Abteilung für
Angewandte Mathematik

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Introduction to Theory and Numerics of PDEs – WiSe 2023/2024

Sheet 7

Ausgabe: 18.12.2023, 12:00 Uhr Abgabe: 08.01.2024, 12:00 Uhr

Homepage to the lecture:

`https://aam.uni-freiburg.de/agsa/lehre/ws23/tun0`

Note: You only have to solve 4 exercises out of 8!

Exercise 1 (2 + 2 points)**.** For $\Omega \subset \mathbb{R}^2$ and $u \in C^2(\Omega)$, let $\tilde{u}(r, \phi) = u(r \cos \phi, r \sin \phi)$.

(i) Show that

$$\nabla u(r \cos \phi, r \sin \phi) = \left[\partial_r \tilde{u}(r, \phi), r^{-1} \partial_\phi \tilde{u}(r, \phi)\right]^T$$

and

$$\Delta u(r \cos \phi, r \sin \phi) = \partial_r^2 \tilde{u}(r, \phi) + r^{-1} \tilde{u}(r, \phi) + r^{-2} \partial_\phi^2 (r, \phi).$$

(ii) Verify that the function $\tilde{u}(r, \phi) = r^{\pi/\alpha} \sin(\phi\pi/\alpha)$ is harmonic.

Exercise 2 (4 points)**.** Let $h \in C(\Omega)$ and assume that

$$\int_\Omega hv \, \mathrm{dx} = 0$$

for all $v \in C^\infty(\Omega)$ with $v = 0$ on $\partial\Omega$. Prove that $h = 0$ in Ω.

Exercise 3 (4 points)**.** Let $a : V \times V \longrightarrow \mathbb{R}$ be symmetric, bilinear and positive semidefinite. Prove that

$$a(v, w) \leq (a(v, v))^{1/2} (a(w, w))^{1/2}.$$

Exercise 4 (4 points)**.** Prove that the set of square summable sequences $\ell^2(\mathbb{N}) = \{(v_j)_{j\in\mathbb{N}} \mid \sum_{j\in\mathbb{N}} v_j^2 < \infty\}$ is a Hilbert space.

Exercise 5 (4 points)**.** Let V be a Banach space and let $a : V \times V \longrightarrow \mathbb{R}$ be bilinear, symmetric and positive semidefinite. Moreover, assume that there exist $c_1, c_2 > 0$ such that

$$c_1 ||v||_V \leq (a(v, w))^{1/2} \leq c_2 ||v||_V$$

for all $v \in V$. Show that a defines a scalar product on V and that V is a Hilbert space with this scalar product.

Exercise 6 (4 points)**.** Let V, W be $n-$ and $m-$dimensional linear spaces. Use the Riesz representation theorem to prove that $L(V, W)$ is isomorphic to $\mathbb{R}^{n\times m}$, i.e. that linear mappings can be identified with matrices.

Exercise 7 (2 + 1 + 1)**.**

(i) Show that the linear operator $A : V \longrightarrow W$ is continuous if and only if it is bounded in the sense that there exists $c > 0$ such that

$$||Av||_W \leq c||v||_V$$

for all $v \in V$.

(ii) Let $A : V \longrightarrow W$ be linear and bounded and let $||A||_{L(V,W}$ be the infimum of all constants $c > 0$. Show that for all $v \in V$ we have

$$||Av||_W \leq ||A||_{L(V,W)} ||v||_V.$$

(iii) Show that $A \mapsto ||A||_{L(V,W)}$ defines a norm on the space of linear and bounded operators $L(V,W)$ such that it is a Banach space.

Exercise 8 (4 points)**.** Determine all matrices $M \in \mathbb{R}^{n\times n}$ such that the bilinear mapping $a : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$a(x,y) = x^T My$$

satisfies the conditions of the (i) Riesz representation theorem and (ii) Lax-Milgram lemma.

We wish you a merry christmas and a happy new year!