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Introduction to Theory and Numerics of PDEs – WiSe 2023/2024

Practical Exercises 2

Ausgabe: 09.11.2023, 12:00 Uhr — Abgabe: 23.11.2023, 12:00 Uhr

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Homepage to the lecture:

`https://aam.uni-freiburg.de/agsa/lehre/ws23/tun0`

Exercise 1 (3+3+4 points)**.** Consider for $T > 0$ the heat equation $\partial_t u - \kappa \partial_x^2 = 0$ in $(0, T) \times (-1, 1)$ with $u_0(x) = \cos((\pi/2)x), u(t, -1) = u(t, 1) = 0$ and $\kappa = 1/200$.

(i) Implement a $\theta-$method to solve the boundary value problem approximately. Choose $\Delta x = 1/20$ and determine Δt such that the method is stable for $\theta = 0$ by trying different values for the time steps Δt.

(ii) The exact solution is given by $u(t, x) = \cos((\pi/2)x)e^{-(\kappa\pi^2/4)t}$. Determine the approximation error at the point $(t, x) = (1, 0)$ for $\theta \in \{(1/2), (3/4), 1\}$ and $\Delta x = \Delta t = 2^{-j}/10, j = 2, 3, 4, 5$. Make a plot of the errors in one windows and use the command `semilogy` instead of `plot` to get a logarithmic scale for the $y-$axis. Interpret your findings in a few words.

(iii) Modify your code such that the inhomogeneous heat equation $\partial_t u - \kappa \partial_x^2 = f$ is solved in $(0, T) \times (-1, 1)$ for $f(t, x) = 0.05x^2$. Compute approximated solutions for homogeneous Dirichlet boundary conditions and initial value $u_0(x) = 1$ if $-0.1 \leq x \leq 0.1$ and $u_0(x) = 0$ otherwise. Compare the approximated solutions for the different values $\theta = 0, \theta = 1/2$ and $\theta = 1$.

Exercise 2.

(i) Consider the $\theta-$method from above and let $\theta = 1/2, T = 5, \kappa = 0.1$ and

$$u_0(x) = \begin{cases} \exp\left(-\frac{1}{4(0.5+x)(0.5-x)}\right), & \text{if } |x| < 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

Modify your program from ex. 1 to approximate the homogeneous heat equation with Neumann boundary conditions, i.e. $\partial_x u(t, -1) = g_l(t)$ and $\partial_x u(t, 1) = g_r(t)$ for $t \in [0, 5)$. In order to do so, use the forward difference quotient for $\partial_x u(t_{k+1}, -1)$ and the backward difference quotient for $\partial_x u(t_{k+1}, 1)$. Test your program for homogeneous Neumann boundary conditions and $\Delta x = \Delta t = 2^{-j}/10, j = 2, 3, 4, 5$. Is this numerical solution meaningful?

Compute the initial mass $\int_{-1}^{1} u_0(x)\,\mathrm{dx}$ using the MATLAB-function `trapz` with $10^3 + 1$ interpolation points. Compare this mass with the mass at time $t = 5$ for the different discrete solutions. Use the display format `long`.

(ii) Now, approximate the Neumann boundary conditions using the central difference quotients $\hat{\partial}_x U_0^{k+1}$ and $\hat{\partial}_x U_J^{k+1}$. In order for this method to be defined at the boundary points, consider so called ghost points $x_{-1} = -1 - \Delta x$ and $x_{J+1} = 1 + \Delta x$. The needed approximations U_{-1}^0 and U_{J+1}^0 can be computed from the known values $(U_j^0)_{j=0,\ldots,J}$ and the discretized Neumann boundary conditions. Compute, as in part (i), the mass at time $t = 5$ for $\Delta x = \Delta t = 2^{-j}/10, j = 2, 3, 4, 5$ and the error compared to the initial mass. Describe your findings and argue why it is better to use the central difference quotient.