University of Freiburg
Department for Applied Mathematics
Prof. Dr. Patrick Dondl

Winter Term 2025

Introduction to the Theory and Numerics for Partial Differential Equations

Series 7

Return: December 3, 2025

Energy

Problem 25 (4 Points). *IBVP for the 1d-wave equation*

(1) Determine functions $u_n(x,t) = v_n(x)w_n(t)$, $n \in \mathbb{N}$, that satisfy the wave equation in $(0,1) \times (0,T)$ subject to homogeneous Dirichlet boundary conditions (i.e., $u_n = 0$ on $(\{0\} \cup \{1\}) \times (0,T)$).

(2) Assume that $u_0, v_0 \in C([0,1])$ satisfy

$$u_0(x) = \sum_{n\in\mathbb{N}} \alpha_n \sin(n\pi x), \quad v_0(x) = \sum_{n\in\mathbb{N}} \beta_n \sin(n\pi x)$$

with given sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$. Derive a representation formula for the solution of the wave equation $u_{tt} - u_{xx} = 0$ in $(0,1)\times(0,T)$ with homogeneous Dirichlet boundary conditions and initial conditions $u(x,0) = g(x)$ and $u_t(x,0) = h(x)$ for all $x \in [0,1]$.

Problem 26 (4 Points). *Neumann*

(1) Prove the energy conservation principle for the wave equation with homogeneous Neumann boundary conditions (i.e., normal derivatives at $\partial\Omega$ vanish.

(2) Deduce uniqueness of solutions for solutions of the wave equation with homogeneous Neumann boundary conditions.

Problem 27 (4 Points). *Energy estimates*

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $f \in C(\overline{\Omega})$.
Suppose $u \in C^2(\overline{\Omega})$ satisfies $-\Delta u = f$ in Ω, $\quad u = 0$ on $\partial\Omega$.

(1) Derive the identity $\|\nabla u\|^2_{L^2(\Omega)} = \int_\Omega f u \,\mathrm{d}x$.

(2) Derive the energy estimate $\|\nabla u\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}$.

Problem 28 (4 Points). *Euler-Lagrange-equation*

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Consider the energy functional

$$I(u) = \int_\Omega \left(\frac{1}{p} |\nabla u|^p + W(u) \right) \mathrm{d}x$$

where $p > 1$ and $W : \mathbb{R} \to \mathbb{R}$ is a smooth function (the potential). Here $|\nabla u|^p = (|\nabla u|^2)^{p/2}$.

(1) For $u \in C^2(\overline{\Omega})$ and $\varphi \in C_c^\infty(\Omega)$, compute the first variation

$$\left.\frac{d}{d\tau}\right|_{\tau=0} I(u + \tau\varphi).$$

(2) Show that if u is a critical point of I (i.e., the first variation vanishes for all test functions φ), then u satisfies the Euler-Lagrange equation

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + W'(u) = 0 \quad \text{in } \Omega.$$

(3) Specialize to the case $p = 2$ and $W(u) = \frac{1}{4}(u^2-1)^2$ (a double-well potential). Write out the resulting equation explicitly.

Hand in the exercise sheets in the box marked “ITaN” on the 2nd floor at Hermann-Herder-Str. 10, next to the entrance to room 201 (CIP). The exercise sheets must be turned in by 12 pm (noon) on the specified date.