University of Freiburg
Department for Applied Mathematics
Prof. Dr. Patrick Dondl

Winter Term 2025

Introduction to the Theory and Numerics for Partial Differential Equations

Series 5

Return: November 19, 2025

Heat

Problem 17 (4 Points). *Scaling*

Suppose u is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.

(1) Show $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.
(2) Show $v(x,t) = x \cdot \nabla u(x,t) + 2tu_t(x,t)$ solves the heat equation as well.

Problem 18 (4 Points). *Subsolutions*

We say $v \in C_1^2(\Omega_T)$ – i.e., v twice continuously differentiable with respect to the spatial variables, once continuously differentiable with respect to the temporal variable – is a subsolution of the heat equation if

$$v_t - \Delta v \leq 0 \quad \text{in } \Omega_T.$$

(1) Prove for a subsolution v that
$$v(x,t) \leq \frac{1}{4r^n} \iint_{E_r(x,t)} v(y,s) \frac{|x-y|^2}{(t-s)^2} \, \mathrm{d}y \mathrm{d}s$$
for all $E_r(x,t) \subset \Omega_T$.
(2) Prove that therefore $\max_{\overline{\Omega}_T} v = \max_{\Gamma_T} v$.
(3) Let $\phi \colon \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u solves the heat equation and $v := \phi(u)$. Prove v is a subsolution.
(4) Prove $v = |\nabla u|^2 + u_t^2$ is a subsolution, whenever u solves the heat equation.

Problem 19 (4 Points). *Discretization*

For $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$, let $A \in \mathbb{R}^{n \times n}$ be the band matrix

$$A = \begin{pmatrix} a & b & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & b \\ & & b & a \end{pmatrix}.$$

Show that A has the eigenvalues $\lambda_p = a + 2b\cos(p\pi/(n+1))$, $p = 1, 2, \ldots, n$.

<u>Hint:</u> Show that for $a = 0$, corresponding eigenvectors $v_p \in \mathbb{R}^n$ are given by $(v_p)_j = \sin(pj\pi/(n+1))$, $j = 1, 2, \ldots, n$.

Problem 20 (4 Points). *Stiff*

(1) Show that the discretization in the space of the heat equation leads to a stiff initial value problem $\partial_t U + AU = 0$, $U(0) = U_0$, which admits a unique solution on every time interval $[0, T]$. A linear initial value problem is called "stiff" if the corresponding matrix has only negative eigenvalues, some of which are of large magnitude.
(2) Show that the explicit Euler scheme is unstable if $\tau > \frac{h^2}{2}$ by constructing appropriate initial data.

Hand in the exercise sheets in the box marked "ITaN" on the 2nd floor at Hermann-Herder-Str. 10, next to the entrance to room 201 (CIP). The exercise sheets must be turned in by 12 pm (noon) on the specified date.