

Introduction to the Theory and Numerics for Partial Differential Equations

Series 1

Return: October 22, 2025

Transport

Problem 1 (4 Points). *Transport and growth*

Write down an explicit formula for a function u solving the initial value problem

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

Problem 2 (4 Points). *Characteristics*

Let u solve the partial differential equation

$$\partial_t u + b(x, t) \partial_x u = 0 \text{ on } \mathbb{R} \times (0, \infty).$$

- (1) Show that u is constant along curves $(y(t), t)$ for solutions of the initial boundary value problems $y'(t) = b(y(t), t)$, $y(0) = x_0$, called characteristics.
- (2) Determine the characteristics for the equation for $b(x, t) = tx$ and for $b(x, t) = 2t$, sketch them, and determine the solution for the initial condition $u(0, x) = u_0(x) = \cos(x)$.

Problem 3 (4 Points). *Finite Differences*

- (1) Prove the following estimates for difference quotients:

$$\begin{aligned} |\partial^\pm u(x_j) - u'(x_j)| &\leq \frac{\Delta x}{2} \|u''\|_{C([0,1])} \\ \left| \widehat{\partial} u(x_j) - u'(x_j) \right| &\leq \frac{\Delta x^2}{6} \|u'''\|_{C([0,1])} \\ |\partial^+ \partial^- u(x_j) - u''(x_j)| &\leq \frac{\Delta x^2}{12} \|u^{(4)}\|_{C([0,1])} \end{aligned}$$

Show that these estimates do not hold if u does not satisfy the required differentiability properties.

- (2) Show that $\partial^+ \partial^- = \partial^- \partial^+$.
- (3) Prove an error estimate for the difference $\partial^+ \partial^+ u(x_j) - u''(x_j)$.

Problem 4 (4 Points). *Finite Differences*

Let $b < 0$ and consider the numerical scheme $\partial_t^+ U_j^k + b \partial_x^+ U_j^k = 0$. Show that the scheme is stable under appropriate conditions on τ and h and prove an error estimate.

Hand in the exercise sheets in the box marked "ITaN" on the 2nd floor at Hermann-Herder-Str. 10, next to the entrance to room 201 (CIP). The exercise sheets must be handed in by 12 pm (noon) on the specified date.