NASDE 25.10.2024

(1)

Today: → Proof of Marsaglia Polar.

→ General normal d-n

→ Integration methods

Proof of Marsaglia Polar:

Take any measurable $g: \mathbb{R}^2 \to \mathbb{R}$

$$\mathbb{E}[g(X)] = \mathbb{E}[(g \circ f)(U)]$$

$$= \frac{1}{\lambda(B(0,1))} \iint_{B(0,1)} g\underbrace{\left(\frac{u_1}{\sqrt{u_1^2+u_2^2}} \cdot \sqrt{-2\ln(u_1^2+u_2^2)}, \frac{u_2}{\sqrt{u_1^2+u_2^2}} \cdot \sqrt{-2\ln(u_1^2+u_2^2)}\right)}_{f(u_1, u_2)} du_1 du_2$$

$\lambda(B(0,1)) = \pi$

Change to Polar coordinates: $(u_1, u_2) = (r\cos\varphi, r\sin\varphi)$, $u_1^2+u_2^2 = r^2$

$du_1 du_2 = r\,dr\,d\varphi$

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$$= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 g\left(\frac{r\cos\varphi}{r} \cdot \sqrt{-2\ln(r^2)}, \frac{r\sin\varphi}{r} \sqrt{-2\ln(r^2)} \right) r\,dr\,d\varphi$$

$$= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 g\left(\cos\varphi \cdot 2 \cdot \sqrt{-\ln(r)}, \sin\varphi \cdot 2\sqrt{-\ln(r)} \right) r\,dr\,d\varphi$$

Change of variables : $s = -\ln(r) \Rightarrow r = e^{-s}$, $dr = -e^{-s} ds \Rightarrow$

$$= \frac{1}{\pi} \int_0^{2\pi} \int_0^{+\infty} g\left(\cos\varphi \cdot 2 \cdot \sqrt{s}, \sin\varphi \cdot 2\sqrt{s} \right) e^{-2s}\, ds\, d\varphi$$

COV: $2\sqrt{s} = \tilde{r} \rightarrow d\tilde{r} = \frac{1}{\sqrt{s}} ds$

$ds = \sqrt{s}\, d\tilde{r} = \frac{\tilde{r}}{2} d\tilde{r}$

$$= \frac{1}{\pi} \int_0^{2\pi} \int_0^{+\infty} g\left(\cos\varphi \cdot \tilde{r}, \sin\varphi \cdot \tilde{r} \right) e^{-\frac{1}{2}\tilde{r}^2} \cdot \frac{\tilde{r}}{2} \cdot d\tilde{r}\, d\varphi$$

$(\tilde{r}\cos\varphi, \tilde{r}\sin\varphi) \rightarrow (x_1, x_2)$

$\tilde{r}\, d\tilde{r}\, d\varphi = dx_1\, dx_2$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} g(x_1, x_2) \cdot e^{-\frac{1}{2}(x_1^2 + x_2^2)} \cdot dx_1\, dx_2 = \mathbb{E}[g(Y_1, Y_2)], \quad (Y_1, Y_2) \sim \mathcal{N}(0, I_{2\times 2})$$

$$\mathbb{E}[g(X)] = \mathbb{E}[g(Y_1, Y_2)]$$

$$(Y_1, Y_2) \sim \mathcal{N}(0, I_{\mathbb{R}^2})$$

$\forall g$ - measurable $\Rightarrow$

$$X = (X_1, X_2) \sim \mathcal{N}(0, I_{\mathbb{R}^2}) \quad \blacksquare$$

Generate normal d-n with general mean and variance:

If $X \sim \mathcal{N}(0, I_{\mathbb{R}^d})$, $b \in \mathbb{R}^d$, $A \in \mathbb{R}^{d \times d}$

$$AX + b \sim \mathcal{N}_{b, AA^T}$$

Assume: $b \in \mathbb{R}^d$, $Q \in \mathbb{R}^{d \times d}$ strictly positive symmetric

Write

$Q = L L^T$ - Cholesky decomps. „chol" in Matlab.

Set $\hat{X} = L \cdot X + b \Rightarrow$

$$\hat{X} \sim \mathcal{N}(b, LL^T) = \mathcal{N}(b, Q)$$

Integration methods:

In the context of SDEs we are interested in approximations

$X^n : [0,T] \times \Omega \to \mathbb{R}$ for

$X : [0,T] \times \Omega \to \mathbb{R}$ sol-n process of SDE.

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- pathwise approximation:
  Fix $\omega \in \Omega$.

$X^n(\omega) : [0,T] \to \mathbb{R}$ to be close to $X(\omega) : [0,T] \to \mathbb{R}$

- approximation of statistical values

$\mathbb{E}[X]$, $Var(X)$, ...

$\Rightarrow$ numerical integration (approximation of integrals)

$\to$ deterministic numerical integration

(usually suffer from the curse of dimensionality)

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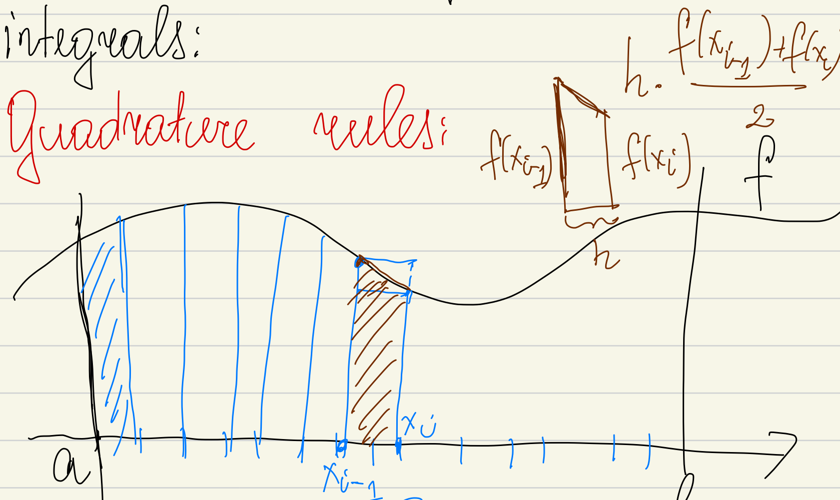
$\to$ Monte Carlo methods

$\to$ usually better alternative in high dim-ds

$\to$ can also be used for deterministic integrals:

Quadrature rules:



Goal: Compute $\int_{[a,b]} f(x)\,dx$

A quadrature rule Q on $A \subseteq \mathbb{R}^d$ is given by

- $X = \{x_i;\ i \in I\}$, $\# I < \infty$ nodes

∘ weights $w = \{w_i, i \in I\}$

We approximate $\int_A f(x)\,dx$ by

$$Q[f] := \sum_{i\in I} w_i \cdot f(x_i) \approx \int_A f(x)\,dx$$

a) Left rectangle method

Equidistant nodes $h = \frac{b-a}{n}, \ n\in\mathbb{N}$

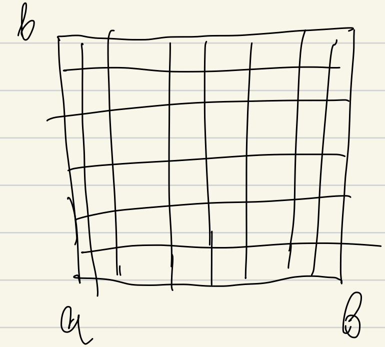
Nodes: $x = \{a, a+h, a+2h, \dots, a+(n-1)\cdot h\}$
$= \{(a+ih), \ i=0,\dots,n-1\}$

[Right rectangle: $x = \{a+h, a+2h, \dots, a+n\cdot h = b\}$]

Equal weights $w_1 = w_2 = \dots = w_n = h \Rightarrow$

$$R^n_{[a,b]}[f] = h\sum_{i=0}^{n-1} f(a+i\cdot h)$$

In 2d: $[a,b]\times[a,b]$



n^2 nodes

Note: in case of d-dimension:

$x = \{\underbrace{(a+i_1\cdot h, a+i_2\cdot h, \dots, a+i_d\cdot h)}_{x_{i_1 i_2 \dots i_d}}, \ i_1, i_2, \dots, i_d \in \{0,1,\dots,n-1\}\}$

Weights:

$w_{i_1,i_2,\dots,i_d} = h^d, \ i_1,i_2,\dots,i_d \in \{0,\dots,n-1\}$

$$R^n_{[a,b]^d} = h^d\left(\sum_{i_1=0}^{n-1}\dots\sum_{i_d=0}^{n-1} f(x_{i_1,i_2,\dots,i_d})\right)$$

B) Trapezoidal rule: (9)

$$\mathbb{T}^{h}_{[a,b]}[f] := \sum_{i=0}^{n-1} h \cdot \frac{f(a+ih)+f(a+(i+1)h)}{2}$$

$$= h \cdot \left(\sum_{i=0}^{n-1} \frac{f(a+ih)+f(a+(i+1)h)}{2} \right)$$

$$= h \left(\frac{f(a)+f(a+h)}{2} + \frac{f(a+h)+f(a+2h)}{2} + \dots \right.$$

$$\left. + \frac{f(a+(n-1)h)+f(a+nh)}{2} \right)$$

$$= h \cdot \left(\frac{f(a)+f(b)}{2} + \sum_{i=1}^{(n-1)} f(a+ih) \right)$$

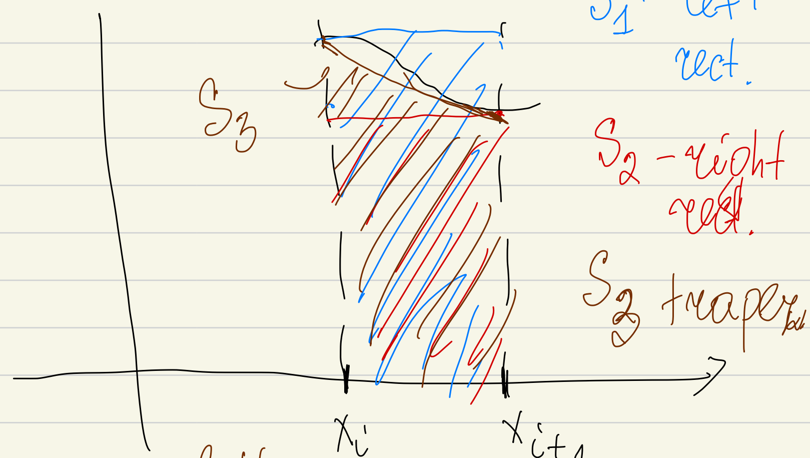
$$h = \frac{b-a}{n}$$

(10)

$$x = \{ a+ih,\ i=0,\dots,n \}$$

$$w_0 = w_n = \frac{h}{2}$$

$$w_1 = w_2 = \dots = w_{n-1} = h$$



S_3 is a better approximation than S_1 and S_2.

To analyze the errors we need (11)
notions of modulus of continuity
and Hölder continuity

Def-n: Let (E, d_E) and (F, d_F) be metric spaces and $f: E \to F$

$\omega_f : [0,\infty] \longmapsto [0,\infty]$ is modulus of continuity of f given by

$$\omega_f(h) = \sup_{\substack{x,y\in E \\ d_E(x,y)\le h}} d_F(f(x), f(y))$$

Remark: f is uniformly continuous

$$\Longleftrightarrow \quad \omega_f(h) \xrightarrow[h\searrow 0]{} 0$$

will be in Ex sheet 3

Def-n: Let $\alpha \in [0,1]$ (12)

$$|f|_{C^\alpha(E,F)} = \sup_{\substack{x,y\in E \\ x\neq y}} \frac{d_F(f(x), f(y))}{|d_E(x,y)|^\alpha} \in [0,\infty]$$

$C^\alpha(E,F) = \{ f : E \to F$, such that $|f|_{C^\alpha(E,F)} < \infty \}$

Space of Hölder cont. f-ns

Remark:

$$|f|_\alpha = \sup_{h\in(0,\infty)} \left[\frac{\omega_f(h)}{h^\alpha} \right]$$

$\alpha = 1$: Lipschitz continuity.

Next time:
Analyze error of $R^n[f]$