Abteilung für
Angewandte Mathematik

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Introduction to Theory and Numerics of PDEs – WiSe 2023/2024

Sheet 3

Ausgabe: 06.11.2023, 12:00 Uhr Abgabe: 13.11.2023, 12:00 Uhr

Homepage to the lecture:

`https://aam.uni-freiburg.de/agsa/lehre/ws23/tun0`

Exercise 1 (4 points)**.** For $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$, let $A \in \mathbb{R}^{n\times n}$ be the bandmatrix

$$A = \begin{bmatrix} a & b & 0 & \dots & 0 \\ b & a & b & \ddots & \vdots \\ 0 & b & a & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b \\ 0 & \dots & 0 & b & a \end{bmatrix}$$

Show that A has the eigenvalues $\lambda_k = a + 2\cos(k\pi/(n+1)), k = 1, 2, \dots, n$.
Hint: Show that for $a = 0$ the eigenvectors $v_k \in \mathbb{R}^n$ are given by $v_{k,j} = \sin(kj\pi/(n+1))$ for $j = 1, 2, \dots, n$.

Exercise 2 (2 + 2 points)**.** Let $J \in \mathbb{N}$ and set $\Delta x = 1/J$.

(i) Prove that vectors $\phi_k \in \mathbb{R}^J, k = 1, \dots, J-1$ given by $\phi_{k,j} = \sqrt{2}\sin(kj\pi\Delta x), j = 0, 1, \dots, J$, define an orthonormal basis for

$$\ell^2_{0,\Delta x} := \{V \in \mathbb{R}^{J+1} | V_0 = V_J = 0\}$$

with respect to the inner product

$$(V, W)_{\Delta x} = \Delta x \sum_{j=0}^{J} V_j W_j.$$

(ii) Show that the vectors ϕ_k are eigenvectors of the operator

$$-\partial_x^+\partial_x^- : \mathbb{R}^{J+1} \to \mathbb{R}^{J+1}, \quad (-\partial_x^+\partial_x^- V)_j = \begin{cases} 0 & \text{for } j = 0, J, \\ -\partial_x^+\partial_x^- V_j & \text{for } j = 1, 2, \dots, J-1. \end{cases}$$

Hint: Use that $\phi_{k,j} = \sqrt{2}\,\mathrm{Im}(\omega^{kj})$ with $\omega = e^{i\pi\Delta x}$.

Exercise 3 (2 + 2 points)**.**

(i) Show that the θ-method is well defined for every choice of θ and every choice of $\Delta x, \Delta t > 0$.

(ii) Show that the θ-method is unstable if $\theta < 1/2$ and $\lambda = \Delta t/\Delta x^2 > 1/2$.