Abteilung für
Angewandte Mathematik

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Introduction to Theory and Numerics of PDEs – WiSe 2023/2024

Sheet 2

Ausgabe: 30.10.2023, 12:00 Uhr — Abgabe: 06.11.2023, 12:00 Uhr

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Homepage to the lecture:

https://aam.uni-freiburg.de/agsa/lehre/ws23/tun0

Exercise 1 (4 points)**.** Let $u \in C^2([0,T]\times[\alpha,\beta])$ solve the heat equation $\partial_t u - \kappa\partial_x^2 u = 0$. Show that for appropriate $\tau, L, x_0 > 0$, the function $\tilde{u}(s,y) = u(\tau s, Ly + x_0)$ solves $\partial_s \tilde{u} - \partial_y^2 \tilde{u} = 0$ in $(0,T') \times (0,1)$.

Exercise 2 (4 points)**.** Let $u \in C^2([0,T]\times[0,1])$ solve the heat equation $\partial_t u - \partial_x^2 u = 0$ with homogeneous Dirichlet boundary conditions. Prove that

$$\frac{\mathrm{d}}{\mathrm{dt}}\frac{1}{2}\int_0^1 (\partial_x u(t,x))^2 \mathrm{dx} \leq 0$$

and deduce the uniqueness of solutions for the heat equation with general Dirichlet boundary conditions.

Exercise 3 (3 + 1 points)**.** The construction of a solution via a *separation of variables* consists in finding functions $u_n(t,x) = v_n(t)w_n(x)$ that solve the heat equation and the prescribed boundary conditions. A solution of the initial value problem is then obtained by determining coefficients $(\alpha_n)_{n\in\mathbb{N}}$ such that

$$u(t,x) = \sum_{n=1}^{\infty} \alpha_n v_n(t) w_n(x)$$

converges in an appropriate sense and satisfies $u(0,x) = u_0(x)$.

(i) Construct pairs (v_n, w_n) such that $u_n(t,x) = v_n(t)w_n(x)$ satisfies $\partial_t u_n - \partial_x^2 u_n = 0$ in $(0,T)\times(0,1)$ and $u_n(t,0) = u_n(t,1) = 0$ for all $t \in (0,T)$.

(ii) Assume that the function $u_0 \in C([0,1])$ is given as

$$u_0(x) = \sum_{n=1}^{\infty} \gamma_n \sin(n\pi x).$$

Construct the solution of the corresponding initial boundary value problem for the heat equation.

Exercise 4 (2 + 1 + 1 points)**.**

(i) Show formally that the function

$$u(t,x) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-|x-y|^2/(4t)} u_0(y) \mathrm{dy}$$

solves the heat equation $\partial_t u - \partial_x^2 u = 0$ in $(0,T)\times\mathbb{R}$ for every $T > 0$.

(ii) Explain why we can expect that $u(t,x) \to u_0(x)$ as $t \to 0$, e.g. for piecewise constant initial data u_0 and $x = 0$.

(iii) Let $u_0(x) = 1$ for $x \geq 0$ and $u_0(x) = 0$ for $x < 0$. Show that $u(t,x)$ is positive for all $t \in (0,T)$ and $x \in \mathbb{R}$, and conclude that information is propagated with infinite speed.