Abteilung für
Angewandte Mathematik

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# Numerik für Differentialgleichungen – SoSe 2023

## Sheet 2

Ausgabe: 10.05.2023, 12:00 Uhr — Abgabe: 17.05.2023, 12:00 Uhr

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**Homepage to the lecture:**

`https://aam.uni-freiburg.de/agsa/lehre/ss23/ndgln`

**Exercise 1** (4 points)**.** Construct an autonomous ODE $y'(t) = f(y(t))$ for which there exists a solution $y \in C^1([0,T])$ but $y \notin C^2([0,T])$.

**Exercise 2** (4 points)**.** Construct infinitely many solutions for the initial value problem $y' = y^{1/3}, y(0) = 0$. Is this a contradiction to the *Picard-Lindelöf theorem*?

**Exercise 3** (2 + 1 + 1 points)**.** Let $A : [0,T] \longrightarrow \mathbb{R}^{n\times n}$ be continuous and consider the system $y'(t) = A(t)y(t)$.

(i) Modify the proof of the Picard-Lindelöf theorem to show that there exists a unique solution with initial condition $y(0) = y_0$ for $y_0 \in \mathbb{R}^n$.

(ii) Show that the set $L$ of solutions for $y'(t) = A(t)y(t)$ forms a vector space.

(iii) Consider the map $E_0 : L \longrightarrow \mathbb{R}^n, E_0(y) := y_0$. Use $E_0$ to prove that $\dim(L) = n$.

**Exercise 4** (1 + 1 + 2 points)**.** Let $y \in C^2(\mathbb{R})$ and $\tau > 0$. For $k \in \mathbb{N}$ define $t_k = k\tau$ and set $y^k = y(t_k)$. Consider the following *difference schemes*

$$d_t^- y^k := \frac{y^k - y^{k-1}}{\tau}, \qquad d_t^+ y^k := \frac{y^{k+1} - y^k}{\tau}, \qquad \hat{d}_t y^k := \frac{y^{k+1} - y^{k-1}}{2\tau},$$

for $k = 1, 2, \dots, K-1$.

(i) Prove that

$$\left| d_t^{\pm} y^k - y'(t_k) \right| \le \frac{\tau}{2} \sup_{t \in t_k + [0, \pm\tau]} |y''(t)| \,.$$

(ii) Can one obtain a better estimate for $\hat{d}_t y^k$?

(iii) Consider the initial value problem $y' = y, y(0) = 1$. Construct an implicit Euler method that leads to a contradiction.