Abteilung für
Angewandte Mathematik

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Introduction to Theory and Numerics of PDEs – WiSe 2023/2024

Sheet 1

Ausgabe: 23.10.2023, 12:00 Uhr | Abgabe: 30.10.2023, 12:00 Uhr

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Homepage to the lecture:

https://aam.uni-freiburg.de/agsa/lehre/ws23/tun0

Exercise 1 (2+2 points)**.** Let u be a solution of the *transport equation*

$$\partial_t u + a(t,x)\partial_x u = 0.$$

(i) Show that u is constant along curves $(t, \gamma(t))$ for solutions of the initial boundary value problems $\gamma'(t) = a(t, \gamma(t)), y(0) = x_0$, called *characteristics*.

(ii) Determine the characteristics for the equation $\partial_t u + tx\partial_x u = 0$, sketch them, and determine the solution for the initial condition $u_0(x) = \sin(x)$.

Exercise 2 (2+1+1 points)**.**

(i) Prove the following estimates for difference quotients:

$$|\partial^{\pm} u(x_j) - u'(x_j)| \le \frac{\Delta x}{2} ||u''||_{C([0,1])},$$

$$|\hat{\partial} u(x_j) - u'(x_j)| \le \frac{\Delta x^2}{6} ||u'''||_{C([0,1])},$$

$$|\partial^+\partial^- u(x_j) - u''(x_j)| \le \frac{\Delta x^2}{12} ||u^{(4)}||_{C([0,1])}.$$

(ii) Show that $\partial^+\partial^- = \partial^-\partial^+$.

(iii) Prove an error estimate for the difference $\partial^+\partial^+ u(x_j) - u''(x_j)$.

Exercise 3 (2+2 points)**.** Let $u_0 \in C^2([0,1])$ and let $\tilde{u}_0$ denote its trivial extension by zero to $\mathbb{R}$.

(i) Find conditions on u_0 that guarantee $\tilde{u}_0 \in C^2(\mathbb{R})$.

(ii) Show that the solution of the transport equation $\partial_t u + a\partial_x u = 0$ with $u(t,0) = 0$ and $u(0,x) = u_0(x)$ satisfies $u \in C^2([0,T] \times [0,1])$, and that

$$||\partial_x^2 u(t,\cdot)||_{C([0,1])} = a^{-2} ||\partial_t^2 u(t,\cdot)||_{C([0,1])} = ||u_0''||_{C([0,1])}.$$

Exercise 4 (2+2 points)**.**

(i) Show by constructing appropriate initial data that the difference scheme

$$U_j^{k+1} = U_j^k + \mu(U_j^{k+1} - U_{j-1}^k)$$

with $\mu = a\Delta t/\Delta x$ is unstable if $\mu > 1$.

(ii) Check the CFL condition and the estimate $\sup_{j=0\dots,J} |U_j^{k+1}| \le \sup_{j=0,\dots,J} |U_j^k|$ of the following difference schemes for the transport equation:

$$\partial_t^+ U_j^k - \partial_x^- U_j^k = 0, \quad \partial_t^+ U_j^k - \hat{\partial}_x U_j^k = 0.$$