NASDE 05.11.2024

Monte Carlo (MC) approximation of variance:

$$\mathrm{Var}(X) = \mathbb{E}\left[\left(X - \mathbb{E}[X]\right)^2\right]$$

$$X \in \mathcal{L}^2(P;\mathbb{R}) \quad \left(\mathbb{E}[|X|^2] < \infty\right)$$

If we know $\mathbb{E}[X]$ then

$$\tilde{X}_n = \left(X_n - \mathbb{E}[X]\right)^2$$

X_n are iid copies of X

$$\mathrm{Var}(X) \approx \frac{1}{N}\sum_{n=1}^{N} \tilde{X}_n$$

$$= \frac{1}{N}\sum_{n=1}^{N}\left(X_n - \mathbb{E}[X]\right)^2$$

If we don't know $\mathbb{E}[X]$: we can do

$$\mathrm{Var}(X) \approx \frac{1}{N}\sum_{n=1}^{N}\Big(X_n - \underbrace{\frac{1}{N}\sum_{j=1}^{N} X_j}_{\approx\, \mathbb{E}[X]}\Big)^2 = Z$$

is biased, i.e.

$$\mathbb{E}[Z] \neq \mathrm{Var}(X)$$

There is an unbiased version:

$$Y = \frac{1}{N-1}\sum_{n=1}^{N}\left(X_n - \frac{1}{N}\sum_{j=1}^{N} X_j\right)^2$$

$$\mathbb{E}[Y] = \mathrm{Var}(X).$$

Proof

$$\mathbb{E}\left[\sum_{n=1}^{N}\left(X_n - \frac{X_1+\dots+X_N}{N}\right)^2\right]$$

$$= \sum_{n=1}^{N}\mathbb{E}\left[\left(X_n - \frac{X_1+\dots+X_N}{N}\right)^2\right] =$$

Note:

$$\mathbb{E}\left[\left(X_n - \frac{X_1+\dots+X_N}{N}\right)\right] = 0 \Rightarrow$$

$$\mathbb{E}\left[\left(X_n - \frac{X_1+\dots+X_N}{N}\right)^2\right] =$$

$$\mathrm{Var}\left(X_n - \frac{X_1+\dots+X_N}{N}\right)$$

$$= \sum_{n=1}^{N}\mathrm{Var}\left(X_n - \frac{X_1+\dots+X_N}{N}\right)$$

$$= \sum_{n=1}^{N}\mathrm{Var}\left(-\frac{X_1}{N} - \frac{X_2}{N} \dots + \frac{(N-1)}{N}\cdot X_n - \dots - \frac{X_N}{N}\right)$$

$X_i, i \in N$ are independent $\Rightarrow$ uncorrelated

$$= \sum_{n=1}^{N}\mathrm{Var}\left(-\frac{1}{N}X_1\right) + \mathrm{Var}\left(-\frac{1}{N}X_2\right) + \dots$$

$$\mathrm{Var}\left(\frac{N-1}{N}X_n\right) + \dots + \mathrm{Var}\left(-\frac{1}{N}X_N\right)$$

$$= \sum_{n=1}^{N}\left[\frac{1}{N^2}\mathrm{Var}(X_1) + \frac{1}{N^2}\mathrm{Var}(X_2) + \dots + \frac{(N-1)^2}{N^2}\mathrm{Var}(X_n) + \dots + \frac{1}{N^2}\mathrm{Var}(X_N)\right]$$

$$= \sum_{n=1}^{N} \mathrm{Var}(X_1)\left[\frac{1}{N^2}\cdot(N-1) + \frac{(N-1)^2}{N^2}\right] = N \cdot \mathrm{Var}(X_1) \cdot \frac{(N-1)\cdot N}{N^2} =$$

$$= (N-1)\cdot \mathrm{Var}(X_1). \qquad \blacksquare$$

Remark: One can show the error estimate

$$\left\| \frac{1}{(N-1)} \sum_{n=1}^{N} \left(X_n - \sum_{j=1}^{N} X_j\right)^2 - \mathrm{Var}(X_1) \right\|_{\mathcal{L}^2} \leq \frac{1}{\sqrt{N}} \cdot \sqrt{\mathbb{E}\left[\left(X - \mathbb{E}[X]\right)^4\right]},$$

$X \in \mathcal{L}^4(\mathbb{P}; \mathbb{R})$

→ same estimate holds for the biased estimator

Stochastic processes: Fix probability space $(\Omega, \mathcal{F}, \mathbb{P})$, time set $\mathbb{T} \subseteq \mathbb{R}$ (usually one takes $\mathbb{T} = [0, T]$, $[0, \infty)$), let $(S, \mathcal{S})$ - be a measurable space ($\mathcal{S}$ is a σ-algebra on S), usually one takes $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $(\mathbb{R}^{d\times m}, \mathcal{B}(\mathbb{R}^{d\times m}))$

$X : \mathbb{T} \times \Omega \to S$ an S-valued stochastic process

If $\forall t \in \mathbb{T}$:

$\Omega \ni \omega \longmapsto X(t,\omega) \in S$ is an

$\mathcal{F}/\mathcal{S}$-measurable.

Remark: A stochastic process X is

(i) a 2 parameter f-n
$(t,\omega) \longmapsto X(t,\omega)$

(ii) a one parameter family $(X_t, t \in \mathbb{T})$ of random variables

(iii) a family $(X(\cdot,\omega), \omega \in \Omega)$ of f-ns
$X(\cdot,\omega) : \mathbb{T} \to S, \ t \mapsto X(t,\omega)$

If the state space is a metric space, X has cont. sample paths if $\forall \omega \in \Omega$

$\mathbb{T} \ni t \longmapsto X(t,\omega) \in S$ is a cont. f-n.

→ Similarly, left, right continuity

Equality types: We say that X and Y are

(i) modifications of each other if
$\forall t \in \mathbb{T}$: $\exists A_t \in \mathcal{F}$ s.t. $P(A_t) = 1$

$$A_t \subseteq \{X_t = Y_t\} = \underbrace{\{\omega \in \Omega : X_t = Y_t\}}_{\text{does not have to be measurable, i.e. } \in \mathcal{F}}$$

(ii) indistinguishable if $\exists A \in \mathcal{F}$: $\mathbb{P}(A) = 1$

$$A \subseteq \bigcap_{t \in \mathbb{T}} \{X_t = Y_t\}$$

Obviously, (ii) $\Rightarrow$ (i)
The reverse is not always true. $\rightarrow$ sheet 4.
If (i) + left or right continuity of X and Y $\Rightarrow$ (ii).

Filtration: A filtration $\mathbb{F} = (\mathbb{F}_t)_{t \in \mathbb{T}}$ is a family of sub-σ-algebras of such that.

$$\mathbb{F}_{t_1} = \sigma(\mathbb{F}_{t_1}) \subseteq \mathbb{F}_{t_2} = \sigma(\mathbb{F}_{t_2}) \subseteq \mathcal{F}$$

$$\forall \quad t_1 \leq t_2$$
$$t_1, t_2 \in \mathbb{T}$$

$(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ - filtered probability space.

Remark: Every stochastic process X induces filtration via

$\mathbb{F}^X_t = \sigma((X_s, s \leq t, s \in \mathbb{T}))$.

$\mathbb{F}^X = (\mathbb{F}^X_t)_{t \in \mathbb{T}}$ is called the filtration generated by X.

X is then $\mathbb{F}^X$-adapted.

[X is $\mathbb{F}$ adapted : X_t is $\mathcal{F}_t / \mathcal{S}$ measurable]

Important stochastic process: Standard Brownian motion (BM)

Def-n: Let $W: [0, T] \times \Omega \to \mathbb{R}^m$ be a stochastic process on $(\Omega, \mathcal{F}, P, \mathbb{F})$. We call it a standard BM if

(i) W is $\mathbb{F}$-adapted.

(ii) W has continuous sample paths

(iii) $W_0 = 0 \in \mathbb{R}^m$ (also P-a.s.)

(iv) $W_{t_2} - W_{t_1} \sim \mathcal{N}(0, (t_2 - t_1) \cdot \mathbb{I}_{\mathbb{R}^m})$ $\forall t_1 \leq t_2$

(stationary increments)

(v) $\sigma(W_{t_2} - W_{t_1})$ and $\mathcal{F}_{t_1}$ are independent $\forall\, 0 \leq t_1 \leq t_2 \leq T$

(independent increments)

Remark: One can define BM without filtration (one removes (i), replaces (v) by $W_{t_n} - W_{t_{n-1}}, W_{t_{n-1}} - W_{t_{n-2}}, \dots, W_{t_1} - W_{t_0}$ are independent)

Goal: define stochastic integral w.r.t. BM W

$$\int_0^T X_s \, dW_s$$

Integrand X has to be predictable + regular (in a certain sense).

Predictability: If X is discrete $X = (X_1, X_2, \dots, X_n, \dots)$

X_{n+1} is $\mathcal{F}_n$-measurable.

In case of cont. time set, $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$

we define

$$\mathrm{Pred}(\mathbb{F}) = \sigma\Big(\{(s,t] \times A : A \in \mathcal{F}_s,\ 0 \le s < t \le T\} \cup \{\{0\} \times B : B \in \mathcal{F}_0\}\Big)$$ predictable σ-algebra

X is $\mathbb{F}$ predictable if

X is $\mathrm{Pred}(\mathbb{F}) / \mathcal{S}$ measurable.

Remarks:

(i) Predictability $\Rightarrow$ Adaptivity

(ii) Adapted + left continuous $\Rightarrow$ predictability

$$\left[\lim_{\varepsilon \to 0} X_{t-\varepsilon} = X_t \right]$$

(iii) $\mathrm{Pred}(\mathbb{F}) = \sigma(X : [0,T] \times \Omega \to S,\ X \text{ is } \mathbb{F}\text{-adapted and left continuous})$

Regularity: $\Big[\int X_s \, dW_s$

$W_s \in \mathbb{R}^m, \ \forall s$

$X_s \in \mathbb{R}^{d \times m}, \ \forall s$

$\int X_s \, dW_s \in \mathbb{R}^{d \times m}$

X is $d \times m$ matrix valued. $\Big]$

We recall the Hilbert-Schmidt norm (or Frobenius norm) for a matrix $A \in \mathbb{R}^{d \times m}$

$$\|A\|_{HS} = \left(= \|A\|_{\mathbb{R}^{d\times m}} \right) = \left(\sum_{i=1}^{d} \sum_{j=1}^{m} |A_{ij}|^2 \right)^{1/2}$$

Next time: Stochastic integral.