NASDE 22.10.2024

Today: more examples on inversion (1)

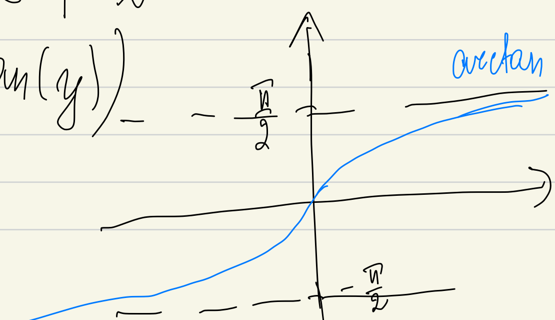
- Acceptance-rejection method
- Methods for normal distribution

(b) Cauchy distribution:

density: $f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$, $x \in \mathbb{R}$

$$F(x) = \int_{-\infty}^{x} f(y)\,dy = \frac{1}{\pi} \int_{-\infty}^{x} \frac{1}{1+y^2}\,dy$$

$$= \frac{1}{\pi} \arctan(y) \Big|_{-\infty}^{x} = \frac{1}{\pi}\left(\arctan(x) - \lim_{y \to -\infty} \arctan(y)\right)$$



(2)

$$= \frac{1}{\pi}\left(\arctan(x) + \frac{\pi}{2}\right)$$

$$= \frac{1}{\pi} \arctan(x) + \frac{1}{2}$$ – injective

$$I_F(y) = F^{-1}(y) = \tan\left(\pi\left(y - \frac{1}{2}\right)\right)$$

$y = \frac{1}{\pi}\arctan(x) + \frac{1}{2}$, $\pi\left(y - \frac{1}{2}\right) = \arctan(x)$

Cauchy d-n $\sim \tan\left(\pi\left(U - \frac{1}{2}\right)\right)$

(c) Discrete d-n:

$\mathbb{P}(X = n) = p_n$, $\forall n = 0, 1, \ldots$

$\sum_{n=0}^{\infty} p_n = 1$, $p_n \in [0, 1]$

$$F(x) = \mathbb{P}(X \le x) \qquad k \le x < k+1$$

$$= \mathbb{P}(X < k+1)$$

$$= \mathbb{P}(\{X=0\} \cup \{X=1\} \cup \dots \cup \{X=k\})$$

$$= P(X=0) + P(X=1) + \dots + P(X=k)$$

$$= \sum_{n=0}^{k} p_n$$

$$F(x) = \sum_{n=0}^{\lfloor x \rfloor} p_n$$

$\lfloor x \rfloor$ - largest integer smaller than x

$\lfloor 1{,}2 \rfloor = 1$, $\lfloor 1{,}9 \rfloor = 1$, $\lfloor 2 \rfloor = 2$

$\lceil x \rceil$; $\lceil 1{,}2 \rceil = 2$ ← round up

$$I_F(y) = \inf\{x \in \mathbb{R} : F(x) \ge y\}$$

F is piecewise constant

$$= \min\{k \in \mathbb{N}_0 : F(k) \ge y\}$$

$$= \min\{k \in \mathbb{N}_0 : \sum_{n=0}^{k} p_n \ge y\}$$

Bernoulli d-n:

$p_0 = 1-p$, $p_1 = p$, $p_2 = p_3 = \dots = 0$

$$I_F(y) = \begin{cases} 0, & \text{if } 0 < y \le p_0 \\ 1, & \text{if } 1-p < y < 1 \end{cases}$$

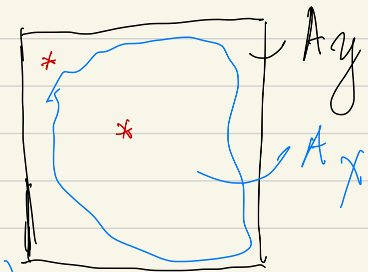
Algoritm:

Generate $U \sim \mathcal{U}(0,1)$

$$X = \begin{cases} 0, & \text{if } U \le 1-p \\ 1, & \underbrace{\text{if } 1-p < U < 1}_{= \text{otherwise}} \end{cases}$$

- Acceptance - rejection method: (5)

Idea - „Generate complex d-n from a simpler d-n"



Let $Z_x \sim \mathcal{U}(A_x)$

$Z_y \sim \mathcal{U}(A_y)$

where $A_x \subseteq A_y$

$$0 < \lambda(A_x) \le \lambda(A_y) < \infty$$

(6)

Algorithm:

- Generate a realization $Z_y \sim \mathcal{U}(A_y)$

If $Z_y \in A_x$

$Z_x = Z_y$ ← accept

if not restart

Lemma (A-R on $A_x \subseteq A_y$)

$$Z_x \sim \mathcal{U}(A_x)$$

Proof: Take $\forall B \in \mathcal{B}(\mathbb{R}^2)$. Then

$$P(Z_x \in B) = P(Z_y \in B \mid Z_y \in A_x) = \frac{P(Z_y \in B \cap A_x)}{P(Z_y \in A_x)}$$

(7)

$$P(\tilde{z}_y \in A_x) = \frac{1}{\lambda(A_y)} \int_{A_x} 1 \, dz \qquad \tilde{z}_y \sim U(A_y)$$

$$= \frac{\lambda(A_x)}{\lambda(A_y)}$$

$$P(z_y \in B \cap A_x) = \frac{1}{\lambda(A_y)} \cdot \int_{B \cap A_x} 1 \, dz$$

$$= \frac{1}{\lambda(A_y)} \cdot \int_{B} 1_{A_x}(z) \, dz$$

here

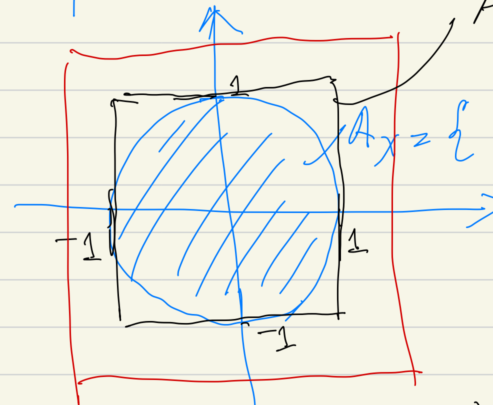
$$1_A(y) = \begin{cases} 1, & \text{if } y \in A \\ 0, & \text{otherwise} \end{cases}$$

$$\frac{P(z_y \in B \cap A_x)}{P(z_y \in A_x)} = \frac{1}{\lambda(A_x)} \int_B 1_{A_x}(z) \, dz$$

(8)

$$= P(U(A_x) \in B) \qquad \blacksquare$$

Example:



$A_y = [-1,1] \times [-1,1]$

$A_x = \{ (x_1, x_2) \in \mathbb{R}^2 \; x_1^2 + x_2^2 \le 1 \}$

Generate $U = (U_1, U_2) \sim U(A_y)$

Check: $U_1^2 + U_2^2 \le 1$

If yes accept, otherwise restart

→ Extensions to non-uniform dists (under certain conditions) also exist.

9

Normally d-d random variables

→ Inversion method

$\Phi^{-1}(u)$ ← no analytic formula

In Matlab „erfinv" – costly to compute

– Central limit theorem

If $(Y_n, n\in\mathbb{N})$ are iid (independent identically d-d) with

$\mathbb{E}[|Y_1|^2] < \infty$ and $Var(Y_1) > 0$

then

$$\frac{\sum_{n=1}^{N} Y_n - N\cdot\mathbb{E}[Y_1]}{\sqrt{N}\cdot\sqrt{Var(Y_1)}} \xrightarrow[N\to\infty]{d-n} \mathcal{N}(0,1)$$

convergence in d-n.

10

$Y_1 \sim \mathcal{U}(0,1)$

Choose N ~ large

$U_1, \ldots, U_N \sim \mathcal{U}(0,1)$

$$\frac{\sum_{n=1}^{N} U_n - N\cdot\frac{1}{2}}{\sqrt{N\cdot\frac{1}{12}}}$$ is close in d-n as $\mathcal{N}(0,1)$

→ However, not exact.

→ Box-Muller method

Let us assume $X_1, X_2 \sim \mathcal{N}(0,1)$ iid

$X = (X_1, X_2) \overset{\text{I}}{\sim} \mathcal{N}(0, I_{\mathbb{R}^2})$

$g : \mathbb{R}^2 \to \mathbb{R}$ measurable: (11)

$$\mathbb{E}[g(X_1, X_2)] = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} g(x,y) \cdot e^{-\frac{x^2+y^2}{2}} dx dy$$

Change coordinates to Polar coordinates

$(x,y) = (r\cos\varphi, r\sin\varphi)$

$dx\,dy = r\,dr\,d\varphi$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} g(r\cos\varphi, r\sin\varphi) \cdot e^{-\frac{r^2}{2}} \cdot r\,dr\,d\varphi$$

Again: $s = \frac{r^2}{2}$, $ds = r\,dr$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} g(\sqrt{2s} \cdot \cos\varphi, \sqrt{2s}\sin\varphi) \cdot e^{-s} ds\,d\varphi$$

Again: (12)

$u_1 = e^{-s}$, $du_1 = -e^{-s} ds$

$s = -\ln(u_1)$ $[e^{-s} \xrightarrow[s\to\infty]{} 0]$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{1} g\left(\sqrt{-2\ln(u_1)} \cdot \cos\varphi, \sqrt{-2\ln(u_1)} \sin\varphi\right) du_1\,d\varphi$$

Again (last one): $u_2 = \frac{1}{2\pi}\varphi$

$du_2 = \frac{1}{2\pi} d\varphi$

$$= \frac{1}{\cancel{2\pi}} \int_0^{1} \int_0^{1} g\left(\sqrt{-2\ln(u_1)} \cdot \cos(2\pi u_2), \sqrt{-2\ln(u_1)} \sin(2\pi u_2)\right) \cancel{2\pi}\, du_1\,du_2$$

13

If $f:(0,1)\times(0,1)\to\mathbb{R}^2$

$$f(U_1,U_2)=\left(\sqrt{-2\ln(U_1)}\cos(2\pi U_2),\ \sqrt{-2\ln(U_1)}\cdot\sin(2\pi U_2)\right)$$

$= \mathbb{E}[g(f(\tilde{U}_1,\tilde{U}_2))]$ where

$\tilde{U}_1,\tilde{U}_2 \sim \mathcal{U}_{(0,1)}$ iid.

We have showed

$$\mathbb{E}[g(X_1,X_2)]=\mathbb{E}[g(f(\tilde{U}_1,\tilde{U}_2))]$$

$\forall g$ measurable.

$\forall B\in\mathcal{B}(\mathbb{R}^2)$, $g=\mathbb{1}_B$ -measurable

14

$\Rightarrow$

$$P((X_1,X_2)\in B)=P(f(\tilde{U}_1,\tilde{U}_2)\in B)\Rightarrow$$

$$\overbrace{(X_1,X_2)}^{\mathcal{N}(0,I_{\mathbb{R}^2})}\sim f(\tilde{U}_1,\tilde{U}_2)$$

Theorem (Box-Muller method)

Let $U_1,U_2:\Omega\to\mathbb{R}$ be iid $\mathcal{U}_{(0,1)}$

Define $X=(X_1,X_2):\Omega\to\mathbb{R}^2$ by

$$X=f(U_1,U_2)$$

Then

$$X\sim\mathcal{N}(0,I_{\mathbb{R}^2})$$

Next method;

Marsaglia Polar: uses the following theorem: let $f:\mathbb{R}^2\to\mathbb{R}^2$ given by

$$f(x_1,x_2) = \begin{pmatrix}x_1\\ x_2\end{pmatrix}\cdot\frac{1}{\underbrace{\sqrt{x_1^2+x_2^2}}_{\|x\|_2}}\sqrt{-2\ln(x_1^2+x_2^2)}$$

let $U:\Omega\to\mathbb{R}^2$ be $\mathcal{U}(\dot{B}(0,1))$ -d-d where

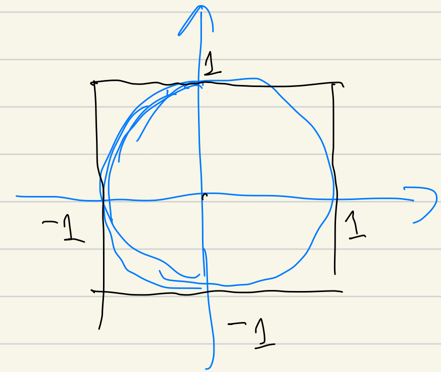
$\dot{B}(0,1)=\{(x_1,x_2)\in\mathbb{R}^2 : x_1^2+x_2^2\in(0,1)\}$

(open ball without the centre)

Then

$$X=f(U)\sim\mathcal{N}(0,I_{\mathbb{R}^2})$$

A-R.



Generate $U_1, U_2\sim\mathcal{U}(0,1)$

$U_1=2U_1-1$, $U_2=2U_2-1$

$(U_1, U_2\sim\mathcal{U}(-1,1))$

$q=U_1^2+U_2^2$

until $q\in(0,1)$

$w=\sqrt{-2\ln q/q}$

$X_1=U_1\cdot w$; $X_2=U_2\cdot w$