Abteilung für
Angewandte Mathematik

JProf. Dr. Diyora Salimova
M.Sc. Mario Keller

Numerik für Differentialgleichungen – SoSe 2023

Sheet 3

Ausgabe: 24.05.2023, 12:00 Uhr — Abgabe: 07.06.2023, 12:00 Uhr

---

Homepage to the lecture:

https://aam.uni-freiburg.de/agsa/lehre/ss23/ndgln

Exercise 1 (4 points)**.** Let $z : [t_k, t_{k+1}] \longrightarrow \mathbb{R}$ be the solution to the initial value problem $z' = f(t, z), z(t_k) = z_k$. Consider the numerical method $z_{k+1} = z_k + \tau\Phi(t_k, z_k, z_{k+1}, \tau)$ and the two concistency functions

$$C(t_k, z_k, \tau) = \frac{z(t_{k+1}) - z_k}{\tau} - \Phi(t_k, z_k, z_{k+1}, \tau),$$
$$\tilde{C}(t_k, z_k, \tau) = \frac{z(t_{k+1}) - z_k}{\tau} - \Phi(t_k, z_k, z(t_{k+1}), \tau),$$

and let Φ be Lipschitz-continuous in the third argument with Lipschitz constant L. Prove that for $\tau \leq 1/(2L)$ it holds that

$$K^{-1}|\tilde{C}(t_k, z_k, \tau)| \leq |C(t_k, z_k, \tau)| \leq K|\tilde{C}(t_k, z_k, \tau)|$$

and compute K explicitly. How is this result related to the statement in Remark 21.3 on page 180 in the book *Numerik 3x9*?

Exercise 2 (2 + 4 + 2 points)**.**

(i) Consider the *implicit midpoint method* with increment function

$$\Phi(t_k, y_k, y_{k+1}, \tau) = f(t_k + \tau/2, (y_k + y_{k+1})/2),$$

see example 21.1 on page 179 in the book *Numerik 3x9*. First, provide an intuitive derivation of Φ. Then, modify your derivation to obtain an explicit version of Φ.

(ii) Prove that the *implicit midpoint method* has concistency order $p = 2$.

(iii) Prove that the implicit Euler method $y_{k+1} = y_k + \tau f(t_{k+1}, y_{k+1})$ has concistency order $p = 1$.

Exercise 3 (4 points)**.** Let $(y_i)_{i=0,\ldots,K}$ be a sequence of non-negative numbers and $\alpha, \beta \geq 0$ such that for $i = 0, 1, \ldots, K$

$$y_i \leq \alpha + \sum_{k=0}^{i-1} \beta y_k.$$

Prove that $y_i \leq \alpha(1 + \beta)^i \leq \alpha e^{K\beta}$ for $i = 0, 1, \ldots, K$. From this result, conclude the discrete version of *Gronwall's lemma*, cf. Lemma 21.1 on page 181 in the book *Numerik 3x9*.