Sheet 5.

You can use the below two lemmas without a proof.

Lemma 0.1. *Let* $g_n\colon \mathbb{R} \to [0,1]$*,* $n \in \mathbb{N}$*, be the functions which satisfy for all* $n \in \mathbb{N}$*,* $x \in \mathbb{R}$ *that*

$$g_1(x) = \begin{cases} 2x & : x \in [0, \frac{1}{2}) \\ 2 - 2x & : x \in [\frac{1}{2}, 1] \\ 0 & : x \in \mathbb{R}\backslash[0,1] \end{cases} \tag{1}$$

and $g_{n+1}(x) = g_1(g_n(x))$*. Then*

(i) it holds for all $n \in \mathbb{N}$*,* $k \in \{0, 1, \ldots, 2^{n-1} - 1\}$*,* $x \in \left[\frac{k}{2^{n-1}}, \frac{k+1}{2^{n-1}}\right]$ *that*

$$g_n(x) = \begin{cases} 2^n(x - \frac{2k}{2^n}) & : x \in \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right] \\ 2^n(\frac{2k+2}{2^n} - x) & : x \in \left[\frac{2k+1}{2^n}, \frac{2k+2}{2^n}\right] \end{cases} \tag{2}$$

and

(ii) it holds for all $n \in \mathbb{N}$*,* $x \in \mathbb{R}\backslash[0,1]$ *that* $g_n(x) = 0$*.*

Lemma 0.2. *Let* $f_n\colon [0,1] \to [0,1]$*,* $n \in \mathbb{N}_0$*, be the functions which satisfy for all* $n \in \mathbb{N}_0$*,* $k \in \{0, 1, \ldots, 2^n - 1\}$*,* $x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$ *that* $f_n(1) = 1$ *and*

$$f_n(x) = \left[\tfrac{2k+1}{2^n}\right] x - \tfrac{(k^2+k)}{2^{2n}}. \tag{3}$$

Then it holds for all $n \in \mathbb{N}_0$*,* $x \in [0,1]$ *that*

$$f_n(x) = x - \left[\sum_{m=1}^{n} \left(2^{-2m} g_m(x)\right)\right] \qquad \text{and} \qquad \left|x^2 - f_n(x)\right| \leqslant 2^{-2n-2}. \tag{4}$$

Problem 1. Let a be the rectifier activation function, let $(A_k, b_k) \in \mathbb{R}^{4\times 4} \times \mathbb{R}^4$, $k \in \mathbb{N} \cap [2, \infty)$, satisfy for all $k \in \mathbb{N} \cap [2, \infty)$ that

$$A_k = \begin{pmatrix} 2 & -4 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ (-2)^{3-2k} & 2^{4-2k} & (-2)^{3-2k} & 1 \end{pmatrix} \quad \text{and} \quad b_k = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ -1 \\ 0 \end{pmatrix}, \tag{5}$$

let $\mathbb{A}_k \in \mathbb{R}^{1\times 4} \times \mathbb{R}$, $k \in \mathbb{N} \cap [2, \infty)$, satisfy for all $k \in \mathbb{N} \cap [2, \infty)$ that

$$\mathbb{A}_k = \left(\left((-2)^{3-2k} \quad 2^{4-2k} \quad (-2)^{3-2k} \quad 1\right), 0\right), \tag{6}$$

let $\phi_k \in \mathbf{N}$, $k \in \mathbb{N} \cap [2, \infty)$, satisfy that

$$\phi_2 = \left(\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{2} \\ -1 \\ 0 \end{pmatrix} \right), \mathbb{A}_2 \right) \tag{7}$$

and

$$\forall\, k \in \mathbb{N} \cap [3, \infty)\colon \phi_k = \left(\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{2} \\ -1 \\ 0 \end{pmatrix} \right), (A_2, b_2), \dots, (A_{k-1}, b_{k-1}), \mathbb{A}_k \right), \tag{8}$$

and let $r_k = (r_{k,1}, r_{k,2}, r_{k,3}, r_{k,4})\colon \mathbb{R} \to \mathbb{R}^4$, $k \in \mathbb{N}$, be the functions which satisfy for all $x \in \mathbb{R}$, $k \in \mathbb{N}$ that

$$r_1(x) = (r_{1,1}(x), r_{1,2}(x), r_{1,3}(x), r_{1,4}(x)) = \mathfrak{M}_{a,4}\big(x, x - \tfrac{1}{2}, x - 1, x\big) \tag{9}$$

and

$$r_{k+1}(x) = (r_{k+1,1}(x), r_{k+1,2}(x), r_{k+1,3}(x), r_{k+1,4}(x)) = \mathfrak{M}_{a,4}\big(A_{k+1} r_k(x) + b_{k+1}\big). \tag{10}$$

Prove by induction that for all $k \in \mathbb{N}$ it holds that

$$\big(\forall\, x \in \mathbb{R}\colon\; 2r_{k,1}(x) - 4r_{k,2}(x) + 2r_{k,3}(x) = g_k(x)\big) \tag{11}$$

and

$$\left(\forall\, x \in \mathbb{R}\colon\; r_{k,4}(x) = \begin{cases} f_{k-1}(x) & : x \in [0,1] \\ \max\{x, 0\} & : x \in \mathbb{R}\backslash[0,1] \end{cases}\right). \tag{12}$$

Problem 2. Show that for all $m \in \mathbb{N} \cap [2, \infty)$, $x \in [0, 1]$ it holds that

$$(\mathcal{R}_a(\phi_m))(x) = f_{m-1}(x), \tag{13}$$

and applying to this Lemma 0.2 that for all $m \in \mathbb{N} \cap [2, \infty)$, $x \in [0, 1]$ it holds that

$$\big|x^2 - (\mathcal{R}_a(\phi_m))(x)\big| \leqslant 2^{-2m}. \tag{14}$$

Problem 3. Let $\varepsilon > 0$. Show that there exists $\Phi \in \mathbf{N}$ such that for all $x \in \mathbb{R}\backslash[0, 1]$ it holds that $(\mathcal{R}_a(\Phi))(x) = a(x)$ and for all $x \in [0, 1]$ we have that $|x^2 - (\mathcal{R}_a(\Phi))(x)| \leqslant \varepsilon$.

Problem 4. Prove that $\mathcal{P}(\Phi) \leqslant \max\{10 \log_2(\varepsilon^{-1}) - 7, 13\}$ and $\mathcal{L}(\Phi) \leqslant \max\{\frac{1}{2} \log_2(\varepsilon^{-1}) + 1, 2\}$