Abteilung für
Angewandte Mathematik

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Introduction to Theory and Numerics of PDEs – WiSe 2023/2024

Sheet 5

Ausgabe: 27.11.2023, 12:00 Uhr | Abgabe: 11.12.2023, 12:00 Uhr

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Homepage to the lecture:

https://aam.uni-freiburg.de/agsa/lehre/ws23/tun0

Exercise 1 (4 points)**.** Let $\Omega = (0,1)^2$ and let $f \in C(\bar{\Omega})$ be given by

$$f(x_1, x_2) = \sum_{m,n\in\mathbb{N}} \alpha_{m,n} \sin(m\pi x_1)\sin(n\pi x_2).$$

Compute $-\Delta u_{m,n}$ for $u_{m,n}(x_1,x_2) = \sin(m\pi x_1)\sin(n\pi x_2)$ and construct the solution of the Poisson problem $-\Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$.

Exercise 2 (2 + 2 points)**.**

(i) Use Gauss's theorem to show that for $u, v \in C^2(\bar{\Omega})$, we have the so called *Green's formulas*:

$$\int_{\partial\Omega} v\,\langle \nabla u, n\rangle \,\mathrm{ds} = \int_{\Omega} (\langle \nabla u, \nabla v\rangle + v\Delta u)\,\mathrm{dx}, \tag{1}$$

$$\int_{\Omega} (u\Delta v - v\Delta u)\,\mathrm{dx} = \int_{\partial\Omega} (u\,\langle \nabla v, n\rangle - v\langle \nabla u, n\rangle)\,\mathrm{ds}. \tag{2}$$

(ii) Let $u_1, u_2 \in C^2(\bar{\Omega})$ be solutions of the boundary value problem $-\Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$. Show that

$$\int_{\Omega} |\nabla(u_1 - u_2)|^2\,\mathrm{dx} = 0$$

Exercise 3 (2 +2 points)**.** Let $x_0 \in \mathbb{R}^d$ for $d \in \{2,3\}, r > 0$ and $u \in C^1(\overline{B_r(x_0)})$.

(i) Show that in polar coordinates with respect to x_0, we have

$$\langle \nabla u, n\rangle = \partial_r u$$

on $\partial B_{a'}(x_0)$ for every $0 < a' \leq r$.

(ii) Show that

$$\lim_{r\to 0} \frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0))} u(s)\,\mathrm{ds} = u(x_0),$$

where $|\partial B_r(x_0)|$ means the surcafe measure of $\partial B_r(x_0)$.

Exercise 4 (4 points)**.** Let $AU = F$ be the linear system of equations corresponding to the discretized Poisson problem $-\Delta u = f$ in $\Omega = (0,1)^2$ with homogeneous Dirichlet boundary conditions. Show that the Richardson scheme for the iterative solution of the linear system can be identified with an explicit discretization of the heat equation.