NASDE 15.10.2024

Famous model in biology:

Lotka-Volterra (predator-prey)

$$\frac{dx}{dt} = x(\alpha-\beta y) \qquad (*)\ \alpha,\beta,\delta,\gamma>0$$
$$\frac{dy}{dt} = y(\delta x-\gamma)$$

x - population density of prey (# of rabbits per km²)

y - - - - - - of predator (fox)

$\frac{dx}{dt}, \frac{dy}{dt}$ - growth rate

* prey reproduce exponentially with α, subject to predation $\beta x y$

* $\delta x y$ - growth of predator subject to $\gamma \cdot y \leftarrow$ loss due to natural causes

→ can also be used in economics

Assum: no environmental changes

However: temperature, flood, earthquake,

→ Hard to model deterministically

→ Some uncertainty, randomness exists ⇒

Stochastic version of (*):

$$dX = x(\alpha-\beta y)dt + C_1 \cdot x\, dW_t^{(1)}$$
$$dY = y(\delta x - y)dt + C_2 \cdot y \cdot dW_t^{(2)}$$

$W = (W^{(1)}\ W^{(2)})$ - 2 dim-l Brownian motion

SDEs in the form

$$dX_t = \underbrace{\mu(X_t)}_{\text{drift parameter}} dt + \underbrace{\sigma(X_t)}_{\text{diffusion parameter}} dW_t$$

First preliminaries from measure and probability theory:

We will work on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$

underlying set

σ-algebra on Ω

probability measure

Def-n $\mathcal{A}$ is a σ-algebra on Ω if

(i) $\mathcal{A} \subset P(\Omega)$ (power set of Ω, set of all subsets), $\emptyset \in \mathcal{A}$

(ii) $B \in \mathcal{A} \Rightarrow B^c = \Omega \setminus B \in \mathcal{A}$ (closed under complementation)

(iii) $B_1, B_2, \dots \in \mathcal{A} \Rightarrow$

$$\bigcup_{i=1}^{\infty} B_i \in \mathcal{A}$$ (closed under countable union)

Eg: $\mathcal{A} = \{\emptyset, \Omega\}$, $\mathcal{A} = P(\Omega)$

$(\Omega, \mathcal{A})$ is called measurable space, each element of $\mathcal{A}$ is a measurable set.

Q: what if $\mathcal{A} \subseteq P(\Omega)$ is not a sigma-algebra?

Def: Sigma-algebra generated by $\mathcal{A}$

$$\sigma_\Omega(\mathcal{A}) = \bigcap_{\substack{B \text{ is a } \sigma\text{-algebra} \\ B \supseteq \mathcal{A}}} B$$

If $\mathcal{A}$ is a σalgebra

$$\sigma_{\Omega}(\mathcal{A}) = \mathcal{A}$$

→ the smallest σ-algebra containing $\mathcal{A}$.

Def-n $(\Omega, \mathcal{A})$ -measurable space, $\mu: \mathcal{A} \to [0, \infty]$. Then μ is a measure if

(i) $\mu(\emptyset) = 0$

(ii) Sigma additivity:
$\forall B_1, B_2, \ldots \in \mathcal{A}$, $B_i \cap B_j = \emptyset, i \neq j$

$$\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i)$$

$(\Omega, \mathcal{A}, \mu)$ - measure space

If, $\mu(\Omega) = 1$ => probability space

Def Given 2 measurable spaces $(\Omega, \mathcal{A})$, $(\tilde{\Omega}, \tilde{\mathcal{A}})$, the function

$X : \Omega \to \tilde{\Omega}$ is $\mathcal{A}/\tilde{\mathcal{A}}$ measurable (or just measurable) if

$\forall S \in \tilde{\mathcal{A}}$:

$$X^{-1}(S) = \{w \in \Omega : X(w) \in S\} \in \mathcal{A}$$

→ Given measurable f-n X one can define σ-algebre generated by X

$$\sigma_{\Omega}(X) = \{X^{-1}(S), S \in \tilde{\mathcal{A}}\}$$

$$\subseteq \mathcal{A}$$

→ If in addition $(\Omega, \mathcal{A}, \mathbb{P})$ prob. space X is called a random variable.

Given a measure space $(\Omega, \mathcal{A}, \mu)$ and measurable f-n

$$X : \Omega \to \mathbb{R}$$

one can define the integral

$$\int_\Omega X \, d\mu$$

[if $\mu = P$, $E_P[X]$] in 3 steps:

(1) $f : \Omega \to [0, \infty)$ meas, simple, i.e. $\# f(\Omega) < \infty$

$$\int_\Omega f \, d\mu := \sum_{y \in f(\Omega)} y \cdot \mu(f^{-1}(y))$$

E.g. $f(\Omega) = \{1, 2\}$

$$\int f \, d\mu = 1 \cdot \mu(f^{-1}(1)) + 2 \cdot \mu(f^{-1}(2))$$

(2) All simple f, s.t. $f \le X$ take sup of their integrals

$$\int X \, d\mu = \sup_{\substack{f \le X \\ f \text{ simple}}} \left(\int f \, d\mu \right), \quad X \geq 0$$

(3) $X = \max\{X, 0\} - \max\{-X, 0\}$

$X \geq 0$: RHS $= X - 0 = X$

$X < 0$: RHS $= 0 - (-X) = X$

Define

$$\int_\Omega X \, d\mu = \underbrace{\int_\Omega \max\{X, 0\} \, d\mu}_{\text{has to be } < \infty} - \underbrace{\int \max\{-X, 0\} \, d\mu}_{\text{has to be } < \infty}$$

Variance

$$\mathrm{Var}_P(X) = \mathbb{E}_P[(X-\mathbb{E}_P[X])^2]$$

$$\mathrm{cov}_P(X,Y) = \mathbb{E}_P[(X-\mathbb{E}_P[X])\cdot(Y-\mathbb{E}_P[Y])]$$

$$\mathrm{cov}(X,X) = \mathrm{Var}(X)$$

If $\mathrm{cov}(X,Y) = 0 \Rightarrow$ The random variables X and Y are uncorrelated

Every RV (random variable) has so-called distribution f-n

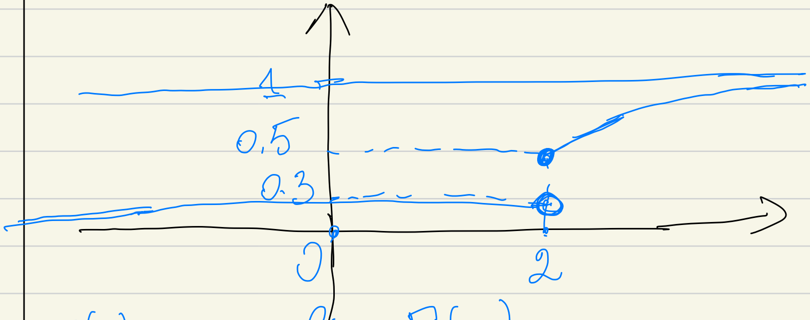
$$F(y) = \mathbb{P}(X \leq y)$$

Def-n: $F: \mathbb{R} \to [0,1]$ is called a distribution f-n

(i) F is non-decreasing ($x \leq y \Rightarrow F(x) \leq F(y)$)

(ii) $\lim\limits_{x\to+\infty} F(x) = 1$, $\lim\limits_{x\to-\infty} F(x) = 0$

(iii) F is càdlàg (i.e. right continuous, left limits)



$$F(2) = 0.5 = \lim_{x\searrow 2} F(x)$$

$$\lim_{x\nearrow 2} F(x) = 0.3 \neq F(2)$$

Examples: Discrete or continuous uniform distribution

Discrete: Finite number of possible events with the same probability

Continuous: $A \in \mathcal{B}(\mathbb{R})$ (Borel set) with

$$0 < \lambda(A) < \infty$$

^ Lebesgue-Borel measure on $\mathbb{R}$

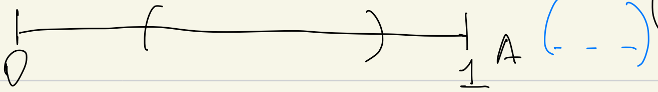
$$\mathcal{U}_A : \mathcal{B}(\mathbb{R}) \to [0,\infty]$$

$$\mathcal{U}_A(B) = \frac{\lambda(B \cap A)}{\lambda(A)}$$

Continuous uniform distribution on A

E.g. $A = (0,1)$, $\lambda(A) = 1$

$B = (\frac{1}{2}, \frac{3}{4})$



$$\mathcal{U}_A(B) = \lambda(B) = \frac{1}{4}$$

$$\mathcal{U}_A(A) = 1$$

② Normal distribution:

$$\mathcal{N}_{0,I_{\mathbb{R}^d}} : \mathcal{B}(\mathbb{R}^d) \to [0,1]$$

$I_{\mathbb{R}^d}$ – Identity matrix on $\mathbb{R}^d$

$$I_{\mathbb{R}^d} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$\mathcal{N}_{0,I_{\mathbb{R}^d}}(B) = \frac{1}{(2\pi)^{d/2}} \int_B e^{-\frac{1}{2}\|x\|^2} dx$$

If $v \in \mathbb{R}^d$ $\varrho \in \mathbb{R}^{d \times d}$ nonnegative symmetric

$$\mathcal{N}_{v,\varrho}(B) = \mathcal{N}_{0, I_{\mathbb{R}^d}}\left(x \in \mathbb{R}^d : \sqrt{\varrho}\, x + v \in B\right)$$

X is $\mathcal{N}_{v,\varrho}$ distributed

$$\mathbb{E}_P[X] = v$$

$X = (X^{(1)}, \dots, X^{(d)})$

$$\mathbb{E}[X] = \begin{pmatrix} \mathbb{E}[X^{(1)}] \\ \mathbb{E}[X^{(2)}] \\ \vdots \\ \mathbb{E}[X^{(d)}] \end{pmatrix} \in \mathbb{R}^d$$

$\mathrm{Var}(X) \in \mathbb{R}^{d \times d}$, $(\mathrm{Var}(X))_{i,j} =$

$= \mathrm{Cov}(X^{(i)}, X^{(j)})$

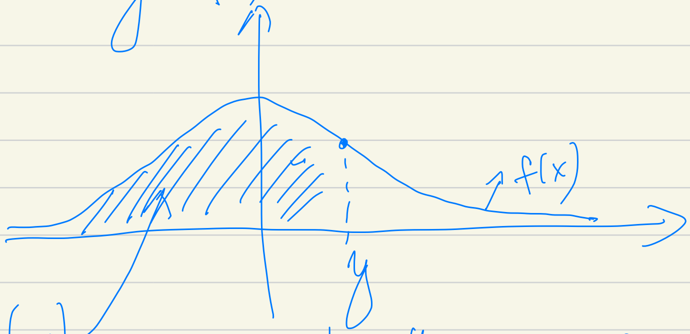
$$\mathrm{Var}(X) = \varrho \in \mathbb{R}^{d \times d}$$

Transformation formula

$A \in \mathbb{R}^{d \times d}$, $b \in \mathbb{R}^d$, $X \sim \mathcal{N}_{v,\varrho}$

$$AX + b \sim \mathcal{N}_{Av+b,\, A \cdot \varrho A^T}$$

Density $f(x)$



$F(y)$ – area under the curve f up to y.

Next time - generation of (pseudo) random numbers