University of Freiburg
Department for Applied Mathematics
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Winter Term 2025

Introduction to the Theory and Numerics for Partial Differential Equations

Series 3

Return: November 5, 2025

Laplace & Poisson

Problem 9 (4 Points)**.**
Let Ω be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending only on Ω, such that

$$\max_{\overline{\Omega}} |u| \leq C \left(\max_{\partial\Omega} |g| + \max_{\overline{\Omega}} |f| \right)$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ \quad\; u = g \text{ on } \partial\Omega. \end{cases}$$

Hint: $-\Delta \left(u + \frac{|x|^2}{2n} \lambda \right) \leq 0$, for $\lambda := \max_{\overline{\Omega}} |f|$.

Problem 10 (4 Points)**.** *Oscillation estimate from Harnack's inequality*
Let u be a function in a ball $B_R(0) \subset \mathbb{R}^n$, such that Harnack's inequality holds, more precisely, assume that there exists $C > 1$ such that

$$\sup_{B_{R/2}(0)} u \leq C \inf_{B_{R/2}(0)} u.$$

The goal is to prove that

$$\operatorname{osc}_{B_{R/2}} u \leq \frac{C-1}{C+1} \cdot \operatorname{osc}_{B_R(0)} u$$

where $\operatorname{osc}_E u = \sup_E u - \inf_E u$ is the oscillation of u on $E \subset B_R(0)$.
Set first $m = \inf_{B_R(0)} u$, $M = \sup_{B_R(0)} u$ and $m_{R/2} = \inf_{B_{R/2}(0)} u$, $M_{R/2} = \sup_{B_{R/2(0)}} u$.

(1) Consider the non-negative functions $v(x) = u(x) - m$ and $w(x) = M - u(x)$. Show that

$$M_{R/2} - m \leq C(m_{R/2} - m) \quad \text{and} \quad M - m_{R/2} \leq C(M - M_{R/2}).$$

(2) Derive

$$(M_{R/2} - m) + (M - m_{R/2}) \leq C(m_{R/2} - m) + C(M - M_{R/2}).$$

(3) Conclude

$$\operatorname{osc}_{B_{R/2}(0)} u = M_{R/2} - m_{R/2} \leq \frac{C-1}{C+1} \cdot (M - m) = \frac{C-1}{C+1} \cdot \operatorname{osc}_{B_R(0)} u.$$

Problem 11 (4 Points)**.** *To Hölder regularity*
Let $\Omega \subset \mathbb{R}^n$ be an open set and $u : \Omega \to \mathbb{R}$ be a given function. Assume that for all $x_0 \in \Omega$ and all $0 < R \leq R_0$ so that $B_{R_0}(x_0) \subset \Omega$, the function u satisfies the oscillation estimate from Problem 10, i.e.,

$$\operatorname{osc}_{B_{R/2}(x_0)} u \leq \theta \cdot \operatorname{osc}_{B_R(x_0)} u$$

for some constant $\theta \in (0, 1)$.

The goal is to prove that u is Hölder continuous on $B_{r/2}(x_0) \subset \Omega$ with $r \leq R_0$, meaning that there exist constants $C > 0$ and $\alpha \in (0,1)$ such that for all $x, y \in B_{r/2}(x_0)$,

$$|u(x) - u(y)| \leq C \cdot \operatorname{osc}_{B_r(x_0)} u \cdot \left(\frac{|x-y|}{r} \right)^\alpha$$

where $\alpha = -\log_2 \theta > 0$.

(1) Fix $x_0 \in \Omega$ and $r \leq R_0$ with $B_r(x_0) \subset \Omega$. By repeatedly applying the oscillation estimate at scales $r, r/2, r/4, \ldots$, show that for any $n \in \mathbb{N}$:

$$\operatorname{osc}_{B_{r/2^n}(x_0)} u \leq \theta^n \cdot \operatorname{osc}_{B_r(x_0)} u$$

(2) Let $x, y \in B_{r/2}(x_0)$ with $|x - y| = \rho \leq r/2$. Show that

$$|u(x) - u(y)| \leq \theta^n \cdot \operatorname{osc}_{B_r(x_0)} u$$

where n satisfies $\frac{r}{2^{n+1}} < \rho \leq \frac{r}{2^n}$.

(3) Show that

$$\theta^n \leq \theta^{-\log_2(\rho/r)} = \left(\frac{\rho}{r} \right)^{-\log_2 \theta} = \left(\frac{\rho}{r} \right)^\alpha$$

where $\alpha = -\log_2 \theta$.

(4) Derive the Hölder estimate

$$|u(x) - u(y)| \leq \operatorname{osc}_{B_r(x_0)} u \cdot \left(\frac{|x-y|}{r} \right)^\alpha$$

for all $x, y \in B_{r/2}(x_0)$.

Problem 12 (4 Points)**.** *Kelvin transform*

The Kelvin transform $\mathcal{K}u = \bar{u}$ of a function $u : \mathbb{R}^n \to \mathbb{R}$ is

$$\bar{u}(x) := u(\bar{x})|\bar{x}|^{n-2} = u(x/|x|)|x|^{2-n} \quad (x \neq 0)$$

where $\bar{x} = x/|x|^2$. Show that if u is harmonic, then so is $\bar{u}$.

Hint: First show that $\nabla_x \bar{x} \left(\nabla_x \bar{x} \right)^T = |\bar{x}|^4 I$. The mapping $x \to \bar{x}$ is conformal, meaning angle preserving.

Hand in the exercise sheets in the box marked “ITaN” on the 2nd floor at Hermann-Herder-Str. 10, next to the entrance to room 201 (CIP). The exercise sheets must be handed in by 12 pm (noon) on the specified date.