes

Introduction to the Theory and Numerics of Partial Differential Equations

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Contents

1 Introduction

In the following let $k \geq 1$ and $\Omega \subset \mathbb{R}^n$ be open.

Definition 1.1.1. *An expression* $F(D^k u(x), D^{k-1}u(x), ..., u(x), x) = 0$ *with* $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times ... \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ *is called partial differential equation of the order* k*, where* $u : \Omega \to \mathbb{R}$ *is the desired solution.*

Definition 1.1.2. *A partial differential equation is called linear, if it has the form*

$$\sum_{|\alpha| \leq k} q_\alpha(x) D^\alpha u = f(x)$$

where α *denotes a multi-index*

Examples. (*i*) *Laplace Equality:* $-\Delta u = -\sum_{i=1}^n u_{x_i x_i} = 0$

(*ii*) *Helmholtz Equality:* $-\Delta u = \lambda u$

(*iii*) *Telegraph Equality:* $u_t + 2du_t - u_{xx} = 0$

(*iv*) *Airy Equality:* $u_t + u_{xxx} = 0$

(*v*) *Balken Equality:* $u_{tt} + u_{xxxx} = 0$

(*vi*) *Eihonal Equality:* $|\nabla u| = 1$

(*vii*) *Burger's Equality:* $u_t + uu_x = 0$

2 Linear partial differential equations

2.1 Transport Equation

Consider $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R},\ u = u(x,t)$ and $u_t + b\nabla u = 0$ on $\mathbb{R}^n \times (0, \infty)$, where $b \in \mathbb{R}^n,\ b = (b_1, ..., b_n)^T$. How do we solve this equation?

Assume u is a solution and $(x,t) \in \mathbb{R}^n \times (0, \infty)$. Set $z(s) = u(x + sb, t + s)$ for $s \in \mathbb{R}$. Then we have $z'(s) = \nabla u(x + sb, t + s) \cdot b + u_t(x + sb, t + s) = 0$, therefore u is constant alogn the line of (x,t) in the direction $(b, 1) \in \mathbb{R}^n \times \mathbb{R}$

2.1.1 Initial Value Problem

Let $b \in \mathbb{R}^n,\ g : \mathbb{R}^n \to \mathbb{R}$ be given and

$$\begin{cases} u_t + b\nabla u = 0 \text{ on } \mathbb{R}^n \times (0, \infty) \\ \qquad\quad u = g \text{ on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Because u is constant on along lines in the direction $(b, 1) \in \mathbb{R}^n \times \mathbb{R}$, we get

$$u(x,t) = u(x - tb, 0) = g(x - tb), \ x \in \mathbb{R}^n, \ t \geq 0.$$

It is not hard to see that a solution of $F(D^k u(x), D^{k-1} u(x), ..., u(x), x) = 0$ in C^1 is always of the form $u(x,t) = u(x - tb, 0) = g(x - tb)$ and conversely $g \in C^1$ is a solution of $F(D^k u(x), D^{k-1} u(x), ..., u(x), x) = 0$.

2.1.2 Finite differentials for the Transport Equation

For $x \in [0,1]$ we consider $x_j = jh, \ j = 0, ..., J, \ h = \frac{1}{J}$ and we approximate $u \in C^1$ by

$$u'(x_j) = \frac{u(x_{j+1}) - u(x_j)}{h} \text{ and } u'(x_j) = \frac{u(x_j) - u(x_{j-1})}{h}$$

Definition 2.1.1. *For a given step lenght $h = \frac{1}{J}, \ J \in \mathbb{N}$ and a sequence $(U_j)_{j=1,...,J}$ the terms*

(i) $\partial_x^+ U_j = \frac{U_{j+1} - U_j}{h}$

(ii) $\partial_x^- U_j = \frac{U_j - U_{j-1}}{h}$

(iii) $\hat{\partial}_x U_j = \frac{U_{j+1} - U_{j-1}}{2h}$

define the forward-, backward- and central differential

Remark. (i) *For $u \in C^2([0,1])$, we get*

$$\frac{u(x+h) - u(x)}{h} = u'(x) + \frac{1}{2} u''(\xi) h$$

with $\xi \in [x, x+h]$ (Taylor)

(ii) *Finite differentials can be iterated, for example*

$$\partial_x^+ \partial_x^- U_j = \partial_x^- \partial_x^+ U_j = \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2}$$

Definition 2.1.2. *The equation in (ii) defines the difference quotient of the second degree.*

Proposition 2.1.3. (i) $|\partial_x^{\pm} u(x_j) - u'(x_j)| \leq \frac{h}{2} \|u''\|_{C([0,1])}$

(ii) $|\hat{\partial}_x u(x_j) - u'(x_j)| \leq \frac{h^2}{6} \|u''\|_{C([0,1])}$

(iii) $\partial_x^+ \partial_x^- u(x_j) - u''(x_j)| \leq \frac{h^2}{12} \|u''\|_{C([0,1])}$

Proof. by Taylor □

2.1.3 Finite Difference Schemes

We consider for $b > 0$ and $[0,1] \times [0,T]$ the initial boundary value problem

$$\begin{cases} u_t(x,t) + bu_x(x,t) = 0 \text{ on } (x,t) \in (0,1) \times (0,T] \\ (0,t) = 0 \text{ for } t \in (0,T] \\ u(x,0) = u_0(x) \text{ for } x \in [0,1] \end{cases}$$

Our domain $[0,1] \times [0,T]$ is discretesized using spatial- and temporal step sizes $h = \frac{1}{J}$ and $\tau = \frac{1}{K}$. We write $(x_j, t_k) = (jh, k\tau)$ for $0 \le j \le J,\ 0 \le k \le K$.

Our finite difference scheme now simply consists of replacing the partial derivatives by finite differences, i.e.

$$\partial_t^+ U_j^k + b\partial_x^- U_j^k = 0$$

or writing out the finite differences

$$\frac{U_j^{k+1} - U_j^k}{\tau} + b\frac{U_j^k - U_{j-1}^k}{h} = 0 \quad (*)$$

For $k = 0$, the values U_j^0 are obtained from the initial condition i.e. $U_j^0 = u_0(x_j),\ j = 0, ..., J$ and for $j = 0$, we use the boundary conditions i.e. $U_0^k = 0,\ k = 0, ..., K$.
Then U_j^k can be iteratively computed using $(*)$.

An important consideration is stability. We know that a solution $u(x,t)$ to the initial boundary value problem satisfies

$$\sup_{x \in [0,1],\ t \in [0,T]} |u(x,t)| \le \sup_{x \in [0,1]} |u_0(x)|$$

This is clear from the solution formula (considering also the boundary condition).

Proposition 2.1.4 (Stability)**.** *Set $\mu = \frac{b \cdot \tau}{h}$. If $0 \le \mu \le 1$ then the numerical solution (U_j^k) computed by the scheme $(*)$ satisfies*

$$\sup_{j=0,...,J,\ k=0,...,K} |U_j^k| \le \sup_{j=0,...,J} |U_j^0|$$

Proof. We have $U_j^{k+1} = U_j^k - \mu(U_j^k - U_{j-1}^k) = (1-\mu)U_j^k + \mu U_{j-1}^k$. For $0 \le \mu \le 1$ this is a convex combination of U_j^k and U_{j-1}^k which, after iteration, proves our statement. $\square$

Remark. *(i) The condition on μ is sharp (see exercises for counterexamples)*

(ii) This uniform boundness of solutions is called stability. We proved it under the

condition $\tau \leq \frac{h}{b}$ (for $b > 0$). That some condition like this must be satisfied is clear.

Proposition 2.1.5 (Convergence). *Set $\mu = \frac{b \cdot \tau}{h}$. If $0 \leq \mu \leq 1$ and $u \in C^2([0,1] \times [0,T])$, then we have*

$$\sup_{j=0,...,J} |u(x_j, t_k) - U_j^k| \leq \frac{t_k}{2}(\tau + bh)\|u\|_{C^2([0,1]\times[0,T])}$$

for all $k = 0, ..., K$ and where U_j^k is computed according to $(*)$.

Proof. (*i*) We define the consistency term $\mathscr{C}_j^k := \partial_t^+ u(x_j, t_k) - b\partial_x^- u(x_j, t_k)$ which measures how much the exact solution fails to satisfy the discrete scheme. Since $\partial_t u_t + b\partial_x u = 0$ on $(0,1) \times (0,T)$ we get from proposition 1.2.2(!)

$$\begin{aligned} |\mathscr{C}_j^k| &\leq |\partial_t^+ u(x_j, t_k) - \partial_t u(x_j, t_k) + b\partial_x^- u(x_j, t_k) - b\partial_x x u(x_j, t_k)| \\ &\leq \frac{\tau}{2} \sup_{t \in [0,T)]} |\partial_t^2 u(x_j, \cdot)| + \frac{bh}{2} \sup_{x \in [0,1]} |\partial_x^2 u(\cdot, t_k)| \quad (1) \end{aligned}$$

(*ii*) We now subtract the equation $u(x_j, t_{k+1}) = u(x_j, t_k) - b\tau\partial_x^- u(x_j, t_k) + \tau\mathscr{C}_j^k$ and $U_j^{k+1} = U_j^k - b\tau\partial_x^- U_j^k$ to deduce for

$$Z_j^k = u(x_j, t_k) - U_j^k \text{ and } Z_j^{k+1} = Z_j^k - b\tau\partial_x^- Z_j^k + \tau\mathscr{C}_j^k$$

For $0 \leq \mu \leq 1$ we get $|Z_j^{k+1}| \leq \sup_{j=0,...,J} |Z_j^k| + \tau|\mathscr{C}_j^k|$ (also using $Z_0^{k+1} \geq 0$). Induction over $k = 0, ..., K-1$ with $Z_j^0 = 0,\ j = 0, ..., J$, using (1), yields the result.

□

Remark. (*i*) *The condition $u \in C^2$ is satisfied if $u_0 \in C^2$ with $u_0(0,0) = 0$.*

(*ii*) *We used consistency and stability of the scheme.*

(*iii*) *The result implies convergence of the approximate solution to an exact solution.*

2.1.4 CFL-Condition

Definition 2.1.6. *An explicit numerical scheme of the form*

$$U_j^{k+1} = \phi(t_k, x_j, U_{j-m_l}^k, ..., U_{j+m_r}^k, \tau, j)$$

satisfies the CFL-Condition if the characteristic through the part (x_j, t_{k+1}) intersects the line $\{(x,t) \mid t = t_k,\ x \in \mathbb{R}\}$ within the convex hull of all grid points $(x_{j-m_l}, t_k), ..., (x_{j+m_r}, t_k)$.

Examples. (*i*) *For our previous scheme with $b \geq 0$ the CFL-Condition is satisfied, if $0 \leq \mu \leq 1$.*

(ii) *The scheme* $\partial_t^+ U_j^k + b\hat{\partial}_x U_j^k = 0$ *satisfies the CFL-Condition for* $|\mu| \leq 1$*, however the scheme is never stable.*

Remark. *The so-called upwinding scheme*

$$U_j^{k+1} = U_j^k - \begin{cases} \mu_j^k (U_j^k - U_{j-1}^k) & \mu_j^k \geq 0 \\ \mu_j^k (U_{j+1}^k - U_j^k) & \mu_j^k < 0 \end{cases}$$

with $\mu_j^k = \frac{b(x_j, t_k)\tau}{h}$ *is stable for* $b : [0,1] \times [0,T]$ *as long as* $\sup b(x,t)\frac{\tau}{h} \leq 1$.

Remark. (i) *If the finite difference scheme is of the form* $U^{k+1} = AU^k$ *(our previous scheme here was of this form), stability, i.e. non-growth of* U^k *as* k *increases, reduces to looking at the spectrum of* A*(and is stable if all eigenvalues of* A *are less than 1).*

(ii) *If the scheme is of the type* $U_j^{k+1} = \sum_{\ell \in \mathbb{Z}} m_\ell U_{j-\ell}^k$ *(a discrete convolution) then a stability analysis in Fourier space can be performed.*

2.2 Laplace's Equation

We consider

(i) $-\Delta u = 0$ (Laplace's equation)

(ii) $-\Delta u = f$ (Poisson's equation)

for $x \in \Omega,\ \Omega \subset \mathbb{R}^n$ open. The unknown is $u : \overline{\Omega} \to \mathbb{R},\ u = u(x)$.

Definition. *A* C^2 *function satisfying the Laplace's equation is called harmonic function.*

For some context for physics assume u denotes the concentration of some quantity in equilibrium. Consider the flux of u through the boundary ∂V of some smooth subdomain $V \subset \Omega$. We have from $\int_{\partial V} F \cdot \nu \; dS = 0$, where F denotes the flux density of u, but $\int_{\partial V} F \cdot \nu \; dS = \int_V \mathrm{div} F \; dx = 0$. Thus $\mathrm{div} F = 0$ in Ω as V is arbitrary.
A very typical model assumes that $F = c \cdot \nabla u$ with $c > 0$. Thus $\mathrm{div} c \cdot \nabla u = c \cdot \Delta u = 0$ is the results equation.

2.2.1 Fundamental solution

Lets consider the Laplace's equation on $\mathbb{R}^n$. The equation is invariant to rotations (Exercise). We thus look for a solution $u(x) = v(|x|) = v(r)$. We get for

$$r = (\sum_{i=1}^n x_j^2)^{1/2} \text{ that } \frac{\partial r}{\partial x_j} = \frac{x_j}{r} \text{ if } x \neq 0.$$

Thus $u_{x_j} = v'(r)\frac{x_j}{r}$, $u_{x_jx_j} = v''(r)\frac{x_j^2}{r^2} + v'(r)(\frac{1}{r} - \frac{x_j^2}{r^3})$ and $\Delta u = v''(r) + \frac{n-1}{r}v'(r)$. Thus if $\Delta u = 0$, we get $v'' + \frac{n-1}{r}v' = 0$. If $v' \neq 0$ we get $\log(|v'|)' = \frac{v''}{v'} = \frac{1-n}{r}$. We obtain $v'(r) = \frac{a}{r^{n-1}}$ for some constant a. Thus if $r > 0$ we get

$$v(r) = \begin{cases} b\log(r) + c & n = 2 \\ \frac{b}{r^{n-2}} + c & n \geq 3 \end{cases}$$

with constants b, c.

Definition 2.2.1. *The function*

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & n = 2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & n \geq 3 \end{cases}$$

defined on $x \in \mathbb{R}^n$, $x \neq 0$ is the fundamental solution of Laplace's equation.

Remark. *Note that $|\nabla\Phi(x)| \leq \frac{C}{|x|^{n-1}}$ and $|\nabla^2\Phi(x)| \leq \frac{C}{|x|^n}$ for $x \neq 0$.*

By construction we have $\Delta\Phi(x) = 0$ for $x \neq 0$. Similarly $x \mapsto \Phi(x-y)$ is harmonic as a function of x for $x \neq y$. Taking $f : \mathbb{R}^n \to \mathbb{R}$, we also get $x \mapsto \Phi(x-y)f(y)$ $(x \neq y)$ is harmonic for any point $y \in \mathbb{R}^n$. By linearity so is the sum of such expressions.

One might get the idea, that

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\ dy$$

solves the Laplace's equation, but this is wrong.
Instead, we get

Theorem 2.2.2 (Solution Poisson's Equation)**.** *Setting*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\ dy = \begin{cases} \frac{1}{2\pi}\int_{\mathbb{R}^n}\log(|x-y|)f(y)\ dy & n = 2 \\ \frac{1}{n(n-2)\alpha(n)}\int_{\mathbb{R}^n}\frac{f(y)}{(x-y)^{n-2}}\ dy & n \geq 3 \end{cases}$$

for $f \in C_c^2(\mathbb{R}^n)$, we get

(i) $u \in C^2(\mathbb{R}^n)$

(ii) $-\Delta u = f$ *in* $\mathbb{R}^n$

Proof. 1. We have $u(x) = \int_{\mathbb{R}^n} \Phi(y)f(x-y)\ dy$, so

$$\frac{u(x+he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y)\frac{f(x+he_i-y) - f(x-y)}{h}\ dy$$

$(h \neq 0,\ e_i = (0, ..., 0, 1, 0, ..., 0))$. But

$$(f(x + he_i - y) - f(x - y))\frac{1}{h} \to f_{x_i}(x - y)$$

uniformly on $\mathbb{R}^n$ as $h \to 0$. Thus

$$u_{x_i}(x) = \int_{\mathbb{R}^n} \Phi(y) f_{x_i}(x - y)\ dy$$

and similary

$$u_{x_i x_j} = \int_{\mathbb{R}^n} \Phi(y) f_{x_i x_j}(x - y)\ dy.$$

As Φ is integrable near the origin (Exercise) and $f_{x_i x_j}$ has compact support, we get continuous second derivatives of u, i.e. $u \in C^2(\mathbb{R}^n)$.

2. Fix $\varepsilon > 0$, we get

$$\Delta u(x) = \int_{B_\varepsilon(0)} \Phi(y) \Delta_x f(x - y)\ dy + \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(y) \Delta_x f(x - y)\ dy = I_\varepsilon + J_\varepsilon$$

Then

$$I_\varepsilon \leq C \cdot \|D^2 f\|_{C^\infty(\mathbb{R}^n)} \int_{B_\varepsilon(0)} |\Phi(y)|\ dy \leq \begin{cases} C\varepsilon^2 |\log \varepsilon| & n = 2 \\ C\varepsilon^2 & n \geq 3 \end{cases}$$

For J_ε, integration by parts yields

$$\begin{aligned} J_\varepsilon &= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(y) \Delta_y f(x - y)\ dy \\ &= -\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \nabla_y \Phi(y) \cdot \nabla_y f(x - y)\ dy + \int_{\partial B_\varepsilon(0)} \Phi(y) \frac{\partial f}{\partial \nu}(x - y)\ dS(y) \\ &= K_\varepsilon + L_\varepsilon, \end{aligned}$$

where ν denotes the inward pointing normal on $\partial B_\varepsilon(0)$. We check

$$|L_\varepsilon| \leq |\nabla f|_{C^\infty(\mathbb{R}^n)} \int_{\partial B_\varepsilon(0)} |\Phi(y)|\ dS(y) \leq \begin{cases} C\varepsilon |\log(\varepsilon)| & n = 2 \\ C\varepsilon & n \geq 3 \end{cases}$$

3. Integrating again by parts, we get

$$K_\varepsilon = \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Delta \Phi(y) f(x - y)\ dy - \int_{\partial B_\varepsilon(0)} \frac{\partial \Phi}{\partial \nu}(y) f(x - y)\ dS(y)$$

since $\Delta\Phi = 0$ away from origin. Using

$$\nabla\Phi(y) = \frac{-1}{n\alpha(n)} \frac{y}{|y|^n} \ (y \neq 0) \text{ and } \nu = \frac{-y}{|y|} = \frac{-y}{\varepsilon}$$

on $\partial B_\varepsilon(0)$, we get

$$\frac{\partial\Phi}{\partial\nu}(y) = \nu \cdot \nabla\Phi(y) = \frac{1}{n\alpha(n)\varepsilon^{n-1}}$$

on $\partial B_\varepsilon(0)$. Taking $\alpha(n)$ to be the volume of the unit ball in n-dimensions, we get

$$K_\varepsilon = -\frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B_\varepsilon(0)} f(x-y)\ dS(y) = ⨍_{\partial B_\varepsilon(0)} f(y)\ dS(y) \to -f(x)$$

as $\varepsilon \to 0$. Combining the estimates and letting $\varepsilon \to 0$, we get $-\Delta u(x) = f(x)$. □

2.2.2 Mean Value Theorems

A Central property of harmonic functions:

Theorem 2.2.3 (Mean Value Formula)**.** *If $u \in C^2(\Omega)$ is harmonic (i.e. $\Delta u = 0$), then*

$$u(x) = ⨍_{\partial B_r(x)} u\ dS = ⨍_{B_r(x)} u\ dy$$

for any ball $B_r(x) \subset \Omega$.

Proof. Set

$$\phi(r) = ⨍_{\partial B_r(x)} u(y)\ dS(y) = ⨍_{\partial B_1(0)} u(x+rz)\ dS(z)$$

Then

$$\begin{aligned} \phi'(r) &= ⨍_{B_1(0)} \nabla u(x+rz) \cdot z\ dS(z) \\ &= ⨍_{\partial B_r(x)} \nabla u(y) \frac{y-x}{r}\ dS(y) \\ &= ⨍_{\partial B_r(x)} \frac{\partial u}{\partial \nu}\ dS(y) \\ &= \frac{r}{n} ⨍_{B_r(x)} \Delta u(y)\ dy = 0. \end{aligned}$$

Thus ϕ is constant, and

$$\phi(r) = \lim_{t\to\infty} \phi(t) = \lim_{t\to\infty} ⨍_{\partial B_t(x)} u(y)\ dS(y) = u(x)$$

The second formula follows by integrating over r. □

Theorem 2.2.4 (Converse to Mean Value Property)**.** *If* $u \in C^2(\Omega)$ *satisfies*

$$u(x) = \fint_{\partial B_r(x)} u \, dS$$

for all $B_r(x) \subset \Omega$*, then* u *is harmonic.*

Proof. By contradiction using the previous theorem □

2.2.3 Properties of Harmonic Functions

Theorem 2.2.5 (Strong Maximum Principle)**.** *Assume* $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *is harmonic in* Ω*.*

(*i*) *Then* $\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x)$ *(Maximum Principle)*

(*ii*) *Furthermore, if* Ω *is connected and there exists* $x_0 \in \Omega$*, s.t.* $u(x_0) = \max_{x \in \overline{\Omega}} u(x)$*, then* u *is constant (Strong Maximum Principle)*

Remark. *Similar statement follows for* $\min u$*, by* $v = -u$

Proof. Assume there exists $x_0 \in \Omega$, $u(x_0) = M = \max_{x \in \overline{\Omega}} u(x)$. Then for $0 < r < \operatorname{dist}(x_0, \partial\Omega)$, we have, by theorem 2.2.4 that

$$M = u(x_0) = \fint_{B_r(x_0)} u \, dy \leq M$$

with equality if and only if $u \equiv M$ in $B_r(x_0)$, i.e. $u(y) = M$ for all $y \in B_r(x_0)$. Thus, the set $\{x \in \Omega \mid u(x) = M\}$ is both open and relatively closed in Ω, thus it equals Ω if Ω is connected. This proves (*ii*) and (*i*) follows immediately, □

Remark. *If* $u \in C^2(\Omega) \cap C(\overline{\Omega})$*,* $\Delta u = 0$ *in* Ω*,* $u = g$ *in* $\partial\Omega$*, and if also* $g(x) > 0$ *for some* $x \in \partial\Omega$*, then* $u > 0$ *on* Ω*.*

Theorem 2.2.6 (Uniqueness)**.** *Let* $g \in C(\partial\Omega)$*,* $f \in C(\Omega)$*, the there exists at most one solution* $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *of the boundary value problem:*

$$\begin{cases} -\Delta u = f \ \textit{in } \Omega \\ \quad u = g \ \textit{on } \partial\Omega \end{cases}$$

Proof. If u, v both solve the boundary value problem apply Theorem 2.2.5 (Strong Maximum Principle) to $w = \pm(u - v)$ □

Theorem 2.2.7 (Smoothness)**.** *If $u \in C(\Omega)$ satisfies the mean value property*

$$u(x) = \fint_{\partial B_r(x)} u(y)\ dS(y)$$

for each ball $B_r(x) \subset \Omega$, then $u \in C^\infty(\Omega)$

Remark. *u may not be smooth (or even continuous) up to the boundary.*

Proof. Consider the standard mollifier

$$\eta_\varepsilon = \begin{cases} e^{-\frac{1}{1-|x|^2/\varepsilon}} & |x| < \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

such that

$$\int_{\mathbb{R}^n} \eta_\varepsilon = 1$$

we know that $u_\varepsilon = u * \eta_\varepsilon$ (defined in $\Omega_\varepsilon = \{\text{dist}(\cdot, \partial\Omega) > \varepsilon\}$) is smooth.

We show that $u(x) = u_\varepsilon(x)$ for $\text{dist}(x, \partial\Omega) < \varepsilon$. Consider

$$\begin{aligned} u_\varepsilon(x) &= \int_\Omega \eta_\varepsilon(x-y)u(y)\ dy \\ &= \frac{1}{\varepsilon^n} \int_{B_\varepsilon(x)} \eta\left(\frac{(x-y)}{\varepsilon}\right) u(y)\ dy \\ &= \frac{1}{\varepsilon^n} u(x) \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) n\alpha(n) r^{n-1}\ dr = u(x) \end{aligned}$$

□

Theorem 2.2.8. *Assume u is harmonic in Ω. Then*

$$|D^\alpha u(x_0)| \leq \frac{Ck}{r^{n+k}} \|u\|_{C^1(B_r(x_0))}$$

for any ball $B_r(x_0) \subset \Omega$ and any multi-index α of order $|\alpha| \leq k$. We have

$$C_0 = \frac{1}{\alpha(n)},\ C_k = \frac{(2^{n+1}nk)^k}{\alpha(n)}$$

for $k = 1, 2, ...$

Proof. We prove the statement by induction on k, with $k = 0$ being obvious from the mean value formula (Quote: I still think there should be a nice value formula).
For $k = 1$, note that u_{x_i} is harmonic (by smoothness and differentiating $\Delta u = 0$). We

obtain

$$|u_{x_i}(x_0)| = \left| \fint_{B_{r/2}(x_0)} u_{x_i} \; dx \right| = \left| \frac{2^n}{\alpha(n) r^2} \int_{\partial B_{r/2}(x_0)} u \cdot \nu_i \; dS \right| \leq \frac{2n}{r} \|u\|_{L^\infty(\partial B_{r/2}(x_0))}.$$

For $x \in \partial B_{r/2}(x_0)$, we have $B_{r/2}(x) \subset B_r(x_0) \subset \Omega$. So, by the $k = 0$ estimate, we get

$$|u(x)| \leq \frac{1}{\alpha(n)} \left(\frac{2}{r} \right)^2 \|u\|_{L^1(B_r(x_0))}$$

This yields

$$|D^\alpha u(x_0)| \leq \frac{2^{n+1} n}{\alpha(n)} \frac{1}{r^{n+1}} \|u\|_{L^1(B_r(x_0))}$$

for $|\alpha| \leq 1$. The higher derivative estimates follow analogously ... □

Theorem 2.2.9 (Liouville's Theorem)**.** *Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded. Then, u is constant.*

Proof. Fix $x_0 \in \mathbb{R}^n$, $r > 0$. By Theorem (2.2.8) we get on $B_r(x_0)$

$$|Du(x_0)| \leq \frac{\sqrt{n} C_1}{r^{n+1}} \|u\|_{L^1(B_r(x_0))} \leq \frac{\sqrt{n} C^1}{r} \|u\|_{L^\infty(\mathbb{R}^n)} \to \infty$$

as $r \to \infty$. Thus $Du \equiv 0$ and u is constant. □

Theorem 2.2.10. *Let $f \in C_c^2(\mathbb{R}^n)$, $n \geq 3$. Then any bounded solution of $-\Delta u = f$ in $\mathbb{R}^n$ has the form*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y) \; dy + C$$

for $x \in \mathbb{R}^n$

Proof. Clearly, since $\Phi(x) \to 0$ as $|x| \to \infty$, the rhs. is bounded and $\tilde{u} = \Phi * f$ is a bounded solution of Laplace's equation. If u is any other solution, then $u - \tilde{u}$ is bounded and harmonic, thus constant due to theorem 2.2.9. □

Remark. *(i) One can even prove analyticity of harmonic functions.*

(ii) There are non-bound harmonic functions on $\mathbb{R}^n$.

Theorem 2.2.11 (Harnack's Inequality)**.** *Consider $V \subset\subset \Omega$ ($\exists K \subset \Omega$, K compact, s.t $V \subset K$ is open). Then there exists C depending only on V, s.t.*

$$\sup_V u \leq C \inf_V u$$

for any non-negative harmonic function u in Ω.

Proof. Take $r = \frac{1}{4}\text{dist}(V, \partial\Omega),\ x, y \in V$, s.t. $|x-y| < r$. Then

$$u(x) = \fint_{B_{2r}(x)} u(z)\ dz \geq \frac{1}{\alpha(n)2^n r^n} \int_{B_r(y)} u(z)\ dz = \frac{1}{2^n} \fint_{B_r(y)} u(z)\ dz = \frac{1}{2^n} u(y)$$

Thus, $2^n u(y) \geq u(x) \geq \frac{1}{2^n} u(y)$ if $x, y \in V,\ |x-y| < r$.

Since V is connected and $\overline{V}$ is compact, we can cover $\overline{V}$ by finitely many balls $\{B_i\}_{i=1}^N$, each with radius $\frac{r}{2}$, and $B_i \cap B_{i-1} \neq \emptyset$ for $i = 1, ..., N$ and thus $u(x) \geq \frac{1}{2^{n(N+1)}} u(y)$ for all $x, y \in V$. $\square$

Now, take $\Omega \subset \mathbb{R}^n$, with $\partial\Omega$ in C^1. We would like to solve

$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ \quad\ u = g \text{ on } \partial\Omega. \end{cases}$$

This is the classic boundary value problem.

Assume $u \in C^2(\overline{\Omega})$ is given, fix $x \in \Omega,\ \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset \Omega$ and apply the divergence theorem on $V_\varepsilon = \Omega \setminus B_\varepsilon(x)$ to $u(y)$ and $\Phi(x-y)$. We compute

$$\int_{V_\varepsilon} u(y)\Delta\Phi(y-x) - \Phi(y-x)\Delta u(y)\ dy = \int_{\partial V_\varepsilon} u(y)\frac{\partial\Phi}{\partial\nu}(y-x) - \Phi(y-x)\frac{\partial u}{\partial\nu}(y)\ dS(y) \quad (2*)$$

Note: $\Delta\Phi(x-y) = 0$ for $x \neq y$ and

$$\left| \int -\partial B_\varepsilon(x)\Phi(x-y)\frac{\partial u}{\partial\nu}(y)\ dS(y) \right| \leq C\varepsilon^{n-1} \max_{\partial B_\varepsilon(0)} |\Phi| = \text{ as } \varepsilon \to 0.$$

We also, from the proof of theorem 2.2.2 have

$$\int_{\partial B_\varepsilon(x)} u(y)\frac{\partial\Phi}{\partial\nu}(y-x)\ dS(y) = \fint_{\partial B_\varepsilon(x)} u(y)\ dS(y) \to u(x)$$

Takin $\varepsilon \to 0$ in $(2*)$ the yields

$$u(x) = \int_{\partial\Omega} \underbrace{\Phi(y-x)\frac{\partial u}{\partial\nu}(y)}_{(3*)} - u(y)\frac{\partial\Phi}{\partial\nu}(y-x)\ dS(y) - \int_\Omega \Phi(y-x)\Delta u(y)\ dy$$

for any $x \in \mathbb{R}$, any $u \in C^2(\overline{\Omega})$. We can compute the rhs. except for $(3*)$. Consider thus a "correcter" $\varphi^x = \varphi^x(y)$ solving

$$\begin{cases} \Delta\varphi^x = 0 \text{ in } \Omega \\ \quad \varphi^x = \Phi(y-x) \text{ on } \partial\Omega. \end{cases}$$

This yields using Gauß-Green again,

$$-\int_\Omega \varphi^x(y)\Delta u(y)\ dy = \int_{\partial\Omega} u(y)\frac{\partial\varphi^x}{\partial\nu}(y) - \varphi^x(y)\frac{\partial u}{\partial\nu}(y)\ dS(y)$$
$$= \int_{\partial\Omega} u(y)\frac{\partial\varphi^x}{\partial\nu}(y) - \Phi(y-x)\frac{\partial u}{\partial\nu}(y)\ dS(y) \quad (4*)$$

Definition 2.2.12. *Green's function for the domain Ω is*

$$G(x,y) = \Phi(y-x) - \varphi^x(y)$$

for $x, y \in \Omega$ with $x \neq y$.

Now adding (3∗) and (4∗), we get

$$u(x) = -\int_{\partial\Omega} u(y)\frac{\partial G}{\partial\nu}(x,y)\ dS(y) - \int_\Omega G(x,y)\Delta u(y)\ dy \quad (5*)$$

where $\frac{\partial G}{\partial\nu}(x,y) = \nabla_y G(x,y)\cdot\nu(y)$. If now $u \in C^2(\overline{\Omega})$ solves

$$(6*) = \begin{cases} -\Delta u = f \text{ in } \Omega \\ \quad u = g \text{ on } \partial\Omega \end{cases}$$

for given continuous functions f, g, we know by plugging into (5∗)

Theorem 2.2.13. *If $u \in C^2(\overline{\Omega})$ solves (6∗) then*

$$u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial\nu}(x,y)\ dS(y) + \int_\Omega f(y)G(x,y)\ dy$$

for x in Ω.

We could say

$$\begin{cases} -\Delta_y G = \delta_x \text{ in } \Omega \\ G(\cdot, y) = 0 \text{ on } \partial\Omega \end{cases}$$

Theorem 2.2.14. *For $x, y \in \Omega,\ x \neq y$, we have*

$$G(y,x) = G(x,y)$$

Proof. Fix $x, y \in \Omega,\ x \neq y$. Set

$$v(z) = G(x,z),\ w(z) = G(y,z),\ z \in \Omega$$

then

$$\Delta v(x) = \Delta w(z) = 0$$

for $z \neq x,\ z \neq y$ respectively. Also, $w = v = 0$ on $\partial\Omega$. Applying Gauß-Green on $V = \Omega \setminus (B_\varepsilon(x) \cup B_\varepsilon(y))$ for $\varepsilon > 0$ sufficiently small, we get

$$\int_{\partial B_\varepsilon(x)} \frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \; dS(z) = \int_{\partial B_\varepsilon(y)} \frac{\partial w}{\partial \nu} v - \frac{\partial v}{\partial \nu} w \; dS(z) \quad (7*)$$

Also, $v(z) = \Phi(z - x) - \Phi^x(z)$, where Φ^x is smooth in Ω. Thus

$$\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(x)} \frac{\partial v}{\partial \nu} w \; dS = \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(x)} \frac{\partial \Phi}{\partial \nu}(x - z) w(z) \; dS(z) = w(x)$$

Thus, the lhs. of (7∗) converges to $w(x)$ as $\varepsilon \to 0$, the rhs. converges to $v(y)$, thus

$$G(y, x) = w(x) = v(y) = G(x, y)$$

□

2.2.4 Green's Function on the Half Space

Consider $\mathbb{R}^n_+ := \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_n > 0\}$, the so-called half space.

Definition 2.2.15. *Green's function for the half space $\mathbb{R}^n_+$ is*

$$G(x, y) = \Phi(y - x) - \Phi(y - \tilde{x}),$$

$x, y \in \mathbb{R}^n_+,\ x \neq y$, where

$$\tilde{x} = (x_1, ..., x_{n-1}, -x_n) \in \mathbb{R}^n_-.$$

We obtain, for $y \in \partial\mathbb{R}^n_+$, that

$$\frac{\partial G}{\partial \nu}(x, y) = -G_{y_n}(x, y) = \frac{-2x_n}{n\alpha(n)} \frac{1}{|x - y|^n}$$

We would thus expect

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x - y|^n} \; dy, \quad x \in \mathbb{R}^n_+ \quad (8*)$$

solves

$$\Delta u = 0 \text{ in } \mathbb{R}^n_+$$

$$u = g \text{ on } \partial\mathbb{R}^n_+$$

(in a limit sense).

The function
$$K(x,y) = \frac{2x_n}{n\alpha(n)} \frac{1}{|x-y|^n}, \quad x \in \mathbb{R}^n_+, y \in \partial\mathbb{R}^n_+$$
is called Poisson-Kernel of the half space, and $(8*)$ is called Poisson's formula of the half space.

Theorem 2.2.16. *Assume $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$ and define u by $(8*)$. Then*

(i) $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$

(ii) $\Delta u = 0$ *in* $\mathbb{R}^n_+$ *and*

(iii) $\lim_{x \to x^0} u(x) = g(x^0)$ *for* $x^0 \in \partial\mathbb{R}^n_+$

Proof. 1. For a fixed x, $y \mapsto G(x,y)$ is harmonic for $x \neq y$. By symmetry of G, $x \mapsto G(x,y)$ is also harmonic for $x \neq y$. Thus
$$x \mapsto -\frac{\partial G}{\partial y_n}(x,y) = K(x,y)$$
is harmonic for $x \in \mathbb{R}^n_+,\ y \in \partial\mathbb{R}^n_n$.

2. We note that
$$\int_{\partial\mathbb{R}^n_+} K(x,y)\ dy = 1\ (9*)$$
for any $x \in \mathbb{R}^n_+$. Thus, as g is bounded, u defined by $(9*)$ is also bounded. As $x \mapsto K(x,y)$ is smooth, we see that $u \in C^\infty(\mathbb{R}^n_+)$ (just take derivatives w.r.t. x in $(9*)$ and note that integration and differentiation can be exchanged here). We get
$$\Delta u(x) = \int_{\partial\mathbb{R}^n_+} \Delta_x K(x,y) g(y)\ dy = 0$$
as $\Delta_x K(x,y) = 0$ for $x \neq y$.

3. Fix now $x^0 \in \partial\mathbb{R}^n_+,\ \varepsilon > 0$, take $\delta > 0$ such that
$$|g(y) - g(x^0)| < \varepsilon \text{ for } |y - x^0| < \delta,\ y \in \partial\mathbb{R}^n_+ \quad (10*)$$
Then, if $|x - x^0| < \delta/2,\ x \in \mathbb{R}^n_+$, we get
$$\begin{aligned}|u(x) - g(x^0)| &= \left| \int_{\partial\mathbb{R}^n_+} K(x,y)(g(y) - g(x^0))\ dy \right| \\ &\leq \int_{\partial\mathbb{R}^n_+ \cap B_\delta(x^0)} K(x,y)|g(y) - g(x^0)|\ dy + \int_{\partial\mathbb{R}^n_+ \setminus B_\delta(x^0)} K(x,y)|g(y) - g(x^0)|\ dy\end{aligned}$$

$$= I + J$$

We note $I < \varepsilon$ by $(9*), (10*)$. Further, if $|x - x^0| < \delta/2$ as $|y - x^0| \geq \delta/2$, we get

$$|y - x^0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}|y - x^0|$$

So, $|y - x| \geq 1/2|y - x^0|$ and we get

$$J \leq 2\|g\|_{L^\infty} \int_{\partial\mathbb{R}^n_+ \setminus B_\delta(x^0)} K(x, y)\ dy \leq \frac{2^{n+2}\|g\|_{L^\infty} x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ \setminus B_\delta(x^0)} \frac{1}{|y - x^0|^n}\ dy \to 0$$

as $x_n \to 0$. This yields

$$|u(x) - g(x^0)| \leq 2\varepsilon$$

for $|x - x^0|$ sufficiently small. $\square$

Remark. *To get Green's function for a ball $B_1(0) \subset \mathbb{R}^n$, use the dual point $\tilde{x}$ to $x \neq 0$ given by $\tilde{x} = x/|x|^2$ and set*

$$\varphi^x(y) = \Phi(|x|(y - \tilde{x}))$$

and note that φ^x is harmonic in $B_1(0)$. The rest follows as before, with

$$G(x, y) = \Phi(y - x) - \Phi(|x|(y - \tilde{x})) \quad x, y \in B_1(0),\ x \neq y.$$

2.2.5 Finite Difference Schemes for Poisson's Equation

Consider

$$\begin{cases} -\Delta u = f \text{ in } \Omega = [0,1]^2 \\ \quad u = 0 \text{ on } \partial\Omega. \end{cases}$$

For $J \geq 1$, set $h = 1/J$ and define grid points

$$x_{j,m} = (jh, mh),\ 0 \leq j, m \leq J$$

and replace the Laplacian by central difference quotients. We thus want to find

$$\{U_{j,m}\}_{j,m=0}^J \subset \mathbb{R}$$

such that

$$-\partial^+_{x_1}\partial^-_{x_1} U_{j,m} - \partial^+_{x_2}\partial^-_{x_2} U_{j,m} = f(x_{j,m}) = (F)_{j,m}$$

for $1 \leq m \leq J$

$$U_{0,m} = U_{J,m} = U_{j,0} = U_{j,J} = 0$$

for $j, m = 0, ..., J$. We see that

$$-\Delta_h U_{j,m} = \frac{-1}{h^2}(U_{j+1,m} + U_{j,m+1} - 4U_{j,m} + U_{j-1,m} + U_{j,m-1})$$

One can also do this for $n > 2$.

We note that the finite difference scheme (above) can be written as a suitable linear system of equations, by setting

$$(j, m) \sim j + (m-1)(J-1) = l$$

for $j, m = 1, ..., J-1,\ l = 1, ..., L = (J-1)^2$.

Setting $X \in \mathbb{R}^{(J-1)\times(J-1)}$ to

$$X = \begin{pmatrix} 4 & -1 & & \\ -1 & 4 & -1 & \\ & -1 & \ddots & \ddots \\ & & \ddots & 4 \end{pmatrix}$$

we can write (the above scheme) as $AU = b$ with

$$A = \begin{pmatrix} X & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & X \end{pmatrix},\ b = h^2 \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_L) \end{pmatrix},$$

where $I \in \mathbb{R}^{(J-1)\times(J-1)}$ is the identity-matrix.

To include boundary conditions $u = g$ on $\partial\Omega$, assume that there exists a function $\tilde{u}_D \in C^2(\overline{\Omega})$, such that $\tilde{u}_D|_{\partial\Omega} = g$, write $u = \hat{u} + \tilde{u}_D$ and note that if u solves the boundary value problem, we have

$$-\Delta\hat{u} = f + \Delta\tilde{u}_D$$

in Ω and $\hat{u} = 0$ on $\partial\Omega$.

For the finite difference scheme, just set

$$(\tilde{U}_D)_{i,j} = g(x_{i,j}) \quad x_{i,j} \in \partial\Omega$$

and solve

$$-\Delta_h \hat{U} = (F)_{i,j} + (\Delta_h \tilde{U}_D)_{i,j} \quad x_{i,j} \in \Omega$$

and $\hat{U}_{i,j} = 0$ for $x_{i,j} \in \partial\Omega$.

Note that one can also impose so-called Newmann-Conditions on part of the boundary, where, instead of u, we prescribe the values of

$$\frac{\partial u}{\partial \nu}.$$

We are looking at

$$\begin{cases} -\Delta u = f \text{ in } \Omega = (0,1)^2 \\ \quad u = 0 \text{ on } \partial\Omega \end{cases}$$

Discretization

$$-\Delta_h U_{j,m} = \frac{-1}{h^2}(U_{j+1,m} + U_{j,m+1} - 4U_{j,m} + U_{j-1,m} + U_{m,j-1})$$

Lemma 2.2.17 (Discrete Maximum Principle). *If $U = (U_{j,m},\ j, m = 0, ..., J)$ satisfies*

$$-\Delta_h U_{j,m} \leq 0$$

for all $j, m = 1, ..., J-1$, then U attains its maximum for $j = 0,\ j = J,\ m = 0$ or $m = J$.

Proof. From $-\Delta_h U_{j,m} \leq 0$, we get

$$U_{j,m} \leq \frac{1}{4}(U_{j-1,m} + U_{j+1,m} + U_{j,m-1} + U_{j,m+1})$$

For $1 \leq j, m \leq J-1$. So $U_{j,m}$ is a convex combination of its surroundings. If, thus, $U_{j,m}$ is a maximum, the estimate has to hold with equality and the maximum is also attained at all of the neighboring points, we can continue this until we reach a boundary. $\square$

Lemma 2.2.18 (Discrete Boundedness). *For all $(Z_{j,m},\ j, m = 0, ..., J)$ with $Z_{j,m} = 0$ for $j = 0,\ j = J,\ m = 0,\ m = J$, we have*

$$\max_{j,m=0,\dots,J} |Z_{j,m}| \leq \frac{1}{2} \sup_{j,m=1,\dots,J-1} |\Delta_h Z_{j,m}|$$

Proof. Write

$$S = \max_{j,m=1,\dots,J-1} |\Delta_h Z_{j,m}|$$

and set $W_{j,m} = (jh)^2 + (mh)^2$, which, on the grid points, coincides with $w(x_1, x_2) =$

$x_1^2 + x_2^2$. Notice $W_{j,m} \geq 0,\ j,m = 0,...,J$ and $\Delta_h W_{j,m} = 4,\ j,m = 1,...,J-1$. Now set

$$V_{j,m} = Z_{j,m} + \frac{S}{4} W_{j,m}$$

and get

$$-\Delta_h V_{j,m} = -\Delta_h Z_{j,m} - S \leq 0$$

The discrete maximum principle implies that $V_{j,m}$ attains its maximum on the boundary, there

$$Z_{j,m} = 0 \text{ and } 0 \leq W_{j,m} \leq 2$$

Therefore

$$Z_{j,m} = V_{j,m} - \frac{S}{4} W_{j,m} \leq \frac{S}{2}$$

Similary, the result

$$-Z_{j,m} \leq \frac{S}{2}$$

holds and we obtain the lemma. $\square$

Proposition 2.2.19 (Error Estimate). *Let* $u \in C^2(\overline{\Omega})$ *and* $U = (U_{j,m},\ j,m = 0,...,J)$ *be the solution to the Poisson boundary value problem*

$$\begin{aligned} -\Delta u &= f \ \textit{in } (0,1)^2 = \Omega \\ u &= 0 \ \textit{on } \partial\Omega \end{aligned}$$

and its discretization respectively, then we have

$$\sup_{j,m=0,...,J} |u(x_{j,m}) - U_{j,m}| \leq \frac{h^2}{24} (\|\partial_{x_1}^4 u\|_{C^0([0,1]^2)} + \|\partial_{x_2}^4 u\|_{C^0([0,1]^2)})$$

Proof. Since $-\Delta u(x_{j,m}) = f(x_{j,m})$ for $j,m = 0,...,J$, the error

$$Z_{j,m} = u(x_{j,m}) - U_{j,m}$$

satisfies

$$\begin{aligned} -\Delta_h Z_{j,m} &= -\Delta_h u(x_{j,m}) + \Delta_h U_{j,m} \\ &= f(x_{j,m}) - f(x_{j,m}) + \Delta u(x_{j,m}) - \Delta_h u(x_{j,m}) \\ &= \partial_{x_1}^2 u(x_{j,m}) - \partial_{x_1}^+ \partial_{x_1}^- u(x_{j,m}) + \partial_{x_2}^2 u(x_{j,m}) - \partial_{x_2}^+ \partial_{x_2}^- u(x_{j,m}) \end{aligned}$$

From our estimates on the difference quotients, we get

$$|-\Delta_h Z_{j,m}| \leq \frac{h^2}{24}(\|\partial_{x_1}^4 u\|_{C^0([0,1]^2 9} + \|\partial_{x_2}^4 u\|_{C^0([0,1]^2)})$$

Together with the discrete boundedness lemma we obtain the result. $\square$

2.3 Heat Equation

We study the equation

$$u_t - \Delta u = 0$$

and its non-homogeneous pendant

$$u_t - \Delta u = f$$

subjected to appropriate initial and boundary conditions. We take $t > 0,\ x \in \Omega,\ \Omega \subset \mathbb{R}^n$ open. The sought after function is

$$u : \overline{\Omega} \times [0, \infty) \to \mathbb{R},\ u = u(x, t)$$

and the laplacian is stable w.r.t the spatial variables x. The function $f : \Omega \times [0, \infty) \to \mathbb{R}$ is given.

For a physical interpretation, consider $V \subset \Omega$, then

$$\frac{d}{dt} \int_V u\ dx = -\int_{\partial V} F \cdot \nu\ dS$$

(if the quantity does not get produced or destroyed within V). Again we assume $F = a \cdot \nabla u$ $(a > 0)$ and the divergence theorem yields

$$u_t = \operatorname{div}(a \cdot \nabla u) = a \cdot \Delta u$$

2.3.1 Fundamental Solution

Consider functions of the form

$$(11*)\ u(x, t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right) \quad x \in \mathbb{R}^n, t > 0$$

with constants α, β, and the function $v : \mathbb{R}^n \to \mathbb{R}$ to be found.

Equation (11*) shows up if we look for solutions to the heat equation, that are invariant under the scaling

$$u(x,t) \mapsto \lambda^\alpha u(\lambda^\beta x, \lambda t)$$

for any $\lambda > 0,\ x \in \mathbb{R}^n, t > 0$. Setting $\lambda = 1/t$ yields (11∗) for $v(y) = u(y, 1)$.

Inserting (11∗) into the heat equation, we get

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot \nabla V(y) + t^{-\alpha+2\beta} \Delta v(y) = 0$$

for $y = t^{-\beta} x$, try $\beta = 1/2$, then our equation reduces to

$$\alpha v + \frac{1}{2} y \cdot \nabla v(y) + \Delta v(y) = 0.$$

Assume u is radial, i.e. $v(y) = w(|y|)$ for some $w : \mathbb{R} \to \mathbb{R}$. This yields

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{u-1}{r} w' = 0,$$

where $r = |y|,\ ' = d/dr$. Setting $\alpha = n/2$, we get

$$(r^{n-1} w')' + \frac{1}{2}(r^2 w)' = 0,$$

i.e.

$$r^{n-1} w' + \frac{1}{2} r^n w = \alpha$$

Assuming $\lim_{r\to\infty} w, w' = 0$, we conclude that $u = 0$, so $w' = -1/2rw$. This has solutions

$$w = b \cdot e^{-\frac{r^2}{4}}$$

for some constant b. Plugging in our choices for α, β, we get then

$$\frac{b}{t^{n/2}} \exp\left(\frac{-|x|^2}{4t}\right)$$

solves the heat equation. This can easily be concluded for $x \in \mathbb{R}^n, t > 0$.

Definition 2.3.1. *The function*

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} \exp\left(\frac{-|x|^2}{4t}\right) & x \in \mathbb{R}^n, t > 0 \\ 0 & x \int \mathbb{R}^n, t < 0 \end{cases}$$

is called the fundamental solution for the heat equation.

Note that we have a singularity a the origin.

Lemma 2.3.2. *For $t > 0$, we have*

$$\int_{\mathbb{R}^n} \Phi(x,t)\ dx = 1$$

Proof. Exercise. $\square$

We now use the fundamental solution to solve the initial value (or Cauchy) problem

$$\begin{cases} u_t - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ \qquad\ \ u = g \text{ on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

Note that $(x,t) \mapsto \Phi(x-y,t)$ solves the heat equation for any given $y \in \mathbb{R}^n, t > 0$. Therefore the convolution

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\ dy = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(\frac{|x-y|^2}{4t}\right) g(y)\ dy,\ x \in \mathbb{R}^n, t > 0,$$

should also be a solution to the heat equation.

Theorem 2.3.3. *Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and define u by*

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\ dy = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(\frac{-|x-y|^2}{4t}\right) g(y)\ dy$$

then

(i) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$

(ii) $u_t(x,t) - \Delta u(x,t) = 0$ *for* $x \in \mathbb{R}^n, t > 0$.

(iii) *For any* $x^0 \in \mathbb{R}^n$ *we have*

$$\lim_{(x,t)\to(x^0,t)} u(x,t) = g(x^0)$$

Proof. 1. Since

$$\frac{1}{t^{n/2}} \exp\left(\frac{-|x|^2}{4t}\right)$$

is infinitely differentiable with uniformly bounded derivatives of all orders on $\mathbb{R}^n \times [\delta,\infty)$ for each $\delta > 0$, we see then $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$. (Exercise)

Also,

$$u_t(x,t) - \Delta u(x,t) = \int_{\mathbb{R}^n} (\Phi_t - \Delta_x \Phi)(x-y,t)\ g(y)\ dy = 0$$

for $x \in \mathbb{R}^n, t > 0$, as Φ solves the heat equation.

2. Fix $x^0 \in \mathbb{R}^n$, $\varepsilon > 0$. Choose $\delta > 0$ s.t. $|g(y) - g(x^0)| < \varepsilon$, if $|y - x^0| < \delta$, for $y \in \mathbb{R}^n$. Then for $|x - x^0| < \delta/2$, we have

$$\begin{aligned}
|u(x,t) - g(x^0)| &\leq \left| \int_{\mathbb{R}^n} \Phi(x-y,t)(g(y) - g(x^0)) \, dy \right| \\
&\leq \int_{\mathbb{R}^n} \Phi(x-y,t)|g(y) - g(x^0)| \, dy \\
&= \int_{B_\delta(x^0)} \Phi(x-y,t)|g(y) - g(x^0)| \, dy + \int_{\mathbb{R}^n \setminus B_\delta(x^0)} \Phi(x-y,t)|g(y) - g(x^0)| \, dy \\
&= I + J
\end{aligned}$$

But

$$I \leq \varepsilon \int_{\mathbb{R}^n} \Phi(x-y,t) \, dy = \varepsilon$$

Further, note that if $|x - x^0| \leq \delta/2$ and $|y - x^0| \geq \delta$ then

$$|y - x^0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}|y - x^0|$$

so, $|y - x| \geq 1/2|y - x^0|$. Thus

$$\begin{aligned}
J &\leq 2\|g\|_{L^\infty} \int_{\mathbb{R}^n \setminus B_\delta(x^0)} \Phi(x-y,t) \, dy \\
&\leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n \setminus B_\delta(x^0)} \exp\left(\frac{-|x-y|^2}{4t} \right) \, dy \\
&\leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n \setminus B_\delta(x^0)} \exp\left(\frac{-|y-x^0|^2}{16t} \right) \, dy \\
&\leq C \int_{\mathbb{R}^n \setminus B_{\delta/\sqrt{t}}(x^0)} \exp\left(\frac{-|z|^2}{16} \right) \, dz \to 0
\end{aligned}$$

as $t \to 0$ from above. Thus, if $|x - x^0| < \delta/2$ and $t > 0$ is small enough,

$$|u(x,t) - g(x^0)| < 2\varepsilon$$

$\square$

Remark. *Consider $g \in C(\mathbb{R}^n)$, supp$(g) \subset B_1(0)$, then*

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y)g(y) \, dy > 0 \quad \forall x \in \mathbb{R}^n, t > 0.$$

→Infinite speed of propagation (different from transport- or wave-equation).

What about the non-homogeneous equation?

Consider

$$(12*) = \begin{cases} u_t - \Delta u = f \text{ in } \mathbb{R}^n \times (0,\infty) \\ \qquad\quad u = 0 \text{ on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

Note that $(x,t) \mapsto \Phi(x-y,t-s)$ solves the heat equation for $y \in \mathbb{R}^n,\ 0 < s < t$.

Thus (for fixed s) the function

$$u = u^s(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\ dy$$

solves

$$u_t^s - \Delta u^s = 0 \text{ in } \mathbb{R}^n \times (0,\infty)$$
$$u^s = f(\cdot,s) \text{ on } \mathbb{R}^n \times \{t=s\}$$

(This is just the initial value problem with starting time $t = s$ instead of $t = 0$ and g replaced by $f(\cdot,s)$.)

Duhamel's principle suspects we can build a solution of (12∗) from this u^s.
Consider

$$u(x,t) = \int_0^t u^s(x,t)\ ds \quad x \in \mathbb{R}^n, t \geq 0,$$

that is

$$u(x,t) = \int_0^t \int_{R^n} \Phi(x-y,t-s) f(y,s)\ dy\ ds.$$

Theorem 2.3.4. *Consider $f \in C^2(\mathbb{R}^n \times (0,\infty))$, with compact support, s.t. $f_t, D^2 f \in C(\mathbb{R}^n \times [0,\infty))$ and set*

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\ dy\ ds$$

Then,

(i) *We have $u_t, D^2u, u, \nabla u$ are in $C(\mathbb{R}^n \times (0,\infty))$*

(ii) $u_t(x,t) - \Delta u(x,t) = f(x,t)$ *for* $x \in \mathbb{R}^n, t > 0$.

(iii) *For $x^0 \in \mathbb{R}^n$ we have*

$$\lim_{(x,t)\to(x^0,t)} u(x,t) = 0$$

Proof. 1. Change variables to get

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y, t-s)\ dy\ ds.$$

The properties of f together with smoothness of $\Phi = \Phi(y,s)$ near $s = t > 0$, we compute

$$u_t(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_t(x-y,t-s)\ dy\ ds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\ dy$$

and

$$u_{x_i x_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_{x_i x_j}(x-y,t-s)\ dy\ ds$$

Thus, $u_t, D^2 u$ are continuous (and so are $u, \nabla u$).

2. We calculate

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \underbrace{\Phi(y,s) \left[\left(\frac{\partial}{\partial t} - \Delta_x \right) (f(x-y,t-s)) \right]}_{=:a}\ dy\ ds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\ dy \\ &= \int_\varepsilon^t a\ dy\ ds + \int_0^\varepsilon a\ dy\ ds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\ dy \\ &= I_\varepsilon + J_\varepsilon + K \end{aligned}$$

We immediatly see that

$$|J_\varepsilon| \leq (\|f_t\|_{L^\infty} + \|D^2 f\|_{L^\infty}) \cdot \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi\ dy\ ds = \varepsilon \cdot C.$$

Also

$$\begin{aligned} I_\varepsilon = &\int_\varepsilon^t \int_{\mathbb{R}^n} \underbrace{\left[\left(\frac{d}{ds} - \Delta_y \right) \Phi(y,s) \right]}_{=0} f(x-y,t-s)\ dy\ ds \\ &+ \int_{R^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\ dy \underbrace{- \int_{R^n} \Phi(y,t) f(x-y,0)\ dy}_{=-K} \\ = &\int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\ dy - K \end{aligned}$$

Thus

$$u_t(x,t) - \Delta u(x,t) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\ dy = f(x,t)$$

for $x \in \mathbb{R}^n, t > 0$ (with this limit being computed as in the proof of theorem 2.3.3.).

Note also that

$$\|u(\cdot,t)\|_{L^\infty} \le t\|f\|_{L^\infty} \to 0$$

as $t \to 0$. $\square$

Remark. *Note that theorem 2.3.3. and 2.3.4. can be combined to yield*

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y)g(y)\ dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)\ dy\ ds$$

as a solution of

$$\begin{aligned} u_t - \Delta u &= f \text{ in } \mathbb{R}^n \times (0,\infty) \\ u &= g \text{ on } \mathbb{R}^n \times \{t=0\}. \end{aligned}$$

2.3.2 Mean Value Formulas

Definition 2.3.5. *For $\Omega \subset \mathbb{R}^n,\ T>0$ define*

- *the parabolic cylinder $\Omega_T = \Omega \times (0,T]$ and*
- *the parabolic boundary $\Gamma_T = \overline{\Omega_T} \setminus \Omega_T$.*

Definition 2.3.6. *For fixed $x \in \mathbb{R}^n,\ t \in \mathbb{R},\ r>0$ we define*

$$E_r(x,t) = \left\{ (y,s) \in \mathbb{R}^{n+1} \mid s \le t,\ \Phi(x-y,t-s) \ge \frac{1}{r^n} \right\}$$

a so-called heat ball.

Theorem 2.3.7 (Mean value property of the heat equation)**.** *Let $u \in C_1^2(\Omega_T)$ (i.e. $u \in C(\Omega_T),\ u_{x_i,x_j} \in C(\Omega_T)$) solve the heat equation, then*

$$u(x,t) = \frac{1}{4r^n} \int\int_{E_r(x,t)} u(y,s)\frac{|x-y|^2}{(t-s)^2}\ dy\ ds$$

for $E_r(x,t) \subset \Omega_T$.

Proof. Assume $x=0,\ t=0,\ E_r = E_r(0,0)$. Set

$$\phi(r) = \frac{1}{r^n}\int\int_{E_r} u(y,s)\frac{|y|^2}{s^2}\ dy\ ds = \int\int_{E_1} u(ry,r^2 s)\frac{|y|^2}{s^2}\ dy\ ds$$

because $\Phi(rx,r^2t) = r^{-n}\Phi(x,t)$. We differentiate with respect to r

$$\phi'(r) = \int\int_{E_1} \frac{|y|^2}{s^2}(y\nabla u(ry,r^2s) + 2rsu_s(ry,r^2s))\ dy\ ds$$

$$
\begin{aligned}
&= \frac{1}{r^{n+1}} \int\int_{E_r} \frac{|y|^2}{s^2} y\nabla u(y,s) + 2u_s(y,s)\frac{|y|^2}{s^2}\ dy\ ds \\
&= \frac{1}{r^{n+1}} \int\int_{E_r} \frac{|y|^2}{s^2} y\nabla u(y,s)\ dy\ ds + \frac{1}{r^{n+1}} \int\int_{E_r} 2u_s(y,s)\frac{|y|^2}{s^2}\ dy\ ds \\
&= A + B
\end{aligned}
$$

Also introduce

$$\psi := -\frac{n}{2}\log(-4\pi s) + \frac{|y|^2}{4s} + n\log(r) \quad (12*)$$

and obtain $E_r = \{(y,s) \mid \psi(y,s) \geq 0$

$$\Rightarrow\ \psi = 0 \text{ on } \partial E_r$$

We use $(12*)$ and write

$$B = \frac{1}{r^{n+1}} \int\int_{E_r} 4u_s y\nabla\psi\ dy\ ds \overset{\text{Green}}{=} -\frac{1}{r^{n+1}} \int\int_{E_r} 4nu_s\psi + 4\psi y\nabla u_s\ dy\ ds$$

Integration by parts in s

$$
\begin{aligned}
B &= \frac{1}{r^{n+1}} \int\int_{E_r} 4nu_s\psi + 4\psi_s y\nabla u\ dy\ ds \\
&= \frac{1}{r^{n+1}} \int\int_{E_r} 4nu_s\psi + 4\left(-\frac{n}{2s} - \frac{|y|^2}{4s^2}\right) y\nabla u\ dy\ ds \\
&= \frac{1}{r^{n+1}} \int\int_{E_r} -4nu_s\psi - \frac{2n}{s} y\nabla u\ dy\ ds - A
\end{aligned}
$$

Consequently, since u solves the heat equation

$$
\begin{aligned}
\phi'(r) = A + B &= \frac{1}{r^{n+1}} \int\int_{E_r} -4n\Delta u\psi - \frac{2n}{s} y\nabla u\ dy\ ds \\
&\overset{\text{Green}}{=} \frac{1}{r^{n+1}} \int\int_{E_r} 4n\nabla u\nabla\psi - \frac{2n}{s} y\nabla u\ dy\ ds \overset{(12*)}{=} 0
\end{aligned}
$$

Thus ϕ is constant, and therefore

$$\phi(r) = \lim_{t\to 0}\phi(t) = u(0,0)\left(\lim_{t\to 0}\frac{1}{t^n}\int\int_{E_t}\frac{|y|^2}{s^2}\ dy\ ds\right) = 4u(0,0)$$

as

$$\frac{1}{t^n}\int\int_{E_t}\frac{|y|^2}{s^2}\ dy\ ds = \int\int_{E_1}\frac{|y|^2}{s^2}\ dy\ ds = 4.$$

$\square$

2.3.3 Properties of Solutions

Theorem 2.3.8 (Strong Maximum Principle)**.** *Assume $u \in C_1^2(\Omega_T) \cap C(\Omega_T)$ solves the heat equation in Ω_T*

(i) Then

$$\max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u$$

(ii) If Ω is connected and there exists a point $(x_0, t_0) \in \Omega_T$ with

$$u(x_0, t_0) = \max_{\overline{\Omega_T}} u$$

then u is constant.

Proof. 1. Suppose $(x_0, t_0) \in \Omega_T$ with

$$u(x_0, t_0) = M := \max_{\overline{\Omega_T}} u$$

for all sufficiently small $r > 0$, we employ for $E_r(x_0, t_0) \subset \Omega$ the mean value property

$$M = u(x_0, t_0) = \frac{1}{4r^n} \int\int_{E_r(x_0,t_0)} u(y, s) \frac{|x_0 - y|^2}{(t_0 - s)^2} \; dy \; ds \leq M$$

since

$$1 = \frac{1}{4r^n} \int\int_{E_r(x_0,t_0)} \frac{|x_0 - y|^2}{(t_0 - s)^2} \; dy \; ds.$$

Equality only holds if u is identically equal to M in $E_r(x_0, t_0)$, therefore $u(y, s) = M \; \forall (y, s) \in E_r(x_0, t_0)$. Let L be a line segment in Ω_T connecting $(x_0, t_0) \in \Omega_T$ with $(y_0, s_0) \in \Omega_T, \; s_0 < t_0$. Consider $r_0 := \min\{s \geq s_0 \mid u(x, t) = M \; \forall (x, t) \in L, \; s \leq t \leq t_0$.

Assume $r_0 > s_0$: $u(z_0, r_0) = M$ for some (z_0, r_0) on $L \cap \Omega_T$, thus $u \equiv M$ on $E_r(z_0, r_0)$ for all sufficiently small $r > 0$. Since

$$E_r(z_0, r_0) \supset L \cap \{r_0 - \sigma \leq t \leq r_0\}$$

for small $\sigma > 0$, we get a contradiction and thus $r_0 = s_0$ and we get $u = M$ on L.

2. Fix any $x \in \Omega, \; 0 \leq t < t_0$. There exists points $\{x_0, x_1, ..., x_m = x\}$ such that the line segments in $\mathbb{R}^n$ connecting x_{i-1} to x_i lie in Ω for $i = 1, ..., m$. (Set of points in Ω which can be connected like this by a polygonal path is non-empty, open and relatively closed on Ω.)

Select times $t_0 > t_1 ... > t_m = t$, then the line segments connecting (x_{i-1}, t_{i-1}) to (x_i, t_i) lie in Ω_T. According to 1. $u \equiv M$ on such a segment and thus $u(x,t) = M$. □

Remark. *If Ω is connected, $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$ satisfies*

$$\begin{cases} u_t - \Delta u = 0 \ \text{in } \Omega_T \\ \quad u = 0 \ \text{on } \partial\Omega \times [0,T] \\ \quad u = g \ \text{on } \Omega \times \{t = 0\} \end{cases}$$

where $g \geq 0$, then u is positive everywhere within Ω_T, if g is positive somewhere on Ω.

Theorem 2.3.9 (Uniqueness on bounded domains)**.** *Let $g \in C(\Gamma_T), f \in C(\Omega_T)$. Then there exists at most one solution $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$ of the initial value problem*

$$(13*) = \begin{cases} u_t - \Delta u = f \ \text{in } \Omega_T \\ \quad u = g \ \text{on } \Gamma_T \end{cases}$$

Proof. If $u, \tilde{u}$ are solutions of $(13*)$, apply theorem 2.3.8. to $w := \pm(u - \tilde{u})$ □

Theorem 2.3.10 (Maximum Principle for Cauchy Problem)**.** *Suppose $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solves*

$$\begin{cases} u_t - \Delta u = 0 \ \text{in } \mathbb{R}^n \times (0,T) \\ \quad u = g \ \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

and satisfies growth estimate

$$u(x,t) \leq Ae^{a|x|^2} \quad (x \in \mathbb{R}^n, 0 \leq t \leq T)$$

for constants $A, a > 0$. Then

$$\sup_{x \in \mathbb{R}^n \times [0,T]} u(x) = \sup_{x \in \mathbb{R}^n} g(x)$$

Proof. 1. Assume first that $4aT < 1$, such that

$$4a(T + \varepsilon) < 1$$

for suitable $\varepsilon > 0$. Fix $y \in \mathbb{R}^n, \mu > 0$ and set

$$v(x,t) = u(x,t) - \frac{\mu}{(T + \varepsilon - t)^{n/2}} \exp\left(\frac{|x - y|^2}{4(T + \varepsilon - t)}\right)$$

for $x \in \mathbb{R}^n, t > 0$. We have

$$v_t - \Delta v = 0$$

in $\mathbb{R}^n \times (0,T)$. Fix $r > 0$, and set $\Omega = B_r(y),\ \Omega_T = B_r(y) \times (0,T]$. Then using theorem 2.3.8., we have

$$\max_{x\in\overline{\Omega_T}} v(x) = \max_{x\in\Gamma_T} v(x)$$

2. Now, for $x \in \mathbb{R}^n$,

$$v(x,0) = u(x,0) - \frac{\mu}{(T+\varepsilon)^{n/2}} \exp\Big(\frac{|x-y|^2}{4(T+\varepsilon)}\Big) \leq u(x,0) = g(x)$$

and, if $|x-y| = r,\ 0 \leq t \leq T$,

$$\begin{aligned}
v(x,t) &= u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} \exp\Big(\frac{r^2}{4(T+\varepsilon-t)}\Big) \\
&\leq Ae^{a|x|^2} - \frac{\mu}{(T+\varepsilon-t)^{n/2}} \exp\Big(\frac{r^2}{4(T+\varepsilon-t)}\Big) \\
&\leq Ae^{a(|y|+r)^2} - \frac{\mu}{(T+\varepsilon-t)^{n/2}} \exp\Big(\frac{r^2}{4(T+\varepsilon-t)}\Big)
\end{aligned}$$

We have

$$\frac{1}{4(T+\varepsilon)} = a + \gamma$$

for some $\gamma > 0$, and thus

$$v(x,t) \leq Ae^{a(|y|+r)^2} - \mu(4(a+\gamma))^{n/2} e^{(a+\gamma)r^2} \leq \sup_{x\in\mathbb{R}^n} g(x)$$

for r sufficiently large (remember $|x-y| = r$), since the negative term "wins" at some point. This, however, implies that

$$v(y,t) \leq \sup_{x\in\mathbb{R}^n} g(x)$$

for all $y \in \mathbb{R}^n,\ 0 \leq t \leq T$ as long as $4aT < 1$. For $\mu \to 0$, we obtain

$$u(x,t) \leq \sup_{y\in\mathbb{R}^n} g(y) \quad x \in \mathbb{R}^n,\ 0 \leq t \leq T$$

3. In the general case repeatedly apply the above result on time intervals $[0,T_1], [T_1, 2T_1], ...$ for $T_1 = 1/(\gamma a)$. $\square$

Theorem 2.3.11 (Uniqueness of the Cauchy-Problem)**.** *Let* $g \in C(\mathbb{R}^n),\ f \in C(\mathbb{R}^n \times$

$[0,T])$, *then there exists at most one solution* $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ *of*

$$\begin{cases} u_t - \Delta u = f \text{ in } \mathbb{R}^n \times (0,T) \\ \qquad u = g \text{ on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

satisfying the growth estimate

$$u(x,t) \le Ae^{a|x|^2} \quad x \in \mathbb{R}^n, \; 0 \le t \le T$$

for some $a, A > 0$.

Proof. Apply theorem 2.3.10 to $w = \pm(u - \tilde{u})$ when both $u, \tilde{u}$ satisfy the assumption. □

Remark. *There are so-called "unphysical" solutions to the heat equation on* $\mathbb{R}^n$ *with very rapid growth and zero initial conditions (so-called Tychonov solutions).*

2.3.4 Finite Difference Schemes for the Heat equation

We simply replace the derivatives in the problem

$$\begin{cases} u_t - \Delta u = 0 \text{ in } (0,1) \times (0,T) \\ \qquad u = g \text{ on } (0,1) \times \{t=0\} \\ \qquad u = 0 \text{ on } (\{0\} \cup \{1\}) \times (0,T] \end{cases}$$

to get the forward Euler scheme

$$\partial_t^+ U_j^k - \partial_x^+ \partial_x^- U_j^k = 0, \; j = 1, ..., J, \; k = 0, ..., K-1$$

$U_0^k = U_J^k = 0, \; k = 0, ..., K-1$ and $U_j^0 = g(x_j), \; j = 0, ..., J$, with spatial discretization size $h = 1/J$, and temporal discretization size $\tau = 1/K$. We have, equivalently,

$$U_j^{k+1} = (1 - 2\lambda)U_j^k + \lambda U_{j-1}^k + \lambda U_{j+1}^k$$

for $\lambda = \tau/h^2$.

This is an explicit scheme, since U_j^{k+1} can be computed explicitly for $\{U_j^k\}_{j=0}^J$

Proposition 2.3.12 (Stability and Convergence)**.** *If* $\lambda \le 1/2$*, then the solution of the forward Euler scheme satisfies*

$$\sup_{j=0,\dots,J} |U_j^k| \le \sup_{j=0,\dots,J} |U_j^0|$$

for all $0 \le k \le K$, *and if in addition, we have* $u \in C^4([0,1] \times [0,T])$, *then*

$$\sup_{j=0,\dots,J} |u(x_j, t_k) - U_j^k| \le \frac{t_k}{2}(\tau + h^2)\left(\|\partial_x^4 u\|_{C([0,1]\times[0,T])} + \|\partial_t^2 u\|_{C([0,1]\times[0,T])}\right)$$

for all $0 \le k \le K$.

Proof. As $1 - 2\lambda \ge 0$, we have that

$$|U_j^{k+1}| \le (1-2\lambda) \sup_{j=0,\dots,J} |U_j^k| + 2\lambda \sup_{j=0,\dots,J} |U_j^k| \le \sup_{j=0,\dots,J} |U_j^k|$$

which implies

$$\sup_{j=0,\dots,J} |U_j^k| \le \sup_{j=0,\dots,J} |U_j^0|$$

For the error estimate, consider

$$\mathscr{C}_j^k = \partial_t^+ u(x_j, t_k) - \partial_x^+ \partial_x^- u(x_j, t_k)$$

which satisfies

$$\begin{aligned} |\mathscr{C}_j^k| &\le |\partial_t^+ u - \partial_t u + \partial_x^2 u - \partial_x^+ \partial_x^- u| \\ &\le |\partial_t^+ u - \partial_t u| + |\partial_x^+ \partial_x^- u - \partial_x^2 u| \\ &\le \frac{1}{2}(\tau + h^2)\left(\|\partial_x^4 u\|_{C([0,1]\times[0,T])} + \|\partial_t^2 u\|_{C([0,1]\times[0,T])}\right) \end{aligned}$$

The error $Z_j^k = u(x_j, t_k) - U_j^k$ satisfies

$$\partial_t^+ Z_j^k - \partial_x^+ \partial_x^- Z_j^k = \mathscr{C}_j^k$$

and thus

$$Z_j^{k+1} = (1-2\lambda) Z_j^k + \lambda Z_{j-1}^k + \lambda Z_{j+1}^k + \tau \mathscr{C}_j^k$$

Again using the convex combination property, we obtain

$$\sup_{j=0,\dots,J} |Z_j^{k+1}| \le \sup_{j=0,\dots,J} |Z_j^k| + \tau \sup_{j=0,\dots,J} \mathscr{C}_j^k$$

By induction, we obtain our estimate. □

Remark. *The bad news is, that* $\lambda \le 1/2$ *requires very small time steps. This is the classical example of a stiff equation.*

We therefore instead use an implicit Euler scheme

$$\partial_t^- U_j^k - \partial_x^+ \partial_x^- U_j^k = 0, \quad j = ,\dots, J-1, \; k = 1, \dots, K$$

$$U_0^k = U_j^k = 0 \quad k = 1, ..., K$$
$$U_j^0 = u_0(x_j), \quad j = 0, ..., J$$

that is,

$$U_j^{k+1} - \lambda(U_{j-1}^{k+1} - 2U_j^{k+1} + U_{j+1}^{k+1}) = U_j^k$$

i.e.

$$AU^{k+1} = U^k$$

with

$$A = \begin{pmatrix} (1+2\lambda) & -\lambda & & & \\ -\lambda & (1+2\lambda) & -\lambda & & \\ & \ddots & \ddots & \ddots & \\ & & -\lambda & (1+2\lambda) & -\lambda \end{pmatrix}$$

Remark. *The Matrix A is regular, so the implicit Euler scheme does admit a solution.*

Proposition 2.3.13. *There exists unique coefficients (U_j^k) that solve the implicit Euler scheme. They satisfy*

$$\sup_{j=0,...,J} |U_j^k| \leq \sup_{j=0,...,J} |U_j^0| \quad k = 0, ..., K$$

(independently of $\lambda = \tau/h^2$). If $u \in C^4([0,1] \times [0,T])$, we have

$$\sup_{j=0,...,J} |u(x_j, t_k) - U_j^k| \leq \frac{t_k}{2}(\tau + h^2)(\|\partial_x^4 u\|_\infty + \|\partial_t^2 u\|_\infty)$$

for $k = 0, ..., K$.

Proof. 1. The matrix from the scheme is strictly diagonally dominant, hence regular.

2. Take $j' \in \{1, ..., J\}$ such that

$$|U_{j'}^{k+1}| \leq \sup_{j=0,...,J} |U_j^{k+1}|,$$

then

$$(1+2\lambda)|U_{j'}^{k+1}| \leq |U_{j'}^k| + \lambda|U_{j'-1}^{k+1}| + \lambda|U_{j'+1}^{k+1}| \leq \sup_{j=1,...,J} |U_j^k| + 2\lambda \sup_{j=0,...,J} |U_j^{k+1}|$$

So,

$$(1+2\lambda) \sup_{j=0,...,J} |U_j^{k+1}| \leq \sup_{j=0,...,J} |U_j^k| + 2\lambda \sup_{j=0,...,J} |U_j^{k+1}|$$

and the discrete maximum principle follows.

3. The error bound follows similarly to the explicit scheme. $\square$

Remark. *The implicit Euler scheme is thus unconditionally stable. The error estimate is the same as for the explicit scheme.*

$\sim$ Regularity of the heat equation.

Theorem 2.3.14. *Suppose $u \in C_1^2(\Omega_T)$ solves the heat equation in Ω_T. Then $u \in C^\infty(\Omega_T)$*

Remark. *This even holds of the boundary values are not smooth.*

Proof. Set
$$C_r(x,t) = \{(y,s) \mid |x-y| \le r,\ t \cdot r^2 \le s \le t\}$$
(a closed cylinder). Fix $(x_0,t_0) \in \Omega_T$ and choose $r > 0$ so that $C = C_r(x_0,t_0) \subset \Omega_T$, define also $C' = C_{3/4 \cdot r}(x_0,t_0),\ C'' = C_{1/2 \cdot r}(x_0,t_0)$.

Pick a smooth cutoff function $\zeta(x,t)$ such that
$$\begin{cases} 0 \le \zeta \le 1,\ \zeta \equiv 1 \text{ on } C' \\ \qquad\qquad \zeta \equiv 0 \text{ near the parabolic boundary of } C. \end{cases}$$
Extend ζ by 0 in $(\mathbb{R}^n \setminus [0,t_0]) \setminus C$.

2. Assume for now that $u \in C^\infty(\Omega_T)$ and set
$$v(x,t) = \zeta(x,t) \cdot u(x,t)$$
then
$$v_t = \zeta u_t + \zeta_t u_t,\ \Delta v = \zeta \Delta u + 2\nabla\zeta\nabla u + u\Delta\zeta$$
Thus, $v = 0$ in $\mathbb{R}^n \times \{t = 0\}$ and
$$v_t - \Delta v = \zeta_t u - 2\nabla\zeta\nabla u - u\Delta f =: \tilde{f}$$
in $\mathbb{R}^n \times (0,t_0)$. Set
$$\tilde{v}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\tilde{f}(y,s)\ dy\ ds$$
Thus
$$\begin{cases} \tilde{v}_t - \Delta\tilde{v} = \tilde{f} \text{ in } \mathbb{R}^n \times (0,t_0) \\ \qquad \tilde{v} = 0 \text{ on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

and theorem 2.3.11. (uniqueness) yields $\tilde{v} = v$.
Now suppose $(x,t) \in C''$. Since $\zeta = 0$ away from C, we get

$$u(x,t) = \int\int_C \Phi(x-y,t-s)\left[(\zeta_s(y,s) - \Delta\zeta(y,s)u(y,s)) - 2\nabla\zeta(y,s)\cdot\nabla u(y,s)\right]\ dy\ ds$$

The expression in the square brackets vanishes near the singularity of Φ. Integrating by parts, we get

$$u(x,t) = \int\int_C \left[\Phi(x-y,t-s)(\zeta_s(y,s) + \Delta\zeta(y,s)) + 2\nabla_y\Phi(x-y,t-s)\nabla\zeta(y,s)\right] u(y,s)\, dy\, ds \quad (14*)$$

We have derived (14∗) assuming $u \in C^\infty$. If u only satisfies the hypothesis of the theorem, we still arrive at (14∗) considering $u^\varepsilon = \eta_\varepsilon * u$ in the calculation and taking $\varepsilon \to 0$.

3. Formula (14∗) is of the form

$$u(x,t) = \int\int_C K(x,t,y,s)u(y,s)\ dy\ ds$$

for $(x,t) \in C''$, where $K(x,t,y,s) = 0$ for all points $(y,s) \in C'$ (since $\zeta = 1$ on C'). Also K is smooth on $C \setminus C'$. We therefore see that $u \in C^\infty$ in $C'' = C_{1/2\cdot r}(x_0,t_0)$. □

Theorem 2.3.15 (Estimates on the derivatives)**.** *For each pair of integers $k,l = 1,2,...$ there exists a constant $C_{k,l}$ such that*

$$\max_{C_{r/2}(x,t)} |D_x^k D_t^l u| \leq \frac{C_{kl}}{r^{k+2l+n+2}} \|u\|_{L^1(C_r(x,t))}$$

for all cylinders $C_{r/2}(x,t) \subset C_r(x,t) \subset \Omega_T$ and all solutions of the heat equation in Ω_T

Proof. 1. Fix a point in Ω_T, we may assume that this point is $(0,0)$. Suppose that $C_1 = C_1(0,0)$ is in Ω_T and that $C_{1/2} = C_{1/2}(0,0)$. Then as in the proof of theorem 2.3.14 we get

$$u(x,t) = \int\int_{C_1} K(x,t,y,s)u(y,s)\ dy\ ds$$

for some smooth function K. Thus

$$|D_x^k D_t^l u(x,t)| \leq \int\int_C |D_x^k D_t^l K(x,t,y,s)||u(y,s)|\ dy\ ds \leq C_{kl}\|u\|_{L^1(C_1)} \quad (15*)$$

for some constant C_{kl}.

2. Now suppose $C_r = C_r(0,0)$ lies in Ω_T, set $C_{r/2} = C_{r/2}(0,0)$. Set $v(x,t) = u(rx, r^2t)$.

Then $v_t - \Delta v = 0$ in C_1 and by (15$*$)

$$|D_x^k D_t^l v(x,t)| \le C_{kl} \|v\|_{L^1(C_1)}.$$

Noting

$$D_x^k D_t^l v(x,t) = r^{2l+k} D_x^k D_t^l u(rx, r^2 t)$$

and

$$\|v\|_{L^1(C_1)\|} = \frac{1}{r^{n+2}} \|u\|_{L^1(C_r)}$$

we get our desired estimate. $\square$

2.4 The Wave Equation

We consider solutions to the wave equation

$$u_{tt} - \Delta u = 0.$$

Note that there is also the non-homogeneous wave equation

$$u_{tt} - \Delta u = f.$$

These equations are again subject to suitable boundary and initial conditions. The unkown is

$$u : \overline{\Omega} \times [0,T] \to \mathbb{R},\ u = u(x,t), \quad x \in \Omega,\ t \in [0,T].$$

Sometimes, the notation

$$\square u = t_{tt} - \Delta u$$

can be seen.

For one physical interpretation, consider an elastic string, membrane or a solid. Denote by $u(x,t)$ the displacement of the material at point x, time t. Then, Newton's second law states (for some $V \subset \Omega$)

$$\frac{d}{dt^2} \int_V u \, dx = \int_V u_{tt} \, dx - \int_{\partial V} F \cdot \nu \, dS.$$

Again we get

$$u_{tt} = -\operatorname{div}(F) \text{ in } \Omega.$$

Take (coming from continuum Mechanics)

$$F = c \cdot \nabla u,$$

and we get

$$u_{tt} - c\Delta u = 0.$$

This suggests also that we need two initial conditions, one for displacement, one for velocity (i.e u_t).

$\sim$ Solution for $n = 1$.

We consider

$$\begin{cases} u_{tt} - u_{xx} = 0 \text{ in } \mathbb{R} \times (0, \infty) \\ u - g, \ u_t = h \text{ in } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Note that

$$\left(\frac{\partial}{\partial_t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = u_{tt} - u_{xx}.$$

Write

$$v(x,t) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u,$$

then $v_t + v_x = 0$ for $x \in \mathbb{R}, \ t > 0$. Thinking back to the transport equation, we get

$$v(x,t) = a(x - t)$$

for some $a(x) = v(x,0)$. Plugging back in, we get

$$u_t(x,t) - u_x(x,t) = a(x - t)$$

This is a non-homogeneous transport equation, with solution

$$u(x,t) = \int_0^t a(x + (t - s) - s) \ ds + b(x + t) = \frac{1}{2}\int_{x-t}^{x+t} a(y) \ dy + b(x + t)$$

for $b(x) = u(x,0) = g(x)$. Noting that

$$a(x) = v(x,0) = u_t(x,0) - u_x(x,0) = h(x) - g'(x)$$

so,

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} h(y) - g'(y) \ dy + g(x+t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y) \ dy.$$

This is called d'Alembert's formula.

Theorem 2.4.1. *Assume $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$ and define $u(x,t)$ by D'Alemberts formula. Then*

(i) $u \in C^2(\mathbb{R} \times [0, \infty))$

(ii) $u_{tt} - u_{xx} = 0$ *in* $\mathbb{R} \times [0, \infty)$

(iii) *For all* $x_0 \in \mathbb{R}$ *we have*

$$\lim_{(x,t)\to(x_0,t)} u(x,t) = g(x_0) \text{ and } \lim_{(x,t)\to(x_0,t)} u_t(x,t) = h(x_0)$$

Proof. By direct calculation. $\square$

Remark. *Note that*

$$u(x,t) = F(x-t) + G(x+t)$$

for appropriate functions F, G and any such expression is a solution to the wave equation.

Remark. *The initial/boundary value problem*

$$\begin{cases} u_{tt} - u_{xx} = 0 \text{ in } \mathbb{R}_+ \times (0, \infty) \\ u = g, \ u_t = h \text{ on } \mathbb{R}_+ \times \{t = 0\} \\ u = 0 \text{ on } \{x = 0\} \times (0, \infty) \end{cases}$$

can be solved by reflection, setting

$$\tilde{g}(x) = \begin{cases} g(x) & x \geq 0 \\ -g(x) & x < 0 \end{cases}, \quad \tilde{h}(x) = \begin{cases} h(x) & x \geq 0 \\ -h(-x) & x < 0 \end{cases}$$

and using d'Alembert's formula (for g, h vanishing at $x = 0$).

$\sim$ Solution for $n = 2, 3$.

Assume $u \in C^m(\mathbb{R}^n \times [0, \infty))$ solves

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times [0, \infty] \\ u = g, \ u_t = h \text{ on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

For given $x \in \mathbb{R}^n, \ t > 0, \ r > 0$ set

$$U(x; r, t) = \fint_{B_r(x)} u(y,t) \, dy$$

and

$$G(x; r) = \fint_{B_r(x)} g(y) \, dy, \quad H(x; r) = \fint_{B_r(x)} h(y) \ y$$

Lemma 2.4.2 (Euler-Poisson-Darboux-Equation). *Fix $x \in \mathbb{R}^n$, and define u, U, G, H as above. Then*

$$U \in C^m(\overline{\mathbb{R}_+} \times [0, \infty))$$

and

$$\begin{cases} U_{tt} - U_{rr} - \dfrac{n-1}{r} U_r = 0 \ \text{in } \mathbb{R}_+ \times (0, \infty) \\ U = G, \ U_t = H \ \text{on } \mathbb{R}_+ \times \{t = 0\} \end{cases}$$

Proof. We get for $r > 0$, that

$$U_r(x; r, t) = \frac{r}{n} \fint_{\partial B_r(x)} \Delta u(y, t)\ dy = \frac{1}{n\alpha(n) r^{n-1}} \int_{B_r(x)} \Delta u(y, t)\ dy. \quad (16*)$$

Note that hus

$$\lim_{r \to 0^+} U_r(x; r, t) = 0$$

and

$$U_{rr}(x; r, t) = \fint_{\partial B_r(x)} \Delta u(y)\ dS(y) + \frac{1-n}{n} \fint_{B_r(x)} \Delta u(y)\ dy$$

Thus,

$$\lim_{r \to 0^+} U_{rr}(x; r, t) = \frac{1}{n} \Delta u(x, t).$$

Similarly, verify that U_{rrr}, etc. are continuous.

Now, from (16∗), we get

$$U_r = \frac{r}{n} \fint_{B_r(x)} u_{tt}\ dy,$$

and so

$$r^{n-1} U_r = \frac{1}{n\alpha(n)} \int_{B_r(x)} u_{tt}\ dy$$

and

$$(r^{n-1} U_r)_r = \frac{1}{n\alpha(n)} \int_{\partial B_r(x)} u_{tt}\ dS(y) = r^{n-1} \fint_{\partial B_r(x)} u_{tt}\ dS(y) = r^{n-1} U_{tt}$$

Which yields the result by product rule and division by r^{n-1}. □

Now take $n = 3$, and set

$$\tilde{U} = rU, \ \tilde{G} = rG, \ \tilde{H} = rH$$

to get

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 \text{ in } \mathbb{R}_+ \times (0, \infty) \\ \tilde{U} = \tilde{G}, \ \tilde{U}_t = \tilde{H} \text{ on } \mathbb{R}_+ \times \{t = 0\} \\ \tilde{U} = 0 \text{ on } \{r = 0\} \times (0, \infty) \end{cases}$$

by direct calculation.

Using d'Alembert's formula, we find for $0 \le r \le t$

$$\tilde{U}(x;r,t) = \frac{1}{2}(\tilde{G}(r+t) - \tilde{G}(r-t)) + \frac{1}{2}\int_{-r+t}^{r+t} \tilde{H}(y)\ dy.$$

Our definition of $\tilde{U}$ implies

$$u(x,t) = \lim_{r \to 0^+} \frac{\tilde{U}(x;r,t)}{r} = \tilde{G}'(t) + \tilde{H}(t).$$

From the definitions of $\tilde{G}, \tilde{H}$, we get

$$u(x,t) = \frac{\partial}{\partial t}\left(t \fint_{\partial B_t(x)} g\ dS\right) + t \fint_{\partial B_t(x)} h\ dS. \quad (17*)$$

Noting

$$\fint_{\partial B_t(x)} g(y)\ dS(y) = \fint_{\partial B_1(0)} g(x+tz)\ dS(z)$$

we get

$$\frac{\partial}{\partial t}\left(\fint_{\partial B_t(x)} g\ dS\right) = \fint_{\partial B_1(0)} \nabla g(x+tz) \cdot z\ dS(z) = \fint_{\partial B_t(x)} \nabla g(y) \left(\frac{y-x}{t}\right)\ dS(y)$$

Plugging into (17∗), we get

$$u(x,t) = \fint_{\partial B_t(x)} th(y) + g(y) + \nabla g(y)(y-x)\ dS(y).$$

This is Kirchhoff's formula for the solution of the initial value problem for the wave equation in 3 dimensions.