Orlicz spaces. $\Omega \subset \mathbb{R}^d$; $\mathcal{M}(\Omega)$ ... measurable

I. Motivation.

To introduce a wider scale of spaces including the L^r spaces, replacing the function s^r by another one to generate the norm.

Definition 1.

Let $\Phi : [0,\infty) \to [0,\infty)$ be measurable. Denote

$$\mathcal{L}^{\Phi} := \{ u \text{ measurable on } \overset{\Omega}{\cancel{(0,\infty)}},\ \int_\Omega \Phi(|u(x)|)\,dx < \infty \}.$$

The set $\mathcal{L}^\Phi$ is called an Orlicz class, one writes

$$\rho(u, \Phi) := \int_\Omega \Phi(|u(x)|)\,dx.$$

Examples.

a) Choosing $\Phi(s) = c s^r$ gives L^r.

b) $\Phi(s) = \begin{cases} s \log^+ s, & s > 0 \\ 0, & s = 0 \end{cases}$ $\quad \log^+ s := \max\{0, \log s\}$.

c) $\Omega = (0,1)$, $\Phi(s) = e^s$. ~~Then~~ Define

$u(x) := -\frac{1}{2} \log x$, $v(x) := 2u(x) = -\log x$.

Then $u \in \mathcal{L}^\Phi$, $v \notin \mathcal{L}^\Phi$.

II. Young functions.

Definition 2. We say that Φ is a Young function if there exists a function $\varphi : [0,\infty) \to [0,\infty)$ such that

$$\Phi(s) = \int_0^s \varphi(s)\,ds, \quad s \geq 0,$$

and φ has the following properties:

(i) $\varphi(0)=0$,

(ii) $\varphi(s)>0$ if $s>0$,

(iii) φ is right-continuous on $[0,\infty)$,

(iv) φ is nonincreasing on $[0,\infty)$,

(v) $\lim\limits_{s\to\infty}\varphi(s)=\infty$.

<u>Lemma 3</u>. A Young function Φ is continuous, nonnegative, strictly increasing and convex on $[0,\infty)$. Moreover, it has the following properties:

(i) $\Phi(0)=0$, $\Phi(\infty)=\infty$,

(ii) $\lim\limits_{s\to 0+}\dfrac{\Phi(s)}{s}=0$,

(iii) $\lim\limits_{s\to\infty}\dfrac{\Phi(s)}{s}=\infty$,

(iv) $\Phi(\alpha s)\le\alpha\Phi(s)$, $\alpha\in[0,1]$, $s\ge 0$,

(v) $\Phi(\beta s)\ge\beta\Phi(s)$, $\beta>1$, $s\ge 0$.

<u>Theorem 4</u>. Let Φ be a Young function. Then $\mathcal{L}^\Phi$ is convex and $\mathcal{L}^\Phi(\Omega)\subset L^1(\Omega)$, provided $|\Omega|<\infty$.

<u>Proof</u>. From (iii) in L.3 one has a $k>0$ such that

$$|u(x)|>k \Rightarrow |u(x)|\le\Phi(|u(x)|).$$

Set $\Omega_k:=\{x\in\Omega,\ |u(x)|>k\}$. Then

$$\int_\Omega|u(x)|\,dx=\int_{\Omega_k}|u(x)|\,dx+\int_{\Omega\setminus\Omega_k}|u(x)|\,dx\le\int_{\Omega_k}\Phi(|u(x)|)\,dx+k|\Omega\setminus\Omega_k|$$
$$\le\rho(u,\Phi)+|\Omega|k<\infty. \qquad\square$$

<u>Remark 5</u>. $\exists\, u\in L^1\setminus\mathcal{L}^\Phi$.

Theorem 6. Let $|\Omega|<\infty$ and $u\in L^1(\Omega)$. Then there exists a Young function Φ such that $u\in \mathcal{L}^{\Phi}(\Omega)$.

Proof. Idea: split Ω into level sets according to u, take a sequence $\alpha_n \uparrow \infty$: $\sum \alpha_n n |\Omega_n| < \infty$, construct Φ by α_n-steps. $\square$

III. Complementary functions.

Definition 7. Let Φ be a Y.f. generated by φ. Put

$$\psi(t) := \sup_{\varphi(s)\le t} s, \quad \Psi(t) := \int_0^t \psi .$$

The function Ψ is then called the <u>complementary function</u> of Φ.

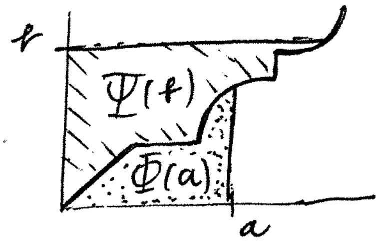
Remark. (i) Ψ is also a Young function.

(ii) If φ is continuous and strictly increasing, then $\psi = \varphi_{-1}$.

Theorem 8 (Young inequality). Let $a, b \in (0,\infty)$. Then

$$ab \le \Phi(a) + \Psi(b).$$

Proof. See the picture.



$\square$

Corollary 9. Let Φ, Ψ be a pair of complementary Young functions, $u\in\mathcal{L}^{\Phi}(\Omega)$, $v\in\mathcal{L}^{\Psi}(\Omega)$. Then

$$\int_\Omega |uv| \le \rho(u,\Phi) + \rho(v,\Psi).$$

IV. The Δ_2-condition.

Definition 10. We say that a Young function Φ satisfies the Δ_2-condition, written $\Phi \in \Delta_2$, if there exists a $k>0$ and a $T>0$ such that for all $s>T$

$$\Phi(2s) \le k\,\Phi(s).$$

Theorem 11. A Young function Φ satisfies Δ_2 if and only if $\limsup\limits_{s\to\infty} \frac{s\varphi(s)}{\Phi(s)} < \infty$.

Proof. Idea: $\frac{\varphi(s)}{\Phi(s)} < \frac{c}{s} \Rightarrow \int_s^{2s} \frac{\varphi}{\Phi} \le \int_s^{2s} \frac{c\,ds}{s} \Rightarrow \log\left(\frac{\Phi(2s)}{\Phi(s)}\right) \le c \log 2$;

$$s\varphi(s) \le \int_s^{2s} \varphi = \Phi(2s) - \Phi(s) \le (k-1)\Phi(s). \qquad \square$$

Remark. For $\Phi \in \Delta_2$ there exists a $T \ge 0$ and a $c>0$ such that $\forall s \ge T$:

$$\frac{\Phi'(s)}{\Phi(s)} \le \frac{c}{s} \Rightarrow (\log \Phi(s))' \le (\log s^c)'.$$

Thus, there are $c_0, T_0 > 0$ such that $\forall s \ge T_0$: $\Phi(s) \le c_0 s^c$.
This condition is, however, only necessary for $\Phi \in \Delta_2$.

Example 12. Define

$$\varphi(s) := \begin{cases} s, & s \in [0,1) \\ (n!), & s \in [(n-1)!, n!), \quad n \ge 2. \end{cases}$$

Then $\Phi \notin \Delta_2$, $\Psi \notin \Delta_2$ and $\Phi(s) \le s^2$, $s \ge 0$.

Theorem 13. A Young function Φ satisfies the Δ_2 condition if and only if $\exists\, k_0 > 1$ and $T_0 > 0$ such that

$$\Psi(s) \le \frac{1}{2k_0} \Psi(k_0 s) \quad \text{for all } s \ge T_0.$$

Proof. Use the fact that if $\Phi_1(s) = a\Phi(bs)$, then $\Psi_1(s) = a\Psi\left(\frac{s}{ab}\right)$. $\Psi_1(s) := \frac{1}{2k_0}\Psi(ks) \Rightarrow \Phi_1(s) = \frac{1}{2k_0}\Phi(2s)$ $\square$

$\Rightarrow \Psi(s) \le \Psi_1(s)\ \forall s \ge T_0 \Rightarrow \exists T_1 : \Phi(s) \ge \Phi_1(s)\ \forall s \ge T_1$.
Reverse analogous.

Remark. For $\Phi, \Psi \in \Delta_2$ the Young inequality can be rescaled. Moreover, one gets $\Psi(\varphi(s)) \approx \Phi(s)$.

V. Comparisons and linearity of the Orlicz classes.

Theorem 14. Let $|\Omega|<\infty$. Then $\mathcal{L}^{\Phi_2} \subset \mathcal{L}^{\Phi_1}$ if and only if there exist $c>0$ and $T>0$ such that for all $s \geq T$

$$\Phi_1(s) \leq c\Phi_2(s).$$

Proof. Sufficiency easy. For necessity, suppose that there is a sequence $\{s_n\}_{n\in\mathbb{N}}$ strictly increasing and such that $\lim_{n\to\infty} s_n = \infty$ and $\Phi_1(s_n) > 2^n \Phi_2(s_n)$. Then there exist a sequence of pairwise disjoint sets $\Omega_n \subset \Omega$ such that $|\Omega_n| = \dfrac{\Phi_2(s_1)|\Omega|}{2^n \Phi_2(s_n)}$. Set

$$u(s) := \begin{cases} s_n, & s \in \Omega_n, \\ 0, & \text{else.} \end{cases}$$

Show that $u \in \mathcal{L}^{\Phi_2} \setminus \mathcal{L}^{\Phi_1}$. □

Theorem 15. (i) Let $|\Omega|<\infty$. Then $\mathcal{L}^{\Phi}$ is a linear set if and only if $\Phi \in \Delta_2$.

(ii) Let $|\Omega| = \infty$ and $\Phi \in \Delta_2$ with $T=0$. Then $\mathcal{L}^{\Phi}$ is a linear set.

Proof. Use convexity and Th. 14. □

VI. Orlicz spaces.

Definition 16. Let Φ be a Y.f., Ψ its complementary, $u \in \mathcal{M}(\Omega)$. The Orlicz norm of u is defined as

$$\|u\|_{\Phi} := \sup\left\{ \int_{\Omega} |uv| ;\ \rho(v,\Psi) \leq 1 \right\}.$$

The set $L^{\Phi} := \{u \in \mathcal{M}(\Omega),\ \|u\|_{\Phi} < \infty\}$ is called an Orlicz space.

Remark. $\mathcal{L}^{\Phi} \subset L^{\Phi}$ since $\|u\|_{\Phi} \leq \rho(v,\Phi)+1$.

Theorem 17. L^{Φ} is a vector space and $\|u\|_{\Phi}$ is a norm on it.

Proof. Rather standard. □

Lemma 18. Let Φ be a Y.f. and $u \in L^\Phi$ such that $\|u\|_\Phi \neq 0$.
Then $$\int_\Omega \Phi\left(\frac{|u(x)|}{\|u\|_\Phi}\right) dx \le 1.$$
Proof. At first, show that
$$\int_\Omega |uv| \le \begin{cases} \|u\|_\Phi, & \text{if } \rho(v,\Psi) \le 1, \\ \|u\|_\Phi \rho(v,\Psi), & \text{if } \rho(v,\Psi) > 1, \end{cases}$$
whenever $u \in L^\Phi$.
Next, suppose $u \in L^\Phi$ is bounded and with compact support. The functions $\Phi\left(\frac{|u|}{\|u\|_\Phi}\right)$ and $\Psi\left(\varphi\left(\frac{|u|}{\|u\|_\Phi}\right)\right)$ belong to $L^1(\Omega)$, thus Corollary 9 gives
$$\int_\Omega \frac{|uv|}{\|u\|_\Phi} = \rho\left(\frac{u}{\|u\|_\Phi}, \Phi\right) + \rho\left(\varphi\left(\frac{|u|}{\|u\|_\Phi}\right), \Psi\right).$$
Use this to show the conclusion.
For a general $u \in L^\Phi$, use an appropriate truncation on a telescopic sequence of sets. □

Theorem 19. Let Φ satisfy the Δ_2-condition. In case that $|\Omega| = \infty$, let the T from the condition equal 0. Then $L^\Phi = \mathcal{L}^\Phi$.
Proof. We know that $\mathcal{L}^\Phi \subset L^\Phi$. To prove the opposite inclusion, set $u \in L^\Phi$, $u \neq 0$. Then $w := \frac{u}{\|u\|_\Phi} \in \mathcal{L}^\Phi$ thanks to L. 18. Since $\Phi \in \Delta_2$, by Th. 15 $\mathcal{L}^\Phi$ is a linear set, thus $u = \|u\|_\Phi w \in \mathcal{L}^\Phi$. □

Remark. L^Φ is the linear hull of $\mathcal{L}^\Phi$.

Theorem 20. Let Φ, Ψ be complementary Y.f.'s. Then
$$\int_\Omega |uv| \le \|u\|_\Phi \|v\|_\Psi \quad \text{for any } u \in L^\Phi, v \in L^\Psi.$$
Proof. Suppose $v \neq 0$. Then L. 18 gives $\int_\Omega \Psi\left(\frac{v}{\|v\|_\Psi}\right) \le 1$, thus
$$\int_\Omega |uv| = \|v\|_\Psi \int_\Omega \left|u \frac{v}{\|v\|_\Psi}\right| \le \|u\|_\Phi \|v\|_\Psi.$$ □

VII. The Luxemburg norm.

Definition 21. Let Φ be a Y.f. and $u \in \mathcal{M}(\Omega)$. Then the Luxemburg norm of u is defined by

$$|||u|||_\Phi := \inf\left\{\lambda > 0, \int_\Omega \Phi\left(\frac{u(x)}{\lambda}\right) dx \leq 1\right\}.$$

Remark. $|||\cdot|||$ is indeed a norm.

VIII. Completeness of Orlicz spaces.

Theorem 22. L^Φ is a Banach space.

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in L^Φ, then for every $\varepsilon > 0$ $\exists n_0$ $\forall n, m \geq n_0$: $\int_\Omega \Phi\left(\frac{|u_n - u_m|}{\varepsilon}\right) \leq 1$.

From L.3(iii) $\exists k > 0$: $|f(x)| > k \Rightarrow \Phi(|f(x)|) \geq |f(x)|$ $\forall f \in \mathcal{M}(\Omega)$.

Let $\Omega = \bigcup_N \Omega_n$, $|\Omega|_n < \infty$ $\forall n$, $\Omega_m \cap \Omega_n = \emptyset$ if $m \neq n$.

Let $m, n \geq n_0$. Define

$$b_{m,n} := (u_m - u_n)\chi_{\{x\in\Omega_1,\, |u_m(x) - u_n(x)| > k\varepsilon\}},$$

$$s_{m,n} := u_m - u_n - b_{m,n}.$$

Then

$$\int_{\Omega_1} |u_m - u_n| = \int_{\Omega_1} |b_{m,n}| + \int_{\Omega_1} |s_{m,n}| \leq \int_{\Omega_1} \Phi\left(\frac{|u_m - u_n|}{\varepsilon}\right)\cdot\varepsilon + \int_{\Omega_1} k\varepsilon \leq \varepsilon(1 + k|\Omega_1|).$$

Thus, $u_n|_{\Omega_1}$ is a Cauchy sequence on $L^1(\Omega_1)$ and one can choose a subsequence $\{u_n^1\} \subset \{u_n\}$ such that $u_n^1 \to u^1 \in L^1(\Omega_1)$ a.e. Similarly, there exists a subsequence $\{u_n^2\} \subset \{u_n^1\}$ such that $u_n^2 \to u^2 \in L^1(\Omega_2)$ and so on. Sequence $\{u_n^n\}$ converges a.e. to $u = \sum_n u^n \chi_{\Omega_n}$. Fatou lemma gives

$$\int_\Omega \Phi\left(\frac{|u - u_n|}{\varepsilon}\right) \leq \liminf_{m\to\infty} \int_\Omega \Phi\left(\frac{|u_m - u_n|}{\varepsilon}\right) \leq 1,$$

thus $u \in L^\Phi(\Omega)$ and $u_n^n \to u$ in $|||\cdot|||_\Phi$. $\square$

IX. Convergence in Orlicz spaces.

Theorem 23. Let Φ be a Young function satisfying the Δ_2-condition, with $T=0$ if $|\Omega|=\infty$. Then

$$\int_\Omega \Phi(|u_n-u|) \xrightarrow{n\to\infty} 0 \quad \text{if and only if} \quad |||u_n-u|||_\Phi \xrightarrow{n\to\infty} 0.$$

Proof. "$\Leftarrow$": Let $|||w|||_\Phi \le 1$. Then $\int_\Omega \Phi(|w|) \le |||w|||_\Phi \int_\Omega \Phi\left(\frac{|w|}{|||w|||_\Phi}\right) \le |||w|||_\Phi$.

"$\Rightarrow$": It is enough to prove that if $\int_\Omega \Phi(|w|) \le k^{-m}$, $m\in\mathbb{N}$, then $|||w|||_\Phi \le \frac{c}{2^m}$ with $c = \begin{cases} 1, & |\Omega|=\infty \\ 2\Phi(T)|\Omega|+2; & |\Omega|<\infty. \end{cases}$

a) $T=0$:

$$\int_\Omega \Phi\left(\frac{|w|}{2^{-m}}\right) \le k^m \int_\Omega \Phi(|w|) \le 1 \implies |||w|||_\Phi \le 2^{-m}.$$

b) $|\Omega|<\infty, T>0$: Let $\Omega_T := \{x\in\Omega,\ w(x) \le 2^{-m}T\}$

$$\int_\Omega \Phi\left(\frac{2^m|w|}{2\Phi(T)|\Omega|+2}\right) \le \int_{\Omega_T} \Phi\left(\frac{2^m|w|}{2\Phi(T)|\Omega|+2}\right) + \int_{\Omega\setminus\Omega_T} \Phi\left(\frac{2^m|w|}{2}\right)$$

$$\le \int_{\Omega_T} \Phi\left(\frac{T}{2\Phi(T)|\Omega|+2}\right) + \frac{1}{2}k^m \int_\Omega \Phi(|w|)$$

$$\le \frac{1}{2}\,\frac{\Phi(T)|\Omega|}{\Phi(T)|\Omega|+1} + \frac{1}{2} \le 1 \implies |||w|||_\Phi \le \frac{2\Phi(T)|\Omega|+2}{2^m}.$$

$\square$

X. Separability.

Theorem 24. Let Φ be a Young function satisfying the Δ_2-condition, with $T=0$ if $|\Omega|=\infty$. Then L^Φ is separable.

Proof. Let $u \in L^\Phi$. Then $u = u^+ - u^-$, where $u^+, u^- \in L^\Phi$, $u^+, u^- \geq 0$. There exists a sequence of $\mathbb{Q}$-valued simple functions based on $\mathbb{Q}$-cubes converging a.e. to u^+ (and u^-) from below. Denote these $\{u_n^+\}$ ($\{u_n^-\}$). Since $0 \leq u_n^+ \leq u^+ \in L^\Phi$, the dominated convergence theorem yields

$$\int_\Omega \Phi(|u^+ - u_n^+|) \xrightarrow{n\to\infty} 0.$$

By Th. 23, this coincides with $|||u^+ - u_n^+|||_\Phi \to 0$. $\square$

Remark. Δ_2 is also a necessary condition.

XI. The maximal closed subspace of $\mathcal{L}^\Phi$.

Definition 25. Let Φ be a Young function. Define

$$E_\Phi := \{u \in \mathcal{L}^\Phi;\ ku \in \mathcal{L}^\Phi\ \forall k>0\}.$$

Remark. (i) The following inclusions are obviously true:

$$E^\Phi \subset \mathcal{L}^\Phi \subset L^\Phi.$$

For a general Y.f. Φ both may be strict.

(ii) E^Φ is the maximal closed linear space contained in $\mathcal{L}^\Phi$, as it will be shown.

Theorem 26. Let Φ be a Y.f. satisfying the Δ_2-condition, with $T=0$ in case $|\Omega|=\infty$. Then

$$E^\Phi = \mathcal{L}^\Phi = L^\Phi.$$

Proof. Let $u \in L^\Phi$. Then $\frac{u}{|||u|||_\Phi} \in \mathcal{L}^\Phi$. The Δ_2-condition yields that for any $k>0$ there exists $C>0$ such that $\Phi(\overset{|||u|||_\Phi}{k}s) \leq C\Phi\left(\frac{s}{|||u|||_\Phi}\right)$ whenever $s \geq T$. WLOG $T=0$. Then $\int_\Omega \Phi(k|u|) \leq C\int_\Omega \Phi\left(\frac{|u|}{|||u|||_\Phi}\right) \leq C$. $\square$

XII. Dual to an Orlicz space.

Theorem 27. Let Φ, Ψ be a pair of complementary Young functions. Let $v \in L^\Psi$ be fixed. Then the formula

$$F(u) := \int_\Omega uv, \qquad u \in L^\Phi,$$

defines a continuous linear functional on L^Φ, whose norm satisfies $\|F\| \approx |||v|||_\Psi$.

Proof. Requires equivalence of the Orlicz and Luxemburg norms, and the Hölder inequality. $\square$

Theorem 28. Let F be a continuous linear functional on E^Φ. Then there exists a unique $v \in L^\Psi$ such that

$$F(u) = \int_\Omega u(x)v(x)dx \quad \text{for all } u \in E^\Phi.$$

Proof. Only existence: Let $E \subset \Omega$ be measurable. Then

$$F(\chi_E) \le \|F\| \, |||\chi_E|||_\Phi = \|F\| \cdot \frac{1}{\Phi_{-1}(1/|E|)}$$

Define a measure ν by $\nu(E) := F(\chi_E)$, $E \in \mathfrak{M}(\Omega)$. By the estimate above, one has $\lim_{|E| \to 0} \nu(E) = 0$, hence ν is absolutely continuous with respect to the Lebesgue measure. Hence, by Radon-Nikodym, there exists a function $v \in \mathcal{M}(\Omega)$ such that $\nu(E) = \int_E v(x)dx$ for all $E \in \mathcal{M}(\Omega)$. Thus, for a simple function u one has $F(u) = \int_\Omega uv$. Let $u \in E^\Phi$. Then $\exists \{u_n\}$ sequence of simple functions, $|||u_n|||_\Phi \le |||u|||_\Phi \; \forall n$, $u_n \to u$ a.e. By Fatou,

$$\left|\int_\Omega uv\right| \le \sup_{n \in \mathbb{N}} \int_\Omega |u_n v| = \sup_{n \in \mathbb{N}} F(|u_n| \operatorname{sgn} v) \le \|F\| \sup_{n \in \mathbb{N}} |||u_n|||_\Phi \le \|F\| \, |||u|||_\Phi$$

Thus, $\|v\|_\Psi = \sup_{|||u|||_\Phi \le 1} \int_\Omega uv \le \|F\|$, i.e. $v \in \Psi$.

Simple functions are dense in E^Φ, therefore $F(u) = \int_\Omega uv$ for all $u \in E^\Phi$. $\square$