University of Freiburg
Department for Applied Mathematics
Prof. Dr. Patrick Dondl

Winter Term 2025

Introduction to the Theory and Numerics for Partial Differential Equations

Series 4

Return: November 12, 2025

Assorted Estimates

Problem 13 (4 Points)**.** *Fractional*

Let u be the solution of

$$\begin{cases} \Delta u = 0 \text{ in } \mathbb{R}^n_+ \\ u = g \text{ on } \partial\mathbb{R}^n_+ \end{cases}$$

given by Poisson's formula for the half-space. Assume g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+$, $|x| \leq 1$. Show ∇u is not bounded near $x = 0$.

Hint: Estimate $\frac{u(\lambda e_n) - u(0)}{\lambda}$.

Problem 14 (4 Points)**.** *Matrix*

Let $J \in \mathbb{N}$, set $L = (J-1)^2$, and let $X \in \mathbb{R}^{(J-1)\times(J-1)}$ and $A \in \mathbb{R}^{L\times L}$ be defined by

$$X = \begin{bmatrix} 4 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{bmatrix}, \quad A = \begin{bmatrix} X & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & X \end{bmatrix},$$

where $I \in \mathbb{R}^{(J-1)\times(J-1)}$ denotes the identity matrix. Show that A is diagonally dominant and irreducible.

Problem 15 (4 Points)**.** *Maximum*

(1) Let $L > 1$ and for α_ℓ, $p_\ell \in \mathbb{R}$, $0 \leq \ell \leq L$, assume that $\alpha_\ell < 0$ for $\ell = 1, 2, \ldots, L$, and

$$\sum_{\ell=0}^{L} \alpha_\ell \geq 0, \quad \sum_{\ell=0}^{L} \alpha_\ell p_\ell \leq 0.$$

Suppose further that $p_0 \geq 0$ or $\sum_{\ell=0}^{L} \alpha_\ell = 0$. Show that $p_0 \geq \max_{1\leq\ell\leq L} p_\ell$ implies $p_0 = p_1 = \cdots = p_L$.

(2) Let $(U_{j,m})_{0\leq j,m\leq J}$ be the finite difference approximation of the Poisson problem $-\Delta u = f$ in $\Omega = (0,1)^2$ and $u = g$ on $\partial\Omega$. Assume that $f \leq 0$ and show that

$$\max_{1\leq j,m\leq J-1} U_{j,m} \leq \max_{x_{j,m}\in\partial\Omega} g(x_{j,m}).$$

Problem 16 (4 Points)**.** *Heavy Heat*

Derive an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f \text{ in } \mathbb{R}^n \times (0,\infty) \\ u = g \text{ on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

where $c \in \mathbb{R}$.

Hand in the exercise sheets in the box marked "ITaN" on the 2nd floor at Hermann-Herder-Str. 10, next to the entrance to room 201 (CIP). The exercise sheets must be turned in by 12 pm (noon) on the specified date.