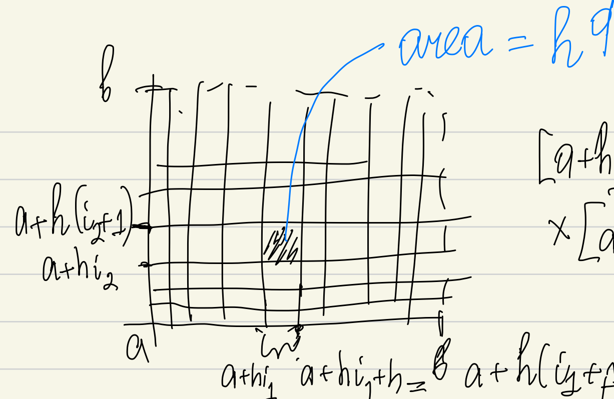
NASDE 29.10.2024

Today: Error of the d-dimensional left rectangle method.



$[a+hi_1, a+h(i_1+1)] \times [a+hi_2, a+h(i_2+1)]$

Prop-n: Let $d \in \mathbb{N}$, $n \in \mathbb{N}$, $\alpha \in (0,1]$, $a < b$, $f \in \mathcal{L}^1([a,b]^d, \mathbb{R})$ (f is measurable and $\int_{[a,b]^d} |f(x)|\,dx < \infty$). Then

$$\left| R^n_{[a,b]^d}[f] - \int_{[a,b]^d} f(x)\,dx \right| \le (b-a)^d \cdot w_f\left(\frac{(b-a)\sqrt{d}}{n}\right) \le \frac{(b-a)^{d+\alpha} \cdot d^{\alpha/2} \cdot \|f\|_{C^\alpha}}{n^\alpha}$$

$f \in C^\alpha \Rightarrow$ convergence rate of $\alpha \in (0,1]$. Convergence rate up to 1.

Proof: $\mathbb{1} \in \mathbb{R}^d$, $\mathbb{1} = (1, \dots, 1)$ and $h = \frac{b-a}{n}$. Then

$$\left| R^n_{[a,b]^d} - \int f(x)\,dx \right| = \left| h^d \sum_{\substack{i=(i_1,\dots,i_d) \\ \in \{0,1,\dots,(n-1)\}^d}} f(a \cdot \mathbb{1} + h \cdot i) - \int_{[a,b]^d} f(x)\,dx \right|$$

$$\int_{[a,b]^d} f(x)\,dx = \sum_{i=(i_1,\dots,i_d)} \int_{[a+hi_1, a+h(i_1+1)] \times \dots} f(x)\,dx$$

$$= \left| \sum_{\substack{i=(i_1,\dots,i_d) \\ \in\{0,1,\dots,(n-1)\}^d}} \int_{[a+hi_1,\,a+h(i_1+1)]\times\dots\times[a+hi_d,\,a+h(i_d+1)]} \left[f(a\cdot\mathbb{1}+h\cdot i) - f(x) \right] dx \right|$$

triangle inequality for $|\cdot|$ $\Rightarrow$

$$\leq \sum_{i=(i_1,\dots,i_d)} \int_{[\,,\,]\times\dots[\,]} \left| f(a\cdot\mathbb{1}+h\cdot i) - f(x) \right| dx$$

$$\leq \sum_i \int w_f(\|a\cdot\mathbb{1}+h\cdot i - x\|)\, dx$$

$|f(x)-f(y)|$

$$w_f(\|x-y\|) = \sup_{\substack{z_1,z_2 \\ \|z_1-z_2\|\leq\|x-y\|}} |f(z_1)-f(z_2)| \geq |f(x)-f(y)|$$

$$a\cdot\mathbb{1}+h\cdot i - x = \begin{pmatrix} a \\ a \\ \vdots \\ a \end{pmatrix} + h\begin{pmatrix} i_1 \\ \vdots \\ i_d \end{pmatrix} - \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} a+hi_1-x_1 \\ \vdots \\ a+h\cdot i_d - x_d \end{pmatrix}$$

$$\|\dots\| = \sqrt{(a+hi_1-x_1)^2+\dots+(a+h\cdot i_d-x_d)^2} \leq \sqrt{(h)^2+\dots+(h)^2} = h\cdot\sqrt{d}$$

ω_f is non-decreasing $\Rightarrow$

$$\le \sum_{i \in G} \int \omega_f(h \cdot \sqrt{d})\, dx = \sum_{i \in G} \omega_f(h \cdot \sqrt{d}) \cdot h^d =$$

$$= n^d \cdot \omega_f(h \cdot \sqrt{d}) \cdot h^d = \cancel{n^d} \cdot \frac{(b-a)^d}{\cancel{n^d}} \cdot \omega_f(h \cdot \sqrt{d}) = (b-a)^d \cdot \omega_f(h \cdot \sqrt{d})$$

$$\le (b-a)^d \cdot h^\alpha \cdot (\sqrt{d})^\alpha \cdot |f|_{C^\alpha} = (b-a)^d \cdot \frac{(b-a)^\alpha}{n^\alpha} \cdot d^{\alpha/2} \cdot |f|_{C^\alpha}$$

$$= \frac{(b-a)^{d+\alpha} \cdot d^{\alpha/2} \cdot |f|_{C^\alpha}}{n^\alpha} \qquad \blacksquare$$

$\alpha \in (0,1]$.

Q: Can we improve the rate 1 by assuming more regularity on f?

Ans.: No, $f \in C^\infty$ given by $f(x_1, \dots, x_d) = x_1$, $f: \mathbb{R}^d \to \mathbb{R}$,

$$\left| R^n[f] - \int_{[0,1]^d} f(x)\, dx \right| = \frac{1}{2n} \qquad \blacksquare$$

Trapezoidal method: the regularity of f improves the rate up to 2:

If $f \in C^1$ then

$$\left| \mathbb{T}^n_{[a,b]}[f] - \int_{[a,b]} f(x)\,dx \right| \leq \frac{(b-a)^{1+\alpha} \cdot |f|_{C^\alpha}}{(2n)^\alpha}, \quad \alpha \in (0,1]$$

$\hookrightarrow$ conv. rate up to order 1.

If $f \in C^1([a,b], \mathbb{R})$ then

$$\left| \mathbb{T}^n[f] - \int f(x)\,dx \right| \leq \frac{(b-a)^{2+\alpha} \cdot |f'|_{C^\alpha}}{2^\alpha \cdot n^{1+\alpha}} \qquad \alpha \in (0,1]$$

$\rightarrow$ conv. rate up to 2.

Proof: Part of Sheet 3.

Can we improve the rate 2? $\Rightarrow$ Part of Sheet 2, No.

$N = n^d$ - number of quadrature nodes $\quad n = N^{\frac{1}{d}}$

$$\left| \mathbb{R}^n[f] - \int f(x)\,dx \right| \leq C \cdot N^{-\frac{\alpha}{d}} \cdot |f|_{C^\alpha}$$

conv. rate $\frac{\alpha}{d}$ wrt N!

→ is small if d is large

N - # of evaluations of f we have to do in order to compute $\mathbb{R}^n$

→ Low speed of convergence if d is large (sparse grids)

→ Monte Carlo methods are better alternative in high dim-ns

Idea: Interpret the integral

$$\int_A f(x)dx \quad , A \subseteq \mathbb{R}^d \text{ bounded}, \ f: \mathbb{R}^d \to \mathbb{R} \text{ integrable}$$

as an expectation of a random variable:

$$\int_A f(x)dx = \lambda(A) \cdot \int f(x) \cdot \underbrace{\frac{1}{\lambda(A)}}_{\text{density of } U \sim U_A} dx = \lambda(A) \cdot \mathbb{E}[f(U)] =$$

$$= \mathbb{E}[\underbrace{\lambda(A) \cdot f(U)}_{= Y: \Omega \to \mathbb{R}}] = \mathbb{E}[Y]$$

Def-n A Random variable $X \in \mathcal{L}^1(\mathbb{P};\mathbb{R})$ $[\mathbb{E}[|X_1|]<\infty]$ is said to be a Monte Carlo approximation of a constant $c \in \mathbb{R}$ if

$\exists N \in \mathbb{N}$, $Z_1, \dots, Z_N \in \mathcal{L}^1$ iid,

$\mathbb{E}[Z_1] = c$ and

$$X = \frac{1}{N} \sum_{i=1}^{N} Z_i$$

Example:

$$\int_{-1}^{1} \exp(\sqrt{|x|})\,dx = 2 \cdot \int_{-1}^{1} \exp(\sqrt{|x|}) \cdot \frac{1}{2}\,dx$$

$$= \mathbb{E}\left[2 \cdot \exp(\sqrt{|U|})\right], \quad U \sim U_{(-1,1)}$$

Let $Y_1, \dots, Y_N \sim U_{(-1,1)}$ iid

then

$$\frac{2}{N} \cdot \sum_{i=1}^{N} \exp(\sqrt{|Y_i|})$$

a Monte Carlo approx. of

$$\int_{-1}^{1} \exp(\sqrt{|x|})\,dx$$

is unbiased (X is unbiased wrt c)

$\mathbb{E}[X] = c$

Remark: MC approximation is unbiased, i.e.

$$\mathbb{E}[X] = c$$

Proof: By the linearity of expectation:

$$\mathbb{E}[X] = \mathbb{E}\left[\frac{1}{N}\cdot\sum_{i=1}^{N} z_i\right]$$

$$= \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[z_i] = \frac{1}{N}\cdot N\cdot c$$

$$= c.$$

Remark: In case of MC app-n of variance one has 2 versions:

1 is biased, 1 is unbiased. However, they have the same conv. rate. ◁

Convergence rate:

Theorem: If $X_n \in \mathcal{L}^2(\mathbb{P};\mathbb{R}),\ n\in\mathbb{N}$ $[\mathbb{E}[|X_n|^2]<\infty]$ are iid

$$\left\|\frac{1}{N}\sum_{i=1}^{N} X_i - \mathbb{E}[X_1]\right\|_{\mathcal{L}^2}$$

$$= \frac{\sqrt{\mathrm{Var}(X_1)}}{\sqrt{N}}$$

$$\left[\|Z\|_{\mathcal{L}^2(\mathbb{P};\mathbb{R})} = \left(\mathbb{E}[|Z|^2]\right)^{1/2}\right]$$

we have conv. rate of $\frac{1}{2}$.

Lemma: Let $X_1, \dots, X_N \in \mathcal{L}^2$.

$$\mathrm{Var}\left(\sum_{i=1}^{N} X_i\right) = \sum_{i=1}^{N}\sum_{j=1}^{N} \mathrm{cov}(X_i, X_j)$$

$$= \sum_{i=1}^{N} \mathrm{Var}(X_i) + \sum_{i=1}^{N}\sum_{j\neq i} \mathrm{cov}(X_i, X_j)$$

If X_i's are uncorrelated $\Rightarrow$ ($\mathrm{cov}(X_i, X_j) = 0$)

$$\mathrm{Var}\left(\sum_{i=1}^{N} X_i\right) = \sum_{i=1}^{N} \mathrm{Var}(X_i)$$

Proof:

$$\mathrm{Var}\left(\sum_i X_i\right) \overset{\text{def}}{=} \mathbb{E}\left[\left(\sum_i X_i - \mathbb{E}\left[\sum_i X_i\right]\right)^2\right]$$

$$= \mathbb{E}\left[\left(\sum_i X_i - \sum_i \mathbb{E}[X_i]\right)^2\right]$$

$$= \mathbb{E}\left[\left(\sum_i (X_i - \mathbb{E}[X_i])\right)^2\right]$$

$$\left(\sum_i a_i\right)^2 = \sum_i \sum_j a_i \cdot a_j$$

$$= \sum_i \sum_j \underbrace{\mathbb{E}\left[(X_i - \mathbb{E}[X_i])\cdot(X_j - \mathbb{E}[X_j])\right]}_{\mathrm{cov}(X_i, X_j)}$$

$\blacksquare$

Proof of the error rate of MC:

$$\mathbb{E}[X_1] = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} X_i\right] \Rightarrow$$

$$\|\cdot\|_{L^2}^2 = \mathbb{E}\left[\left|\frac{1}{N}\sum X_i - \mathbb{E}[X_1]\right|^2\right] =$$

$$\mathbb{E}\left[\left|\frac{1}{N}\sum X_i - \mathbb{E}\left[\frac{1}{N}\sum X_i\right]\right|^2\right] = \mathrm{Var}\left(\frac{1}{N}\sum X_i\right)$$

X_i, iid $\Rightarrow$ $\frac{1}{N}X_i$, $i \in \mathbb{N}$ are also iid $\Rightarrow$ uncorrelated

$$= \sum_{i=1}^{N} \mathrm{Var}\left(\frac{1}{N}X_i\right) = N \cdot \mathrm{Var}\left(\frac{1}{N}\cdot X_1\right) = N \cdot \frac{1}{N^2} \cdot \mathrm{Var}(X_1)$$

$$= \frac{1}{N}\cdot \mathrm{Var}(X_1) \quad \Rightarrow \quad \|\text{error}\|_{L^2} = \frac{1}{\sqrt{N}} \cdot \sqrt{\mathrm{Var}(X_1)}$$

→ Next time: MC approx. of variance + Stochastic processes