Abteilung für
Angewandte Mathematik

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Introduction to Theory and Numerics of PDEs – WiSe 2023/2024

Sheet 6

Ausgabe: 11.12.2023, 12:00 Uhr — Abgabe: 18.12.2023, 12:00 Uhr

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Homepage to the lecture:

https://aam.uni-freiburg.de/agsa/lehre/ws23/tun0

Exercise 1 (4 points)**.** Write the initial boundary value problem for the wave equation as an abstract boundary value problem $F(u) = 0$ in U and $G(u) = 0$ and ∂U by defining appropriate mappings F and G.

Exercise 2 (2 + 2 points)**.**

(i) Let $A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n, c \in \mathbb{R}$, and $f \in C(\overline{U})$. Assume that $u \in C^2(\overline{U})$ satisfies

$$\sum_{i,j=1}^{n} a_{ij}\partial_{z_i}\partial_{z_j}u(z) + \sum_{j=1}^{n} b_j\partial_{z_j}u(z) + c\,u(z) = f(z)$$

for all $z \in U$. Suppose that $A = Q^T\Lambda Q$ is diagonalizable and define $\tilde{u}(\xi) = u(Q\xi)$. Determine the PDE satisfied by $\tilde{u}$.

(ii) Determine the type of the following PDEs

$$\begin{aligned} \partial_t u + \Delta u &= f \qquad \text{in } (0,T)\times\Omega \subset \mathbb{R}_{\geq 0}\times\mathbb{R}^d, \\ \partial_{x_1}^2 u - 3\partial_{x_1}\partial_{x_2}u + \partial_{x_2}^2 u &= 0 \qquad \text{in } \Omega \subset \mathbb{R}^2, \\ \partial_t u - \partial_{x_1}^2 + \partial_{x_2}u &= f \qquad \text{in } (0,T)\times\Omega \subset \mathbb{R}_{\geq 0}\times\mathbb{R}^2. \end{aligned}$$

Exercise 3 (6 + 2 points)**.**

(i) Let $J \in \mathbb{N}$ and $\Delta x = 1/J$. Let $(\phi_p \,|\, p = 1,\ldots,J-1)$ be the eigenvectors of $-\partial_x^+\partial_x^-$ given by $\phi_{p,j} = \sqrt{2}\sin(pj\pi\Delta x)$, $0 \leq j \leq J$. Show that the vectors $\psi_{(p,q)} \in \mathbb{R}^{(J+1)^2}$, defined by

$$\psi_{(p,q),(j,m)} = \phi_{p,j}\phi_{q,m} = 2\sin(pj\pi\Delta x)\sin(qm\pi\Delta x)$$

are eigenvectors of the discrete Laplace operator $-\Delta_h = -\partial_{x_1}^+\partial_{x_1}^- - \partial_{x_2}^+\partial_{x_2}^-$ and that they define an orthonormal basis of the space of grid functions with vanishing boundary conditions with respect to the inner product

$$(V,W)_{\Delta x} = \Delta x^2 \sum_{j,m=0}^{J} V_{j,m}W_{j,m}.$$

(ii) Carry out a stability analysis of the $\theta-$method for approximating the two dimensional heat equation.