Abteilung für
Angewandte Mathematik

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Introduction to Theory and Numerics of PDEs – WiSe 2023/2024

Practical Exercises 3

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Homepage to the lecture:

`https://aam.uni-freiburg.de/agsa/lehre/ws23/tun0`

Exercise 1 (10 points)**.**

(i) Implement the unconditionally stable method

$$\partial_t^+ \partial_t^- U_j^n = \frac{1}{4} \partial_x^+ \partial_x^- (U_j^{n+1} + 2U_j^n + U_j^{n-1})$$

to numerically approximate the wave equation $\partial_t^2 u - \partial_x^2 u = 0$ in $(0,T) \times (0,1)$ with homogeneous Dirichlet boundary conditions and initial conditions $u(x,0) = u_0(x)$ and $\partial_t u(0,x) = v_0(x)$ for $x \in (0,1)$ and $u_0, v_0 \in C([0,1])$. To discretize the initial value $\partial_t u(0,x)$ use the central difference quotient and the ghost point $-\Delta t$, such that you can expect quadratic convergence. Test your program for the exact solution $u(t,x) = \cos(\pi t)\sin(\pi x)$. Make plots that show the dependence of the quadratic convergence and the discretizing parameters.

(ii) Check experimentally that the discrete energy

$$\Gamma^k = \frac{\Delta x}{2} \sum_{j=1}^{J-1} |\partial_t^+ U_j^k|^2 + \frac{\Delta x}{2} \sum_{j=1}^{J} |\partial_x^- U_j^{k+1/2}|^2$$

is independent of $k = 0, \dots K-1$. Here, $U_j^{k+1/2} = (U_j^{k+1} + U_j^k)/2$.

Exercise 2 (10 points)**.** The sound of a stringed instrument is characterized by the occurrence of different overtones. In order to experimentally verify that the wave equation describes effects of this kind, we consider a string of length $\ell > 0$ that is plucked at a point $x_p \in (0, \ell)$ at time $t = 0$ and thereby deflected from its initial position by a distance $H > 0$. The state of the string is therefore described by

$$u_0(x) = \begin{cases} Hx/x_p & \text{für } x \le x_p, \\ H(\ell - x)/(\ell - x_p) & \text{für } x > x_p. \end{cases}$$

We assume that the sound is picked up at a point $x_s \in (0, \ell)$, for example by the sound hole in the sound box of an acoustic guitar or an electromagnetic pickup in the case of an electric guitar. Assuming that the initial velocity of the string is equal to 0, the exact solution is obtained by separating the variables in the wave equation

$$u(t,x) = \sum_{m=1}^{\infty} \alpha_m \cos(\omega_m t) \sin(m\pi x)$$

with $\omega_m = m\pi c$ and $c = (\sigma/\varrho)^{1/2}$, where ϱ describes the density and σ the tension of the string. Use your program from Project 1 to determine the numerical solution of the wave equation with $c = 2$, $T = 2$, $x_p = 1/8$ and $H = 1/100$. Use your approximations to determine coefficients α_m, $m = 1, 2, \dots, K$, for the discrete cosine transformation using the MATLAB function `dct`, such that

$$U_{j_s}^k = \sum_{m=1}^{K} \alpha_m \cos(\omega_m t_k),$$

where j_s is the corresponding index to the grid point $x_s = 1/4$. Create plots of the oscillations $w_m(t) = \alpha_m \cos(\omega_m t)$, $m = 1, 2, \ldots, 6$, as functions of $t \in [0, T]$. Visualize the dominant overtones by graphing the distribution of the amplitudes in the form of the function $m \mapsto |\alpha_m|$. Compare these results with the corresponding results for other values of x_p and x_s.