Abteilung für
Angewandte Mathematik

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Introduction to Theory and Numerics of PDEs – WiSe 2023/2024

Practical Exercises 1

Ausgabe: 26.10.2023, 12:00 Uhr — Abgabe: 09.11.2023, 12:00 Uhr

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Homepage to the lecture:

`https://aam.uni-freiburg.de/agsa/lehre/ws23/tun0`

Exercise 1 (4+4+2 points)**.**

(i) Consider the transport equation $\partial_t u + \partial_x u = 0$ in $(0,T) \times (0,1)$ for $T = 1$ with boundary condition $u(t,0) = u(t,1)$ and initial condition

$$u_0(x) = \begin{cases} 1 & \text{falls } 0.4 \le x \le 0.6, \\ 0 & \text{sonst.} \end{cases}$$

Implement a numerical method to solve this equation by approximating the time derivative with the forward difference quotient and the spatial derivative with the backward difference quotient. Let $(\Delta t, \Delta x) \in \{(1/80)(2,2), (1/80)(2,1), (1/80)(1,2)\}$. Check for all pairs $(\Delta t, \Delta x)$ if the CFL Condition is satisfied and compare your numerical solutions to the exact solution.

(ii) Modify your code from (i) to numerically solve the equation

$$\partial_t u + a(x)\partial_x u = 0$$

with $a : (0,1) \longrightarrow \mathbb{R}_{\ge 0}$. What should the CFL Condition be for non-constant functions a? Test your code for $a(x) = \sqrt{1 + 4x^2}$ and initial values $u_0(x) = 1$ for $0.05 \le x \le 0.25$ and $u_0(x) = 0$ otherwise.

(iii) Compute a solution for $a(x) = -1$ with u_0 as in (i).

(iv) Modify your code from (i) again to approximate the spatial derivative with the forward difference quotient. Derive a CFL condition and test 2 different pairs $(\Delta t, \Delta x)$.

Exercise 2 (10 points)**.** Let us consider the *Upwind method* for the transport equation, which is defined by

$$U_j^{k+1} = \begin{cases} (1 - \mu a_j^k) U_j^k + \mu a_j^k U_{j-1}^k, & a_j^k \ge 0, \\ (1 + \mu a_j^k) U_j^k - \mu a_j^k U_{j-1}^k, & a_j^k < 0, \end{cases}$$

where $\mu = \Delta t / \Delta x$ and $a_j^k = a(t_k, x_j)$. Implement this method and test it for $a(x) = \sin(x)$ and the boundary values $u(t,0) = u(t,1) = 0$ with two different initial values u_0.