NASDE 12.11.2024

Today: SDEs. (1)

We fix the setting: $T \in (0,\infty)$, $d, m \in \mathbb{N}$, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$ – filtered probability space, $W : [0,T] \times \Omega \to \mathbb{R}^m$ Brownian motion.

We also fix the coefficient fns and initial value

$\mu : \mathbb{R}^d \to \mathbb{R}^d$ (drift)

$\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ (diffusion)

$\xi : \Omega \to \mathbb{R}^d$, $\mathbb{F}_0$-measurable.

We consider the SDE:

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \quad t\in[0,T]$$
$$X_0 = \xi \qquad (*)$$

We say that stochastic process $X : [0,T]\times\Omega \to \mathbb{R}^d$ is a sol-n of $(*)$

(2)

if

(1) X is $\mathbb{F}$-adapted (X_t is $\mathbb{F}_t$-meas.) and has cont. sample paths

(2) $\mathbb{P}\left(\int_0^T \|\mu(X_s)\|_{\mathbb{R}^d} + \|\sigma(X_s)\|^2_{HS}\, ds < \infty\right) = 1$

weaker than $\left[\int_0^T \mathbb{E}\left[\|\sigma(X_s)\|_{HS}\right] < \infty\right]$,

One can define $\int Y_s\, dW_s$ for Y with $P\left(\int_0^T \|Y_s\|^2_{HS}\, ds\right) < \infty$, however $\int_0^T Y_s\, dW_s \in L^0(P, \mathbb{R}^d)$

(3) $X_t = \xi + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s$, $\forall t\in[0,T]$ $\mathbb{P}$-a.s.

Note: X is an Itô process with drift $\mu(X_s)$ / diff. $\sigma(X_s)$

Uniqueness: sol-ns of $(*)$ are not typically unique in the pathwise sense. ③

- However, under suitable assumptions we can get uniqueness up to indistinguishability

[$\exists A \in \mathcal{F}$: $\mathbb{P}(A) = 1$ and

$$A \subseteq \bigcap_{t\in[0,T]} \{X_t = Y_t\}$$ - X and Y indistinguish.

However, if X and Y are cont. stoch. pr.

$$\bigcap_{t\in[0,T]} \{X_t = Y_t\} \in \mathcal{F}.$$

$$\mathbb{P}(\forall t\in[0,T]: \{X_t = Y_t\}) = 1 \;]$$

Thm: Uniqueness of sol-ns: ④

Let μ, σ be locally Lipschitz cont., i.e. $\forall K \subseteq \mathbb{R}^d$ compact:

$$\sup_{\substack{x,y\in K \\ x\neq y}} \left\{ \frac{\|\mu(x)-\mu(y)\| + \|\sigma(x)-\sigma(y)\|}{\|x-y\|} \right\} < \infty.$$

and let X and Y be sol-ns of $(*)$. Then X and Y are indistinguishable from each other, i.e.

$$\mathbb{P}(\forall t\in[0,T]: X_t = Y_t) = 1.$$

Example of locally Lipschitz cont. f-n:

$f(x) = x^2$, $f: \mathbb{R} \to \mathbb{R}$

$$|f(x) - f(y)| = |x^2 - y^2| = |x-y| \cdot |x+y|$$

If $x,y\in[a,b]\subset\mathbb{R}$

$|x+y| \le |x| + |y| \le 2\max\{|a|,|b|\} \Rightarrow$

$|f(x)-f(y)| \le 2\max\{|a|,|b|\}\cdot|x-y|$

$\forall x,y\in[a,b]$ ✓

However, there is no $L>0$ s.t.

$$|f(x)-f(y)|\le L|x-y| \quad ,\forall x,y\in\mathbb{R}$$

f is not globally-Lipschitz.

Theorem: Existence & uniqueness:

Let $p\in[2,\infty)$, $\xi\in L^p(\mathcal{F}_0,\mathbb{R}^d)$, μ,σ are globally Lipschitz continuous, i.e. $\exists C\in[0,\infty)$, $\forall x,y\in\mathbb{R}^d$

$$\|\mu(x)-\mu(y)\|_{\mathbb{R}^d} + \|\sigma(x)-\sigma(y)\|_{HS} \le C\|x-y\|_{\mathbb{R}^d}$$

$$\left[\sup_{\substack{x,y\in\mathbb{R}^d\\ x\neq y}} \frac{\|\mu(x)-\mu(y)\|+\|\sigma(x)-\sigma(y)\|}{\|x-y\|}<\infty\right]$$

Then

(i) there exists up to indistinguishability unique sol-n X

(ii) $\sup_{t\in[0,T]} \|X_t\|_{L^p(P;\|\cdot\|_{\mathbb{R}^d})}<\infty$

Time dependent coeffients: (**)

$$X_t = \xi + \int_0^t \mu(s,X_s)\,ds + \int_0^t \sigma(s,X_s)\,dW_s$$

$s+d\cdot X_s$

One can autonomize (**) to

$$\tilde X_t = \tilde\xi + \int_0^t \tilde\mu(\tilde X_s)\,ds + \int_0^t \tilde\sigma(\tilde X_s)\,dW_s$$

Existence & uniqueness for (**): (7)

$$\|\mu(t_1,x_1)-\mu(t_2,x_2)\|+\|\sigma(t_1,x_1)-\sigma(t_2,x_2)\| \le C\left(|t_1-t_2|+\|x_1-x_2\|\right)$$

C is independent of t_1, t_2, x_1, x_2

$\Rightarrow$ global Lipschitz cont of $\tilde{\mu}, \tilde{\sigma} \Rightarrow \exists! \, \tilde{X} \Rightarrow \exists! \, X$.

Example: Geometric BM.

$$dX_t = \alpha X_t dt + \beta X_t dW_t, \quad t\in[0,T], \quad X_0 = \xi$$

$\mu(x) = \alpha \cdot x$, $\sigma(x) = \beta \cdot x$

μ and σ are globally Lipschitz $\Rightarrow$ we have existence & uniqueness.

Last time we showed

$$X_t = \exp\left(\left(\alpha - \frac{\beta^2}{2}\right)t + \beta W_t\right)\cdot \xi$$ (8)

is a sol-n $\Rightarrow$ it is the sol-n.

X - geometric Brownian motion.

$$\frac{d}{dt}\mathbb{E}[X_t] = \alpha \cdot \mathbb{E}[X_t]$$

$$\mathbb{E}[X_t] = e^{\alpha t}\cdot \xi.$$

Ex: Black-Scholes model.

$d=2$, $m=1$

$$\mu(x) = \mu\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r x_1 \\ \alpha x_2 \end{pmatrix}, \quad \sigma(x) = \begin{pmatrix} 0 \\ \beta x_2 \end{pmatrix}$$

$r, \alpha \in \mathbb{R}$, $\beta \in (0,\infty)$

$$dX_t = \begin{pmatrix} r X_t^{(1)} \\ \alpha X_t^{(2)} \end{pmatrix} dt + \begin{pmatrix} 0 \\ \beta X_t^{(2)} \end{pmatrix} dW_t$$

$$\left.\begin{aligned} dX_t^{(1)} &= r X_t^{(1)} dt \\ dX_t^{(2)} &= \alpha X_t^{(2)} dt + \beta X_t^{(2)} dW_t \end{aligned}\right\} \; X_0 = \xi.$$ BS-SDE

Obviously, μ and σ are globally Lipschitz $\Rightarrow \exists!$ sol-n

$$X : [0,T] \times \Omega \to \mathbb{R}^2$$

$$\sup_{t \in [0,T]} \mathbb{E}[\|X_t\|^p] < \infty.$$

$$\Rightarrow \begin{cases} X_t^{(1)} = e^{rt} \cdot \xi^{(1)}, & \text{(deterministic)} \\ X_t^{(2)} = \exp\left(\left(\alpha - \frac{\beta^2}{2}\right)t + \beta W_t\right) \cdot \xi^{(2)}. \end{cases}$$

$X_t^{(1)}$ - models the price of a „riskfree" asset. (e.g. bank account with fixed interest rate r).

$(X_t^{(2)})$ - models the price of an underlying (e.g. stock, commodity, currency, index, S&P 500...) with

- with expected rate $\alpha \in \mathbb{R}$,
- with volatility $\beta > 0$
- with initial price $\xi^{(2)} > 0$.

$\to$ Can be used in derivative pricing

A financial derivative: a product which is derived via one or several underlyings

Simple examples:
European put and call options:

European put option: Contract between 2 parties, gives the holder of the option the right but no obligation to sell fixed underlying, at a fixed time T, for the fixed price K. (11)

value at time T:

$$\max\{K - X_T, 0\}$$

European call: Gives right to buy ...

$$\max\{X_T - K, 0\}$$

American put:

$$\max\{K - x_t, 0\}, \quad t \in [0, T]$$

Derivative pricing: Fixing „fair" price for the finan. derivative at time $t=0$ „today".

→ Assume at time T we have gain (loss) $f(X)$, $f: C([0,T], \mathbb{R}) \to [0,\infty)$

In the case of European call (12)

$$f(X) = \max\{X_T - K, 0\}$$

The fundamental theorem of asset pricing tells us under certain ass. on the market we have

$\exists$ $\mathbb{F}$-adapted cont. process

$D: [0,T] \times \Omega \to \mathbb{R}$ s.t.

$$D_T = f(X^{(2)}) \quad \text{and}$$

$$d\tilde{X}_t = r\tilde{X}_t + \beta \tilde{X}_t \, dW_t, \quad \tilde{X}_0 = X_0^{(2)}$$

$$D_0 = \frac{\mathbb{E}[f(\tilde{X})]}{X_T^{(1)}} = e^{-rT} \cdot \xi^{(1)} \cdot \mathbb{E}[f(\tilde{X})]$$

European call:

$$e^{-rT} \xi^{(1)} \mathbb{E}\left[\max\left\{e^{(r-\frac{\beta^2}{2})T + \ln(x_0^{(2)}) + \beta W_T} - K, 0\right\}\right]$$