University of Freiburg
Department for Applied Mathematics
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Winter Term 2025

Introduction to the Theory and Numerics for Partial Differential Equations
Computer Practical

Series 3
Return: November 26, 2025
Computer Heat

Project 5 (16 Points)**.** *Schemes for the Heat Equation*

(1) Implement the explicit and the implicit finite difference Euler scheme for the initial boundary value problem

$$u_t = \kappa \Delta u \quad \text{in} \quad (0,1) \times (0,T)$$

for $T = 1$ and $\kappa = \frac{1}{100}$, $u(x,0) = \sin \pi x$ for $x \in (0,1)$, and $u(0,t) = u(1,t) = 0$ for $t \in [0,T]$. Experimentally determine stability of the schemes for different values of the spatial and temporal discretization parameters h and τ.

Also consider initial conditions of the form $u(x,0) = \sin \pi n x$ for $n \in \mathbb{N}$. Compare the resulting solutions, in particular their rate of decay.

(2) One can try to improve the approximation to use a scheme that interpolates between the forward and backward Euler time-stepping scheme.

$$\begin{cases} \partial_t^- U_j^{k+1} - (1-\theta)\partial_x^+ \partial_x^- U_j^k - \theta \partial_x^+ \partial_x^- U_j^{k+1} = 0, & j = 1,2,\ldots,J-1, \\ & k = 0,1,\ldots,K-1, \\ U_0^{k+1} = U_J^{k+1} = 0, & k = 0,1,\ldots,K-1, \\ U_j^0 = u_0(x_j) & j = 0,1,2,\ldots,J. \end{cases}$$

for $\theta \in [0,1]$. For $\theta = \frac{1}{2}$, the scheme is called "Crank-Nicolson method".

Implement this θ-midpoint scheme to approximately solve the initial boundary value problem from (1). Experimentally analyze its stability.

(3) Verify that the exact solution of the problem is given by

$$u(t,x) = \sin(\pi x)\exp(-\kappa \pi^2 t).$$

For $\theta = \frac{1}{2}$, $\theta = \frac{3}{4}$, and $\theta = 1$, determine the approximation error using the at the point $(t,x) = (1, 0.5)$ for $h = \tau = \frac{2^{-j}}{10}$, for $j = 2,3,\ldots,5$. Plot the errors in a suitable format What is your conclusion?

(4) Modify your code to allow for a right-hand side f, i.e., the partial differential equation

$$\partial_t u_t - \kappa \Delta u = f,$$

and solve the initial boundary value problem in $(0,1) \times (0,T)$ with $T = 2$, $f(x) = (x - \frac{1}{2})^2$, homogeneous (i.e., zero) Dirichlet boundary conditions at $x = 0$ and $x = 1$, and the initial condition defined by $u_0(x) = 1$ if $0.45 \leq x \leq 0.55$, zero otherwise. Compare the numerical solutions for various discretization parameters and $\theta = 0, \frac{1}{2}, 1$.