Abteilung für
Angewandte Mathematik

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Introduction to Theory and Numerics of PDEs – WiSe 2023/2024

Practical Exercises 5

Ausgabe: 21.12.2023, 12:00 Uhr
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Homepage to the lecture:

`https://aam.uni-freiburg.de/agsa/lehre/ws23/tun0`

Exercise 1 (15 Punkte)**.** Due to difficult exercises, a group of students exceptionally have no time and have to eat frozen pizza. The interior of their oven is given by the area $\Omega = (0, 0.4) \times (0, 0.3) \times (0, 0.4)$. We denote the temperature distribution in the oven by θ. At the back $y = 0$ of the oven, the temperature is a constant 200°C. The flap is located at the front $y = 0.3$ of the oven. If this is open, the temperature there corresponds exactly to the room temperature (20°C). If it is closed, the oven is perfectly isolated there, just like on the other two sides $x = 0$ and $x = 0.4$, i.e. $\partial_n\theta = 0$. In addition, the oven is well preheated: at the time $t = 0$ the temperature is 200°C everywhere. However, the students do not agree on which of the procedures makes more sense if you want to use as little energy as possible.

(i) The oven is open for 30 seconds, then closed for 30 seconds and then open again for 30 seconds;
(ii) The oven is first closed for 30 seconds and then open for 60 seconds at a time.

The students know that a mathematical model is obtained by exploiting that the heat density w is proportional to the temperature density θ, i.e. $w = \rho c_p \theta$, and the heat flux is proportional to the temperature gradient, i.e. $q = -\kappa\nabla\theta$, and the total thermal energy is conserved, i. h. $\partial_t w + \operatorname{div} q = 0$. Using the internet, research that reasonable parameters for the model are approximately given by the corresponding constants for air, namely the density $\rho = 1.2041\,\mathrm{kg/m^3}$, the thermal conductivity coefficient $\kappa = 0.0262\,\mathrm{W/(m\,K)}$ and the specific heat capacity $c_p = 1.005 \times 10^3\,\mathrm{J/(kg\,K)}$. Together, they also show that a dimensional reduction can be performed by replacing θ with the mean value

$$\widetilde{\theta}(t, x, y) = 0.4^{-1} \int_0^{0.4} \theta(t, x, y, z)\mathrm{d}z$$

is replaced. However, none of them is able to answer the question with the help of this information.

Formulate an initial boundary value problem to describe the average temperature distribution $\widetilde{\theta}$ in $\widetilde{\Omega} = (0, 0.4) \times (0, 0.3)$. Implement a Crank-Nicolson method to solve the problem and use it to simulate scenarios (i) and (ii). Based on your simulation, decide whether it is more energy efficient to open the oven once for a longer period or twice for a shorter period. Explain the weaknesses of the model and the numerical method.

Exercise 2 (5 Punkte)**.** Numerically compute the $W^{1,2}$ norm of the functions $u(x) = \log\log|x|$ and $u(x) = |x|^{1/2}$ in the domain $\Omega = (-1/2, 1/2)^2$. Visualize the functions and their gradients.