University of Freiburg
Department for Applied Mathematics
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Winter Term 2025

Introduction to the Theory and Numerics for Partial Differential Equations

Series 6

Return: November 26, 2025

Wave

In the following three problems, you may assume sufficient differentiability of all involved quantities.

Problem 21 (4 Points). *Stokes' rule*

Assume u solves the initial-value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = 0,\ \ u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Show that $v := u_t$ solves

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ v = h,\ v_t = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

This is Stokes' rule.

Problem 22 (4 Points). *Equations from Electrodynamics and Continuum Mechanics*

In the following, operators that normally act on scalar quantities are used row-wise.

(1) Assume $\mathbf{E} = \left(E^1, E^2, E^3\right)$ and $\mathbf{B} = \left(B^1, B^2, B^3\right)$ solve Maxwell's equations

$$\begin{cases} \mathbf{E}_t = \operatorname{curl} \mathbf{B}, \quad \mathbf{B}_t = -\operatorname{curl} \mathbf{E} \\ \operatorname{div} \mathbf{B} = \operatorname{div} \mathbf{E} = 0 \end{cases}$$

Show

$$\mathbf{E}_{tt} - \Delta \mathbf{E} = 0, \quad \mathbf{B}_{tt} - \Delta \mathbf{B} = 0.$$

(2) Assume that $\mathbf{u} = \left(u^1, u^2, u^3\right)$ solves the evolution equations of linear elasticity

$$\mathbf{u}_{tt} - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla (\operatorname{div} \mathbf{u}) = \mathbf{0} \quad \text{in } \mathbb{R}^3 \times (0,\infty).$$

Show $w = \operatorname{div} \mathbf{u}$ and $\mathbf{w} = \operatorname{curl} u$ each solve wave equations, but with differing speeds of propagation.

Problem 23 (4 Points). *Equipartition of energy*

Let u solve the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = g, u_t = h & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

Suppose g, h have compact support. The kinetic energy is $k(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x,t)dx$ and the potential energy is $p(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x,t)dx$. Show

(1) $k(t) + p(t)$ is constant in t,
(2) $k(t) = p(t)$ for all large enough times t.

Problem 24 (4 Points)**.** *Convolution*
Let $K \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with $\operatorname{supp}(K) \subset \mathbb{R}^n \times B_R(0)$ for some $R > 0$, i.e., with compact support in the second variable. For $f \in L^1_{loc}(\mathbb{R}^n)$, define

$$(Kf)(x) = \int_{\mathbb{R}^n} K(x,y) f(y) \, \mathrm{d}y$$

Show that $Kf \in C^\infty(\mathbb{R}^n)$ and that $\partial_x^\alpha (Kf) = \int (\partial_x^\alpha K)(x,y) f(y) \, \mathrm{d}y$.

Hand in the exercise sheets in the box marked "ITaN" on the 2nd floor at Hermann-Herder-Str. 10, next to the entrance to room 201 (CIP). The exercise sheets must be turned in by 12 pm (noon) on the specified date.