Abteilung für
Angewandte Mathematik

JProf. Dr. Diyora Salimova
M.Sc. Mario Keller

Introduction to Theory and Numerics of PDEs – WiSe 2023/2024

Practical Exercises 4

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Homepage to the lecture:

`https://aam.uni-freiburg.de/agsa/lehre/ws23/tun0`

Exercise 1 (10 points)**.** Calculate approximate solutions $u_m \in \mathcal{P}_m([0,1])$[1] of the one-dimensional Poisson problem $-u'' = f$ in $\Omega = (0,1)$ with boundary conditions $u(0) = u(1) = 1$, by solving the system of equations

$$-u_m''(x_i) = f(x_i),\ i = 1,2,\ldots,m-1, \quad u_m(x_0) = u_m(x_m) = 1,$$

with $x_i = i/m$ for $i = 0,1,\ldots,m$. Test the method for $f(x) = 1$ and $f(x) = \operatorname{sign}(x-1/2)$. Investigate the behavior of the error $\max_{i=0,\ldots,m} |u(x_i) - u_m(x_i)|$ and the condition of the system of equations for $m \to \infty$.

Exercise 2 (5 + 5 points)**.**

(i) Define functions f, g and u_D such that $u(x,y) = \sin(\pi x)\sin(\pi y)$ is a solution to the boundary value problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega = (0,1)^2, \\ u &= u_D \quad \text{on } \Gamma_{\mathrm{D}} = \{0\} \times [0,1], \\ \partial_n u &= g \qquad \text{on } \Gamma_{\mathrm{N}} = \partial\Omega \setminus \Gamma_{\mathrm{D}} \end{aligned}$$

Solve the problem numerically using a finite difference method. Realize the Neumann boundary conditions by introducing suitable ghost points and approximating the derivatives in the normal direction to Γ_{N} with central difference quotients. Compare your numerical solutions with the exact solution u and verify the quadratic convergence of the method.

(ii) Write a program for the numerical solution of the boundary value problem

$$\begin{aligned} -\Delta u &= 1 \quad \text{in } \Omega = B_1(0), \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Use the discretization $r_i = (i - \frac{1}{2})\Delta r$, $i = 1,\ldots,J+1$, $\vartheta_m = (m-1)\Delta\vartheta$, $m = 1,\ldots,K+1$, the circular disk in polar coordinates for $\Delta r = \frac{2}{2J+1}$, $\Delta\vartheta = \frac{2\pi}{K}$. Express the Laplace operator in polar coordinates and discretize the occurring first-order derivatives with the central difference quotient. Compare your approximation with the exact solution $u(x) = (|x|^2 - 1)/4$.

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[1] $\mathcal{P}_m([0,1]) = \{p \in C([0,1]) \,|\, p(x) = a_0 + a_1 x + \ldots + a_m x^m \text{ where } a_0,\ldots,a_m \in \mathbb{R}\}$