NASDE 18.10.2024

Random number generators: ①

Def-n: Let $U_n : \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of independent $U_{(0,1)}$ distributed random variables

$U_{(0,1)}$ pseudo random numbers – sequence of numbers calculated via deterministic algorithm which are, in certain case, similar to $U_n, n \in \mathbb{N}$

- Marsaglia's Mother
- Mersenne Twister („rand" in Matlab)

X=rand$(10^6, 1)$ – 10^6 samples of $U_{(0,1)}$

$\text{mean}(X) \approx \frac{1}{2}$

Other distributions ②

$X \sim U_{(0,1)}$

$2 \cdot X \sim U_{(0,2)}$

$X + 2 \sim U_{(2,3)}$

Today:

Inversion method:

Def-n: Let $F : \mathbb{R} \to [0,1]$ be a distribution f-n. Then $I_F : (0,1) \to \mathbb{R}$ is given by

$$I_F(y) = \inf\{x \in \mathbb{R} : F(x) \geq y\}$$

$$= \inf(F^{-1}[y,1])$$

called the generalized inverse distribution f-n associated to F

Well-definedness of I_F:

$\lim_{x\to+\infty} F(x) = 1$, $y < 1$

$\Rightarrow \exists \tilde{x} \in \mathbb{R} : 1 \ge F(\tilde{x}) > y$

$\Rightarrow \{x \in \mathbb{R} : F(x) \ge y\} \ne \emptyset$

$\lim_{x\to-\infty} F(x) = 0$, $y > 0 \Rightarrow$

$\{x \in \mathbb{R} : F(x) \ge y > 0\}$ is bounded from below. ✓

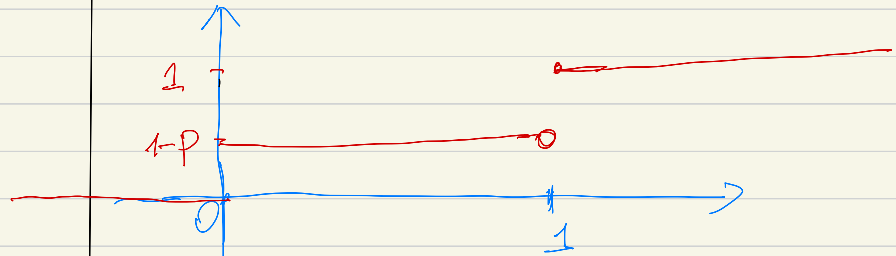
I_F – quantile function (in statistics)

Examples:

(1) Bernoulli d-n with parameter $p \in [0,1]$

$\mathbb{P}(X=1) = p$

$\mathbb{P}(X=0) = 1-p$



$F(y) = \mathbb{P}(X \le y)$

$\mathbb{P}(0 \le X < 1) = \mathbb{P}(X=0) = 1-p$

$$F(x) = \begin{cases} 0, & x < 0 \\ 1-p, & 0 \le x < 1 \\ 1, & x \ge 1. \end{cases}$$

$$I_F(y) = \begin{cases} 1 : & y \in (1-p, 1) \\ 0 : & 0 < y \le 1-p \end{cases}$$

$I_F(y) = \inf\{x \in \mathbb{R} : F(x) \ge y > 1-p\} = 1$

Lemma (Properties of I_F): (5)

(1) I_F is non-decreasing, i.e.

$$y_1 \leq y_2 \Rightarrow I_F(y_1) \leq I_F(y_2)$$

(2) $\forall y \in (0,1)$: $F(I_F(y)) \geq y$

$$I_F(y) = \min\left(F^{-1}[y,1]\right)$$

(3) $\forall x \in F^{-1}((0,1)) = \{z \in \mathbb{R} : F(z) \in (0,1)\}$

$$I_F(F(x)) \leq x$$

(4) Switching formula:

$$I_F(y) \leq x \iff y \leq F(x)$$

(5) $I_F(y) > x \iff y > F(x)$

(6) let $D \subseteq \mathbb{R}$ be open $F|_D : D \to [0,1]$ is injective, then (6)

$$\forall y \in F(D) \Rightarrow F|_D^{-1}(y) = I_F(y)$$

Theorem (Inversion method)

let $X : \Omega_1 \to \mathbb{R}$ be a RV with distribution F, let $\mathcal{U} \sim \mathcal{U}_{(0,1)}$ with $\mathcal{U}(\Omega) \subseteq (0,1)$

$$\Big[\, P(\underbrace{\{\omega \in \Omega_1 : \mathcal{U}(\omega) \in (2,3)\}}_{\neq \emptyset \text{ (does not have to be } \emptyset)}) = 0 \,\Big]$$

let

$$\hat{X} \doteq I_F \circ \mathcal{U} : \Omega_1 \to \mathbb{R}$$

Then

X and $\hat{X}$ have the same dis-n!

$$X \overset{d}{\sim} \hat{X}$$

Proof of the inversion method: (7)

$$\hat{F}(y) = \mathbb{P}(\hat{X} \leq y) = \mathbb{P}(I_F(U) \leq y)$$

(4) switching formula

$$\overset{(4)}{=} \mathbb{P}(U \leq F(y)) \overset{U \sim \mathcal{U}_{(0,1)}}{=} F(y)$$

$\blacksquare$

Algorithm: (1) Calculate I_F

(2) Generate $U \sim \mathcal{U}_{(0,1)}$

$I_F(U) \sim X$ (with dist-n F)

Proof of Lemma:

(1) Let $y_1 \leq y_2 \Rightarrow$

$$[F(x) \geq y_2] \Rightarrow [F(x) \geq y_1]$$

$$\{x \in \mathbb{R} : F(x) \geq y_1\} \supseteq \{x \in \mathbb{R} : F(x) \geq y_2\}$$

$$\underbrace{\inf\{x \in \mathbb{R} : F(x) \geq y_1\}}_{I_F(y_1)} \leq \underbrace{\inf\{x \in \mathbb{R} : F(x) \geq y_2\}}_{I_F(y_2)}$$

$\blacksquare$

(2) Let us take non-decreasing (8)

$\{x_n\}_{n \in \mathbb{N}} \subseteq F^{-1}([y,1])$

$$\lim_{n \to \infty} x_n = \inf\{F^{-1}([y,1])\}$$

$x_n \in F^{-1}([y,1])$, F is non-decr.

$$F(x_n) \geq y \quad \Rightarrow$$

$$y \leq \lim_{n \to \infty} F(x_n) = F(\lim_{n \to \infty} x_n)$$

limit of F from right, F is right cont.

$$= F(I_F(y))$$

$I_F(y) = \min\{F^{-1}[y,1]\}$ follows from righ cont of F.

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③ $\forall x \in F^{-1}((0,1))$

$$I_F(F(x)) = \inf\{z \in \mathbb{R};\ F(z) \le F(x)\}$$

$z = x$, $x \in$

$\le x$

④ Switching formula:

$$I_F(y) \le x$$

$$y \overset{(2)}{\le} F(I_F(y)) \le F(x) \Rightarrow$$

F is non-decr.

$$y \le F(x)$$

Now let $y \le F(x)$

$$I_F(y) \overset{①}{\le} I_F(F(x)) \overset{③}{\le} x \Rightarrow$$

$$I_F(y) \le x$$

$$\{I_F(y) \le x\} \Leftrightarrow \{y \le F(x)\}$$

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⑤ is similar to ④

⑥ Try yourself.

Remark: I_F is not, in general, contin. However, càglàd (left continuous, with right limits)

Examples:

ⓐ Exponential dist-n with parameter λ:

$$X \sim Exp_\lambda, \quad \lambda \in (0,\infty)$$

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-\lambda x}, & \text{if } x \ge 0 \end{cases}$$

$F|_{(0,\infty)}$ is injective

$$y = 1 - e^{-\lambda x} \Rightarrow e^{-\lambda x} = 1 - y \Rightarrow$$

$$x = -\frac{\ln(1-y)}{\lambda}$$

$$F\big|_{(0,\infty)}^{-1}(y) = -\frac{\ln(1-y)}{\lambda}$$ 11

6 in Lemma $\Rightarrow$ $\forall y \in F((0,\infty)) = (0,1)$

$$I_F(y) = -\frac{\ln(1-y)}{\lambda} \quad \Rightarrow$$

$$X \sim I_F(u) = -\frac{1}{\lambda}\ln(1-u)$$

$$\sim -\frac{1}{\lambda}\ln(u)$$