FEM

1. Assume we have ↓ variational problem (weak for-m)
Find $u \in H_0^1((a,b))$ s.t.

$$\underbrace{\int_a^b u'(x)\, v'(x)\, dx}_{=:a(u,v)} = \underbrace{\int_a^b f(v)\cdot v(x)\, dx}_{=:b(v)} \quad \forall v \in H_0^1((a,b))$$

The strong form is then

$$-u'' = f \quad \text{in } (a,b)$$
$$u(a) = u(b) = 0$$

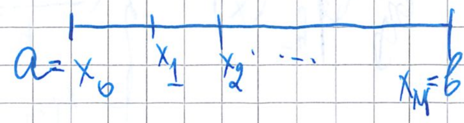
No Neumann BC

Idea underlying FEM: approximate u by means of continuous, piecewise polynomial f-ns
→ When talking about piecewise polynomials one has to fix a partitioning on the domain (a,b) first.
→ We equip $\Omega = [a,b]$ with $M+1$ nodes:

$$\mathcal{V}(M) := \{a = x_0 < x_1 < \dots < x_{M-1} < x_M = b\}$$

Then we have mesh/grid:

$$\mathcal{M} := \{(x_{j-1}, x_j) : 1 \le j \le M\}$$



A special case: equidistant mesh:

$$x_j = a + jh, \quad h = \frac{b-a}{M}$$

Meshwidth:

$$h_{\mathcal{M}} := \max |x_j - x_{j-1}| = \max h_j$$

Recall Prop 2.10 (Continuous piecewise diff. f-n).
Assume that $(\Omega_j)_{j=1,\dots,J}$ is an open partition of Ω, i.e,
$\overline{\Omega} = \overline{\Omega}_1 \cup \dots \cup \overline{\Omega}_J$, Ω_j is open for $j=1,\dots,J$ and $\Omega_j \cap \Omega_\ell = \emptyset, j \ne \ell$.
Let $u \in C(\overline{\Omega})$ be s.t. $u|_{\Omega_j} \in C^1(\overline{\Omega}_j)$ for $j = 1,2,\dots,J$ ⇒
u is weakly diff., i.e. $u \in H^1(\Omega)$

The simplest space of contin. $\mathcal{M}$-piecewise polynomial f-ns in $H_0^1(\Omega)$:

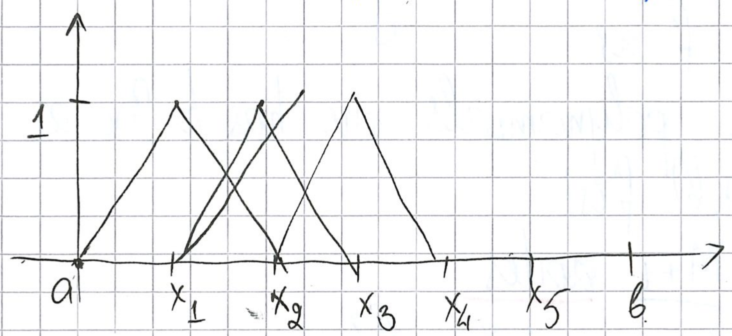
$$S^{0}_{1,0}(\mathcal{M}) := \{ v \in C^0([a,b]),\ v|_{[x_{i-1}, x_i]} \text{ linear},\ i=1,\dots,M, \quad v(a)=v(b)=0 \} = V_h$$

C^0 — 0 BC — 1st degree

Now we need do specify a basis for $S^0_{1,0}(M)$

- We use 1D „tent (hat) f-ns":

$$\{\varphi_h^1, \dots, \varphi_h^{M-1}\}$$



$$\varphi_h^j(x) = \begin{cases} \frac{(x - x_{j-1})}{h_j}, & x_{j-1} \le x \le x_j \\ \frac{x_{j+1} - x}{h_{j+1}}, & x_j \le x \le x_{j+1} \\ 0, & \text{elsewhere.} \end{cases} \Rightarrow$$

$$\varphi_h^j(x_i) = \delta_{ij} := \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i \ne j \end{cases}$$

$$\operatorname{Supp}(\varphi_h^j) = [x_{j-1}, x_{j+1}], \quad j=1,\dots,M-1$$

We compute weak derivatives:

$$\frac{d\varphi_h^j}{dx}(x) = \begin{cases} \frac{1}{h_j}, & \text{if } x_{j-1} \le x \le x_j, \\ -\frac{1}{h_{j+1}}, & \text{if } x_j < x \le x_{j+1} \\ 0, & \text{elsewhere.} \end{cases}$$

Obviously, derivatives are piecewise constant and discontinuous

Now we expand

$$u_h = \mu_1 \varphi_h^1 + \dots + \mu_N \varphi_h^N \;, \quad \text{test weak formulation}$$

$$N = M-1$$

with all tent f-ns $\Rightarrow$ We get:

seek $\mu_\ell \in \mathbb{R}$, $\ell = 1, \dots, N$:

$$\underbrace{\int_a^b \sum_{\ell=1}^{N} \mu_\ell \frac{d\varphi_h^\ell}{dx}(x) \frac{d\varphi_h^k}{dx}(x)\, dx}_{=a(u_h,\, \varphi_h^k)} = \underbrace{\int_a^b f(x)\, \varphi_h^k(x)\, dx}_{=b(\varphi_h^k)} \quad k=1,\dots,N \quad \Rightarrow$$

$$\sum_{\ell=1}^{N} \underbrace{\left(\int_a^b \frac{d\varphi_h^\ell}{dx}(x) \frac{d\varphi_h^k}{dx}(x) \right)}_{A_{k\ell}} \mu_\ell = \underbrace{\int_a^b f(x)\, \varphi_h^k(x)\, dx}_{B_k} \;, \quad k = 1, \dots, N \quad \Rightarrow$$

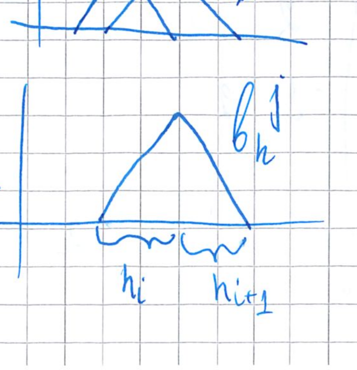
We get linear system: $A\mu = B$

A - Galerkin matrix, B - RHS vector.

Compute entries of A:

$$\frac{d\varphi^j}{dx}(x) = \begin{cases} \frac{1}{h_j}, & x_{j-1} \le x \le x_j \\ -\frac{1}{h_{j+1}}, & x_j < x \le x_{j+1} \\ 0, & \text{elsewhere} \end{cases} \quad \Rightarrow$$

$$\int_a^b \frac{d\varphi^j}{dx}(x) \frac{d\varphi^i}{dx}(x)\, dx = \begin{cases} 0, & \text{if } |i-j| \ge 2 \\ -\frac{1}{h_{i+1}}, & \text{if } j = i+1 \\ -\frac{1}{h_i}, & j = i-1 \\ \frac{1}{h_i} + \frac{1}{h_{i+1}}, & 1 \le i = j \le M-1 \end{cases}$$



$$\Rightarrow A = \begin{bmatrix} \frac{1}{h_1}+\frac{1}{h_2} & -\frac{1}{h_2} & 0 & \cdots & \\ -\frac{1}{h_2} & \frac{1}{h_2}+\frac{1}{h_3} & & & \\ 0 & & & & -\frac{1}{h_{M-1}} \\ & & -\frac{1}{h_{M-1}} & \frac{1}{h_{M-1}} & +\frac{1}{h_M} \\ 0 & & & & \end{bmatrix} \in \mathbb{R}^{N,N}$$

A is symmetric, strictly diagonally dominant, tridiagonal $\rightarrow$ invertible

Special case: equidistant mesh: $h_i = h \Rightarrow$

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & \ddots & 2 \end{bmatrix}$$

How to calculate RHS? numerical quadrature, e.g. trapezoidal rule (with given mesh $\mathcal{M}$):

$$\int_a^b \varphi(t)\,dt \approx \sum_{l=1}^{M} \frac{1}{2} h_l \left(\varphi(x_{l-1}) + \varphi(x_l)\right),$$

In our case:

x_{k-1} x_k x_{k+1}

$$\int_a^b f(x)\varphi^k(x)\,dx \approx \frac{1}{2}h_k\left(f(x_{k-1})\varphi^k(x_{k-1}) + f(x_k)\varphi^k(x_k)\right) + \frac{1}{2}h_{k+1}\left(f(x_k)\varphi^k(x_k) + f(x_{k+1})\varphi^k(x_{k+1})\right)$$

$$= \frac{1}{2}(h_k + h_{k+1})\, f(x_k)$$

$\Rightarrow$ Equidistant mesh: Solve

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & \\ & & & 2 \end{bmatrix}\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_N \end{bmatrix} = h\begin{bmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix}$$

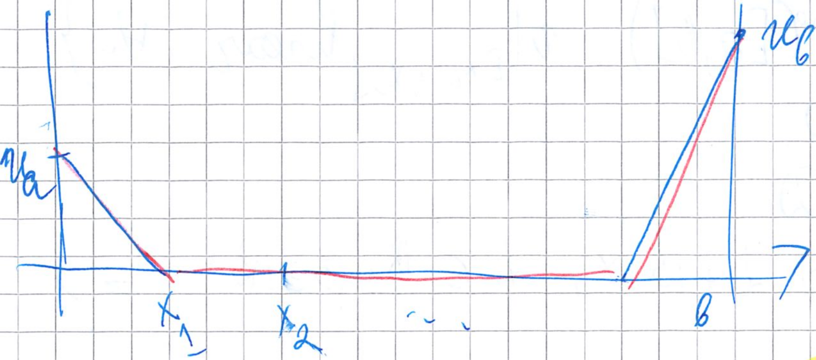
Remark: Offset f-n for finite element Galerkin d-n.

Assume we have

$$u(a) = u_a \quad , \quad u(b) = u_b .$$

In this case we can use piecewise linear offset f-n

$$u_{0,h}(x) = \begin{cases} u_a\left(1 - \frac{x-a}{h_1}\right), & \text{if } a \le x \le x_1 \\ u_b\left(1 - \frac{b-x}{h_M}\right), & \text{if } x_{M-1} \le x \le b \\ 0 & \text{, elsewhere.} \end{cases}$$



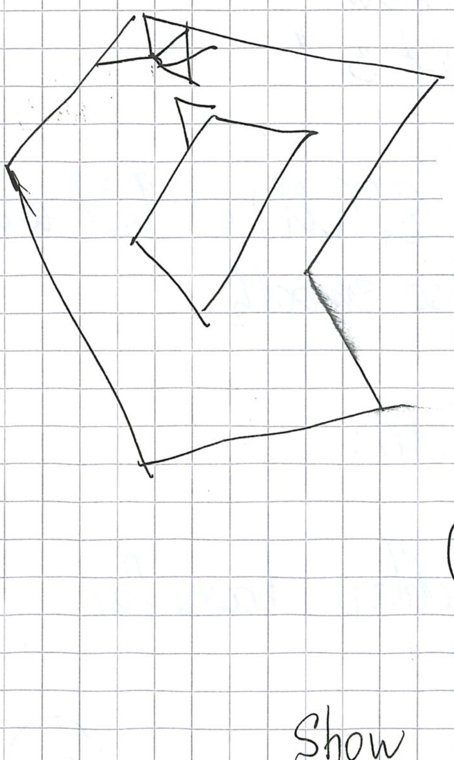
$u = \tilde{u} + u_{0,h}(x)$

$\tilde{u}$ is sol-n with homogeneous Dirichlet BC.

Triangular FEM in 2D:

For the sake of simplicity, assume Ω is a polygon

Q: What is the 2D counterpart of the 1D mesh/grid $\mathcal{M}$?



Many more possibilities in higher dim-n!

We opt for triangulations

A triangulation ① $\mathcal{M} = \{K_i\}_{i=1}^M$, $M \in \mathbb{N}$, K_i - open triangle

② disjoint interiors: $i \neq j \Rightarrow K_i \cap K_j = \emptyset$

③ partition property

$$\bigcup_{i=1}^M \overline{K_i} = \overline{\Omega}.$$

④ intersection $\overline{K_i} \cap \overline{K_j}$, $i \neq j$:

- either $\emptyset$
- or an edge of both triangles
- or a vertex of both triangles

Show → „ Pict 1 & 2.

We call:
vertices of Δ's : nodes of mesh $(= \mathcal{V}(\mathcal{M}))$
Δ's = cells or elements of mesh $(= \mathcal{M})$
$\mathcal{M} = \{K_1, \dots, K_M\}$, $\mathcal{V}(\mathcal{M}) = \{x^1, \dots, x^N\}$, $N \in \mathbb{N}$

Linear Finite element space

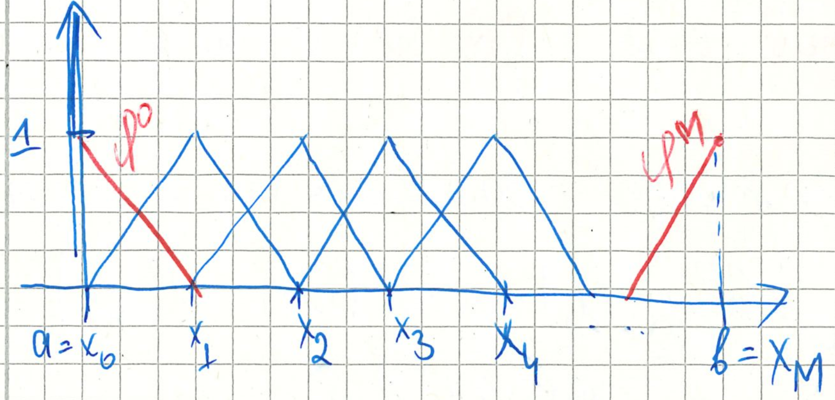
Recall in 1D we used the space
$$S^0_{1,0}(\mathcal{M}) := \{v \in C^0([a,b]) : v|_{[x_{i-1}, x_i]} \text{ linear}, v(a) = v(b) = 0\}$$
If no Dirichlet BC we can use
$$S^0_1(\mathcal{M}) := \{v \in C^0([a,b]) : v|_{[x_{i-1}, x_i]} \text{ linear } \forall i\}$$
$\Rightarrow$ Generalize to $d = 2$:
Linear f-n in 2D
$$x \in \mathbb{R}^2 \longmapsto \alpha + \beta \cdot x, \quad \alpha \in \mathbb{R}, \beta \in \mathbb{R}^2 \Rightarrow$$
in 2D:
$$V_{0,h} = S^0_1(\mathcal{M}) := \{v \in C^0(\overline{\Omega}) : \forall K \in \mathcal{M} : v|_K(x) = \alpha_K + \beta_K \cdot x, \ \alpha_K \in \mathbb{R}, \beta_K \in \mathbb{R}^2, x \in K\}$$

Notation
S^0_1: $0 \leftarrow$ cont. f-n, $C^0(\Omega)$; $1 \leftarrow$ locally, 1st degree polynomials

$\subset H^1(\Omega)$ "Show $\longrightarrow$ pict 3"

Similarly, as in 1D case
$S^0_1(\mathcal{M}) \subset H^1(\Omega)$, because $S^0_1(\mathcal{M}) \subset C^0(\overline{\Omega})$ and piecewise smooth

Next: Basis fns: In 1D:
Recall the basis ("tent f-ns") for $S^0_{1,0}(\mathcal{M})$:
$$\{\varphi^1_h, \dots, \varphi^{M-1}_h\}$$
By adding 2 more "half-tent" f-n we obtain basis for $S^0_1(\mathcal{M})$:
$$\{\varphi^0_h, \dots, \varphi^M_h\}$$



We have „nodal property":

$$\varphi_h^j(x_i) = \delta_{ij} := \begin{cases} 1 & , \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

$\rightsquigarrow$ use it in 2D

Idea: Define the basis f-n φ_h^x, $x \in V(M)$ by „nodal conditions"

$$\varphi_h^x(y) = \begin{cases} 1 & , \text{if } y = x \\ 0 & , \text{if } y \in V(M) \setminus \{x\} \end{cases}$$ $\to$ „show pict 4."

Is this possible? Yes, because:

(i): There is exactly one plane through three non-collinear pts in $\mathbb{R}^3$ ($\Rightarrow$ Sheet 10)

(ii) The graph of a linear f-n $\mathbb{R}^2 \mapsto \mathbb{R}$ is a plane.

$v_h \in S_1^0(M)$ is uniquely determined by $\{v_h(x), x \in V(M)\}$

$$\Rightarrow \dim S_1^0(M) = \# V(M) = N$$

If $V(M) = \{x_1, \ldots, x_N\}$, we define the nodal basis

$$\{\varphi_h^1, \ldots, \varphi_N^1\} \text{ by}$$

$\varphi_h^i \in S_1^0(M)$,

$$\varphi_h^i(x_j) = \begin{cases} 1 & , \text{if } i = j \\ 0 & , \text{else} \end{cases}$$

$i, j \in \{1, \ldots, N\}$ $\to$ „show pict 5."

Linear finite element space for homogeneous Dirichlet BC:

$u|_{\partial\Omega} = 0$ corresponding Sobolev space $H_0^1(\Omega)$ =>

Galerkin space: $V_{0,h} = S_{1,0}^0(\mathcal{M}) := S_1^0(\mathcal{M}) \cap H_0^1(\Omega)$.

$S_{1,0}^0$ ← zero on $\partial\Omega$

We can obtain $S_{1,0}^0(\mathcal{M})$ by dropping basis f-ns on the boundary

$$S_{1,0}^0(\mathcal{M}) = \mathrm{Span}\{\varphi_h^j : x_j \in \Omega \text{ (interior node)}\} \Rightarrow$$

$$\dim(S_{1,0}^0(\mathcal{M})) = \#\{x \in V(\mathcal{M}),\ x \notin \partial\Omega\} \rightarrow \text{show "pict6"}$$

Sparsity of the Galerkin matrix:

Recall that in 1D the Galerkin matrix was tridiagonal, in particular, sparse. Since

$$A_{ij} = a(\varphi_h^i, \varphi_h^j):$$

$\{\mathrm{Vol}(\mathrm{supp}(\varphi_h^i) \cap \mathrm{supp}(\varphi_h^j)) = 0 \iff$ Nodes $x_i, x_j \in V(\mathcal{M})$ not connected by an edge$\}$

$\Rightarrow A_{ij} = 0.$ → show "pict 7".

Computation: For the sake of simplicity we consider the bilinear form for the Poisson problem:

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v\, dx, \quad u, v \in H_0^1(\Omega).$$

Galerkin discretization is based on:

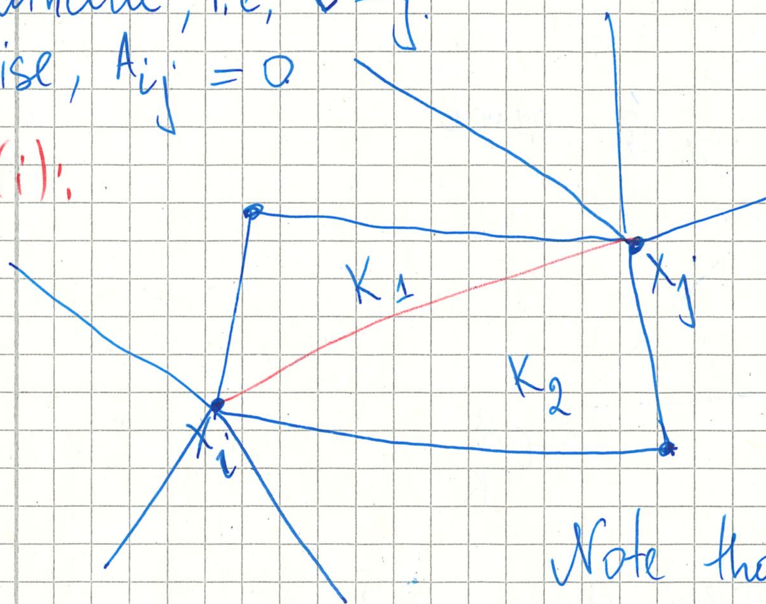
(i) triangular mesh, set of vertices $\{x_i\} = V(\mathcal{M})$,

(ii) discrete trial/test space $S_1^0(\mathcal{M}) \subset H^1(\Omega)$

(iii) the nodal basis $\{\varphi_h^i\}$

Then
$$A_{ij} = a(\varphi_h^i, \varphi_h^j) = \int_{\Omega} \nabla \varphi_h^i \, \nabla \varphi_h^j \, dx \implies$$
We need only deal with the sit-ns, where $x_i, x_j \in V(\mathcal{M})$:
(i) are connected by an edge of the triangulation
(ii) coincide, i.e, $i = j$.
otherwise, $A_{ij} = 0$

Case (i):

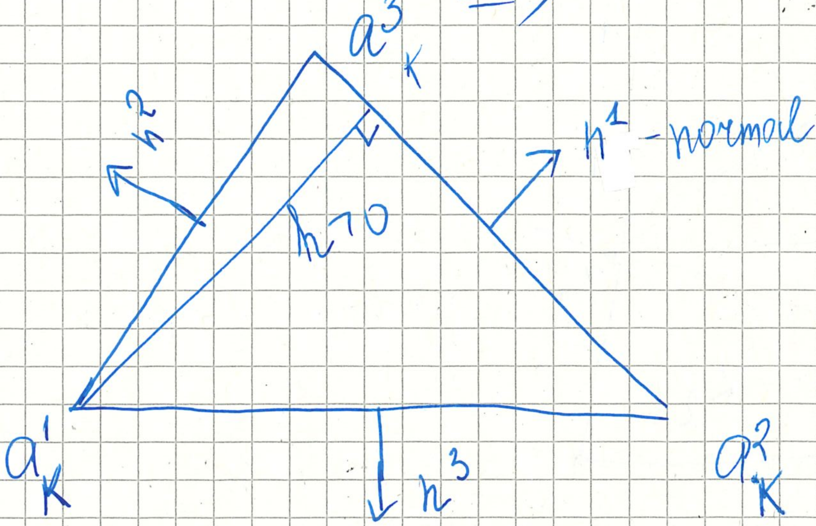


Note that $\mathrm{supp}(\varphi_h^i) \cap \mathrm{supp}(\varphi_h^j)$
$$\subseteq K_1 \cup K_2 \implies$$
$$A_{ij} = \int_{K_1} \nabla \varphi_h^i|_{K_1} \nabla \varphi_h^j|_{K_1} dx + \int_{K_2} \nabla \varphi_h^i|_{K_2} \nabla \varphi_h^j|_{K_2} dx.$$
This motivates us to calculate locally:
$$a_K(\varphi_h^i, \varphi_h^j) := \int_K \nabla \varphi_h^i \, \nabla \varphi_h^j \, dx$$
where $K \in \mathcal{M}$ is a triangle, x_i, x_j are vertices of K
Assume a_K^1, a_K^2, a_K^3 are the vertices of K
$$a_K^1 = \begin{bmatrix} a_1^1 \\ a_2^1 \end{bmatrix}, \quad a_K^2 = \begin{bmatrix} a_1^2 \\ a_2^2 \end{bmatrix} \quad a_K^3 = \begin{bmatrix} a_1^3 \\ a_2^3 \end{bmatrix}$$
Then $\forall i \in \{1, 2, 3\}$
$$a_K^i = x_j \quad , j \in \{1, \dots, N\}$$
i - local indexing
j - global indexing

We write
$$\lambda_i := \varphi_h^j|_K \quad , i \in \{1, 2, 3\}$$

The f-ns $\lambda_1, \lambda_2, \lambda_3$ on the triangle K are also known as barycentric coordinate f-ns $\longrightarrow$ "show pict 8 & 9"

Note that $\lambda_i(a^j_K) = \delta_{ij}$ and $\lambda_i : \mathbb{R}^2 \to \mathbb{R}$ affine linear. $\Rightarrow$



e_i - edge opposite vertex a^i_K $\Rightarrow$

$$\lambda_1(x) = \frac{-|e_1|}{2|K|}(x - a^2_K) \cdot n^1$$

[$\lambda_1(a^2_K) = 0$, $\lambda_1(a^3_K) = 0$ since $a^3_K - a^2_K \perp n^1$,

$$\lambda_1(a^1_K) = \frac{|e_1|}{2|K|} \cdot \underbrace{(a^1_K - a^2_K) \cdot (-n^1)}_{= h} = \frac{|e_1| \cdot |h|}{2} \cdot \frac{1}{|K|} = 1 \text{]}$$

There is exactly 1 plane through 3 noncollinear pts in $\mathbb{R}^3$

Similarly,

$$\lambda_2(x) = \cancel{\ldots} - \frac{|e_2|}{2|K|} \cdot (x - a^3_K) \cdot n^2$$

$$\lambda_3(x) = -\frac{|e_3|}{2|K|} \cdot (x - a^1_K) \cdot n^3.$$

Let us now compute the gradients

$$\operatorname{grad} \lambda_1 = -\frac{|e_1|}{2|K|} n^1$$

$$\operatorname{grad} \lambda_2 = -\frac{|e_2|}{2|K|} n^2 \longrightarrow \text{constant gradients}$$

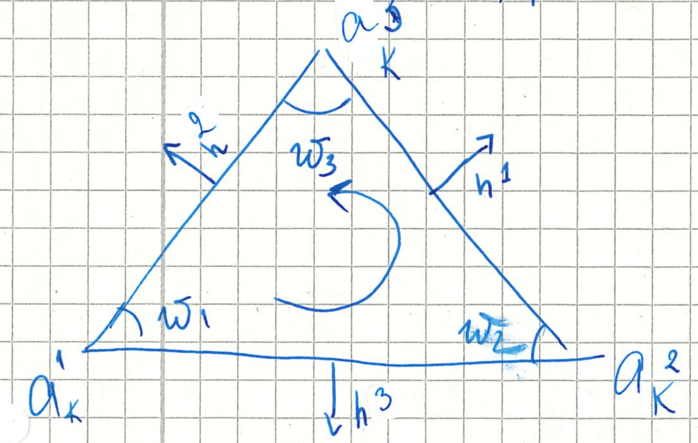
$$\operatorname{grad} \lambda_3 = -\frac{|e_3|}{2|K|} \cdot n^3$$

Now we can compute the contributions to entries of the Galerkin matrix:

$$A_K = \left[\int_K \operatorname{grad}(\lambda_i) \cdot \operatorname{grad}(\lambda_j) \, dx \right]_{i,j=1}^{3} \in \mathbb{R}^{3\times 3}$$

$$\parallel$$

$$\frac{1}{4|K|} |e_i||e_j| \cdot n_i \cdot n_j$$



If $i \neq j$ we have

* $n_i \cdot n_j = \cos(\pi - \omega_k)$
* $|K| = \frac{1}{2} |e_i||e_j| \cdot \sin \omega_k$ $\Rightarrow$

$$(A_K)_{i,j} = \frac{1}{2|e_i||e_j| \cdot \sin(\omega_k)} \cdot |e_i||e_j| \cdot \cos(\pi - \omega_k)$$

$k \neq i, \; k \neq j$

$$= -\frac{1}{2} \cot(\omega_k)$$

$i = j$: $\sum_{i=1}^{3} \lambda_i = 1 \Rightarrow \operatorname{grad}\left(\sum_{i=1}^{3} \lambda_i\right) = 0 \Rightarrow$

$$\sum_{i=1}^{3} (A_K)_{i,j} = 0 \Rightarrow (A_K)_{i,i} = -(A_K)_{i,j} - (A_K)_{i,\ell}$$

$$\Rightarrow (A_K)_{i,i} = \frac{1}{2}(\cot(\omega_k) + \cot(\omega_j)), \quad i \neq j, \; i \neq k, \; j \neq k \Rightarrow$$

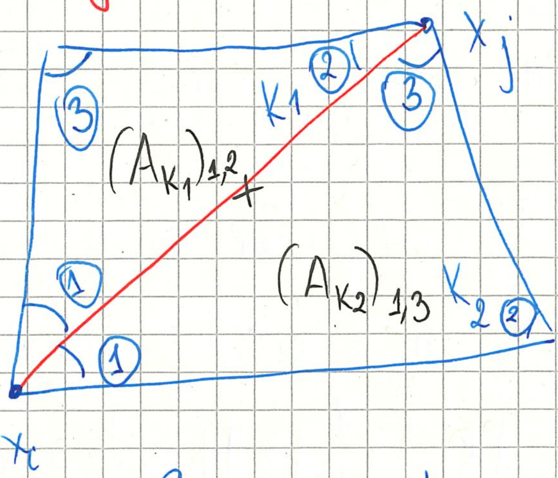
$$A_K = \frac{1}{2}\begin{bmatrix} \cot(w_3)+\cot(w_2) & -\cot(w_3) & -\cot(w_2) \\ -\cot(w_3) & \cot(w_3)+\cot(w_1) & -\cot(w_1) \\ -\cot(w_2) & -\cot(w_1) & \cot(w_2)+\cot(w_1) \end{bmatrix} \quad (*)$$

Another formula one can prove:

$$A_K = |K| \cdot \begin{bmatrix} \beta_1^1 & \beta_1^2 & \beta_1^3 \\ \beta_2^1 & \beta_2^2 & \beta_2^3 \end{bmatrix}^T \cdot \begin{bmatrix} \beta_1^1 & \beta_1^2 & \beta_1^3 \\ \beta_2^1 & \beta_2^2 & \beta_3^3 \end{bmatrix}$$

$\lambda_i(x) = \alpha_i + \beta^i \cdot x \quad , \quad \text{grad}\,\lambda_i = \beta^i$ $\qquad\uparrow$ easier to implement

Assembly of full Galerkin matrix:



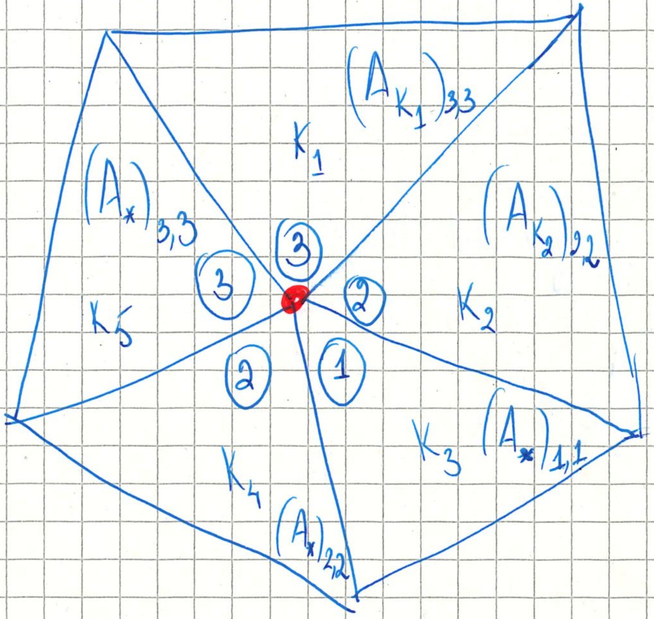
Note that for $i \neq j$ we have:

$$(A)_{ij} = \int_{K_1} \text{grad}(\varphi^j_h|_{K_1})\, \text{grad}(\varphi^i_h|_{K_1}) \cdot dx + \int_{K_2} \text{grad}(\varphi^j_h|_{K_2})\, \text{grad}(\varphi^i_h|_{K_2})\, dx$$

Then we can use $(*)$:

$$(A)_{i,j} = (A_{K_1})_{1,2} + (A_{K_2})_{1,3} \quad , i \neq j.$$

$$A_{ii} = \int_{\Omega_i} (\text{grad}\,\varphi_h^i)(\text{grad}\,\varphi_h^i)\, dx$$



• x_i

- We should sum corresponding diagonal entries of element matrices belonging to triangles adjacent to node x_i:

$$(A)_{ii} = (A)_{K_1 3,3} + (A_{K_2})_{2,2} + (A_{K_3})_{1,1} + (A_{K_4})_{2,2} + (A_{K_5})_{3,3}$$

=> In conclusion to build the $S_1^0(\mathcal{M})$-Galerkin matrix we will need:

* a numbering of nodal basis f-ns <-> numbering of mesh vertices $\in \mathcal{V}(\mathcal{M})$
* a numbering of triangles (cells) of mesh $\mathcal{M} = \{K_1, \dots, K_M\}$, $M := \#\mathcal{M}$
* a local numbering of the three vertices of every triangle $K \in \mathcal{M}$. =>

We get ↓ pseudo-code:

<u>foreach</u> $e \in \mathcal{E}(\mathcal{M})$ ($\mathcal{E}(\mathcal{M})$ = set of edges of $\mathcal{M}$)
  $(i,j) \equiv$ vertex numbers of endpoints of e
  $A_{ij} = 0$, $A_{ji} = 0$,
<u>foreach</u> triangle K adjacent to e
  find local numbers $l, m \in \{1,2,3\}$ of endpoints e
    $A_{ij} = A_{ij} + (A_K)_{lm}$
    $A_{ji} = A_{ji} + (A_K)_{m,l}$
<u>endfor</u>
<u>endfor</u>

foreach $v \in \mathcal{V}(\mathcal{M})$

  j = number of vertex v

  $A_{jj} = 0$

  foreach triangle K adjacent to v

    ℓ = local number of v in K

    $A_{jj} = A_{jj} + (A_K)_{\ell,\ell}$

  endfor

endfor