Abteilung für
Angewandte Mathematik

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Introduction to Theory and Numerics of PDEs – WiSe 2023/2024

Sheet 4

Ausgabe: 13.11.2023, 12:00 Uhr — Abgabe: 20.11.2023, 12:00 Uhr

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Homepage to the lecture:

`https://aam.uni-freiburg.de/agsa/lehre/ws23/tun0`

Exercise 1 (2 + 2 points)**.**

(i) Determine functions $u_n(t,x) = v_n(t)w_n(x), n \in \mathbb{N}$, that satisfy the wave equation in $(0,T) \times (0,1)$ subject to homogeneous Dirichlet boundary conditions.

(ii) Assume that $u_0, v_0 \in C([0,1])$ satisfy

$$u_0(x) = \sum_{n\in\mathbb{N}} a_n \sin(n\pi x), \qquad v_0(x) = \sum_{n\in\mathbb{N}} b_n \sin(n\pi x)$$

with given sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$. Derive a representation for the solution of the wave equation $\partial_t^2 u - c^2\partial_x^2 u = 0$ in $(0,T)\times(0,1)$ with homogeneous Dirichlet boundary conditions and initial conditions $u(0,x) = u_0(x)$ and $\partial_t(0,x) = v_0(x)$ for all $x \in [0,1]$.

Exercise 2 (2 + 2 points)**.** Let $(z_k)_{k\in\mathbb{N}}$ be a sequence of real numbers that for $\alpha, \beta \in \mathbb{R}$ and all $k \in \mathbb{N}$ satisfies the recursion

$$\begin{pmatrix} z_k \\ z_{k+1} \end{pmatrix} = A \begin{pmatrix} z_{k-1} \\ z_k \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}.$$

(i) Show that if the eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ satisfy $|\lambda_i| < 1, i = 1,2$, then there exists $c > 0$ such that $|z_k| \leq c$ for all $k \in \mathbb{N}$.

(ii) Show that if the eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ coincide and satisfy $|\lambda_i| = 1, i = 1,2$, or if $\max_{i=1,2}|\lambda_i| > 1$, then there exist unbounded sequences $(z_k)_{k\in\mathbb{N}}$ that satisfy the recursion.

Exercise 3 (2 + 2 points)**.** For $J \in \mathbb{N}$, let $\Delta x = 1/J$ and let $V; W \in \mathbb{R}^{J+1}$.

(i) Prove the discrete product rule

$$\partial_x^-(W_jV_j) = W_j(\partial_x^- V_j) + (\partial_x^+ W_{j-1})V_{j-1}.$$

(ii) Deduce the summation-by-parts formula

$$\Delta x \sum_{j=0}^{J-1} (\partial_x^+ W_j)V_j = -\Delta x \sum_{j=0}^{J} W_j(\partial_x^- V_j) + W_J V_J - W_0 V_0,$$

and explain its relation to the integration-by-parts formula.

Exercise 4 (2 + 2 points)**.**

(i) Prove that the implicit difference scheme for the wave equation is well defined, i.e., leads to regular linear systems of equations in all time steps.

(ii) Show that the implicit difference scheme for the wave equation has a consistency error $\mathcal{O}(\Delta t^2 + \Delta x^2)$.