NASDE 08.11.2024

$p \in [0,\infty)$ (1)

$$L^p(\mathrm{Pred}(\mathbb{F}); HS(\mathbb{R}^m, \mathbb{R}^d))$$
$$= \Big\{ X : [0,T] \times \Omega \to \mathbb{R}^{d\times m} \text{ is } \mathbb{F}\text{-predictable and } \int_0^T \mathbb{E}\big[\|X_s\|_{HS}^p\big]\, ds < \infty \Big\}$$

We define $\int_0^T X_s \, dW_s$ for $p=2$

- First, we define for simple predictable processes:

$Y : [0,T] \times \Omega \to \mathbb{R}^{d\times m}$ is simple if

- $\exists n \in \mathbb{N}$, $0 \le t_1 < t_2 \dots < t_n \le T$ and
- $H_k : \Omega \to \mathbb{R}^{d\times m}$ is $\mathbb{F}_{t_k}$-meas.

$\forall k \in \{1, \dots, n-1\}$ s.t. (2)

$$Y_t = \sum_{k=1}^{n-1} \underbrace{H_k}_{\mathbb{F}_{t_k}\text{-meas.}} \cdot \underbrace{\mathbb{1}_{(t_k, t_{k+1}]}(t)}_{\text{left-cont.}} \quad \forall t \in [0,T]$$

$t \in [0, t_1]$ $Y_t = 0$

$t \in (t_1, t_2]$ $Y_t = H_1$ is $\mathbb{F}_{t_1}$-meas

$t \in (t_2, t_3]$: $Y_t = H_2$, . $\mathbb{F}_{t_2}$-meas

$\Rightarrow$ Y is $\mathbb{F}$-adapted + left-cont

$\Rightarrow$ Y is $\mathbb{F}$-predictable.

Assume that $Y \in L^2(\mathrm{Pred}(\mathbb{F}), HS)$:

$$\int_0^T \mathbb{E}\big[\|Y_s\|_{HS}^2\big]\, ds < \infty$$

(3)

$$\sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}[\|Y_s\|_{HS}^2]\, ds$$

$\forall$ H_k

$$= \sum_{k=1}^{n-1} \underbrace{(t_{k+1} - t_k)}_{\geq 0} \cdot \mathbb{E}[\|H_k\|_{HS}^2] < \infty$$

$$\iff \mathbb{E}[\|H_k\|_{HS}^2] < \infty \quad \forall k=1,\dots,n-1$$

$$H_k \in \mathcal{L}^2(P; HS)$$

If $W : [0,T] \times \Omega \to \mathbb{R}^m$ BM, we define

$$I_{0,T}^W(Y) := \sum_{k=1}^{n-1} H_k \cdot (W_{t_{k+1}} - W_{t_k})$$

$$\left(\int_0^T Y_s\, dW_s\right)$$

(4)

If $0 \leq a \leq b \leq T$:

$$I_{a,b}^W(Y) := \sum_{k=1}^{n-1} H_k \cdot \left(W_{\min(t_{k+1}, b)} - W_{\min\{t_{k+1}, b, \max\{t_k, a\}\}}\right)$$

Theorem: Let

$$V = \{ Y \text{ simple},\ Y \in L^2(\text{Pred}, HS) \}$$

Then

$$I_{a,b}^W : V \to L^2(P, \mathbb{R}^d)$$

is a well-defined linear operator and for $Y \in V$ we have

(i) $\|I_{0,T}^W(Y)\|_{L^2(P,\mathbb{R}^d)} = \|Y\|_{L^2(\text{Pred}, HS)}$

Itô's isometry

(ii)

$$\| I^W_{a,b}(Y) \|_{L^2} \leq \| Y \|_{L^2}$$

$\Rightarrow$ $I^W_{a,b}$ is bounded

$$I^W_{a,b} : V \longrightarrow L^2(\mathbb{P}, \mathbb{R}^d)$$

One shows that V is dense in

$L^2(\mathrm{Pred}(\mathbb{F}), HS)$ $\Rightarrow$

We can cont. extend it to

$$I^W_{a,b} : L^2(\mathrm{Pred}(\mathbb{F}), HS) \to L^2(\mathbb{P}, \mathbb{R}^d).$$

$$\int_a^b Y_s \, dW_s$$

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Main properties of the stochastic integral: 6

If $0 \leq a \leq b \leq T$:

1. $\left(\int_a^t X_s \, dW_s \right)_{t \in [a,b]}$ is $(\mathbb{F}_t)_{t \in [a,b]}$-adapted.

2. linearity:
$$\int_a^b [\alpha X_s + \beta Y_s] \, dW_s = \alpha \cdot \int_a^b X_s \, dW_s + \beta \int_a^b Y_s \, dW_s$$

3. If $a \leq c \leq b$
$$\int_a^b X_s \, dW_s = \int_a^c X_s \, dW_s + \int_c^b X_s \, dW_s$$

(4) $\mathbb{E}\left[\int_a^b X_s\,dW_s\right] = 0$ (7)

(5) Martingale property:

$$\mathbb{E}\left[\int_0^t X_s\,dW_s \,\Big|\, \mathbb{F}_u\right] = \int_0^u X_s\,dW_s$$

$0 \le u \le t \le T$

i.e.

$\left(\int_0^t X_s\,dW_s\right)_{t\in[0,T]}$ is $(\mathbb{F}_t)_{t\in[0,T]}$-martingale

Proof: Idea: Prove for $X \in V$ simple. Use the density of $V \subset L_i^2(\mathrm{Pred}(\mathbb{F}), HS)$ to extend it to $L_i^2(\mathrm{Pred}(\mathbb{F}), HS)$.

→ We use stochastic integral to define Itô processes (8)

In classical analysis: the fundamental thm of calculus tells us

$x : [0,T] \to \mathbb{R}$ is in C^1 iff

$\exists\, y : [0,T] \to \mathbb{R}$ cont. such that

$$x(t) = x(0) + \int_0^t y(s)\,ds \quad \forall t \in [0,T]$$

and in this case

$$y = x'.$$

In stochastic analysis: An analogue of C^1 is Itô process.

Def-n: An $(\mathbb{F}_t)$-adapted process

$X : [0,T] \times \Omega \to \mathbb{R}^d$ with cont.

sample paths is called an **Itô process** if

$$X_t = \underset{\in\mathbb{R}^d}{X_0} + \int_0^t \underset{\in\mathbb{R}^d}{Y_s}\,ds + \int_0^t \underset{\in\mathbb{R}^{d\times m}}{\mathcal{Z}_s}\,\underset{\in\mathbb{R}^m}{dW_s}$$

for some $Y \in L_1^2$, $\mathcal{Z} \in L_1^{2,0}$

Y - drift, $\mathcal{Z}$ - diffusion

Remark on notation: One also writes

$$dX_t = Y_t\,dt + \mathcal{Z}_t\,dW_t, \quad t\in[0,T]$$

In classical analysis: We have **chain rule**: $f\in C^1$, then

$f\circ x : [0,T] \to \mathbb{R}$, $t \mapsto f(x(t))$ is also in C^1 and

$$(f\circ x)'(t) = f'(x(t))\cdot x'(t)$$

or equivalently

$$f(x(t)) = f(x(0)) + \int_0^t f'(x(s))\,y(s)\,ds$$

where

$$x(t) = x(0) + \int_0^t y(s)\,ds$$

→ The stochastic alternative of the chain rule is Itô formula

Theorem: If $f\in C^2(\mathbb{R}^d,\mathbb{R})$ then

$$f(X_t) = f(X_a) + \int_a^t \nabla f(X_s)^T Y_s\,ds + \frac{1}{2}\int_a^t \operatorname{trace}\left(\mathcal{Z}_s^T (D^2 f)(X_s)\,\mathcal{Z}_s\right)ds + \int_a^t \nabla f(X_s)^T \mathcal{Z}_s\,dW_s \quad P\text{-a.s.}$$

i.e. $(f(X_t))_{t\in[0,T]}$ is also an Itô process with drift

$$\left(\nabla f(X_t)^T Y_t + \frac{1}{2}\operatorname{trace}\left(Z_t^T (D^2 f)(X_t) Z_t\right)\right.$$

and diffusion

$$\left(\nabla f(X_t)^T Z_t\right)_{t\in[0,T]}.$$

Remark: Using this you can deduce time dependent version of Itô formula

$$f(t, X_t) = \dots$$

(sheet 4)

→ One can also have $f \in C^2(\mathbb{R}^d, \mathbb{R}^n)$

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Example: Geometric BM

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$$\tilde{X}_t = \xi \cdot \exp\left(\left(\alpha - \frac{\beta^2}{2}\right)t + \beta W_t\right)$$

$\xi, \alpha, \beta \in \mathbb{R}$ are constants

Denote

$$X_t = \left(\alpha - \frac{\beta^2}{2}\right)t + \beta \cdot W_t$$

$$= \int_0^t \left(\alpha - \frac{\beta^2}{2}\right) ds + \int_0^t \beta \, dW_s$$

$$= X_0 + \int_0^t \underbrace{\left(\alpha - \frac{\beta^2}{2}\right)}_{Y_s} ds + \int_0^t \underbrace{\beta}_{Z_s} dW_s$$

X is an Itô process. $f(y) = \xi \cdot e^y$

$$\tilde{X}_t = f(X_t) = f(X_0) + \int_0^t f'(X_s) \cdot \left(\alpha - \frac{\beta^2}{2}\right) ds$$

(13)

$$+\frac{1}{2}\int_0 \beta^2 \cdot f''(X_s)\,ds + \int_0^t f'(X_s)\,\beta\,dW_s$$

$$= f(X_0) + \int_0^t \left[\xi \cdot e^{X_s}\left(\alpha - \cancel{\frac{\beta^2}{2}}\right) + \cancel{\frac{1}{2}\beta^2 \xi \cdot e^{X_s}}\right] ds$$

$$+ \int_0^t \xi \cdot e^{X_s} \cdot \beta \cdot dW_s$$

$$= \xi + \int_0^t \xi \cdot e^{X_s}\alpha\,ds + \int_0^t \xi \cdot e^{X_s} \cdot \beta\,dW_s$$

$$= \xi + \int_0^t \alpha \cdot \tilde{X}_s\,ds + \int_0^t \beta \cdot \tilde{X}_s\,dW_s$$

$$\Rightarrow \tilde{X}_t = \xi + \int_0^t \alpha\,\tilde{X}_s\,ds + \int_0^t \beta \cdot \tilde{X}_s\,dW_s$$

$\tilde{X}$ is a sol-n of the stochastic differential eq-n (SDE)

$$dY_t = \alpha\,Y_t\,dt + \beta\,Y_t\,dW_t,$$

(14)

$$Y_0 = \xi, \quad t \in [0, T]$$

Next time: SDEs, existence & uniqueness, examples.