NASDE 15.11.2024

→ Some comments on financial modeling (1)

(1) There are more general models:
e.g. Stochastic Ginzburg Landau

$$dX_t = \underbrace{[\alpha X_t - \delta X_t^3]}_{\text{not globally Lipschitz,}} dt + [\beta X_t + \bar{\beta}] dW_t$$

however the existence & uniqueness has been shown

- CIR (Cox-Ingersoll-Ross) process
used in interest rates

$$dX_t = [\delta - \alpha \cdot X_t] dt + \beta \sqrt{X_t} dW_t.$$

. . . . . Stochastic volatility models.

(2) Noise can be general Lévy process, might have discontinuities, i.e. jumps

(3) Many real world conditions might not be satisfied: (2)

- transaction costs
- default risk (default of asset) counterparty default, . . .

→ Generalize to take into account these factors.

In conclusion we will have some modeling error. → difficult to quantify.

Approximation error: Error incurred through approximating the sol-n via some scheme.

Let us start with numerical approximations of SDEs

1 Euler-Maruyama (EM) scheme:

The SDE

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0 = \xi, \quad t\in[0,T]$$

$$\left[\; X_t = \xi + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s \right]$$

Step size $h = \frac{T}{N}$ for some $N\in\mathbb{N}$

$Y: \{0,1,\dots,N\}\times\Omega \to \mathbb{R}^d$ discrete stochastic process, which is given iteratively

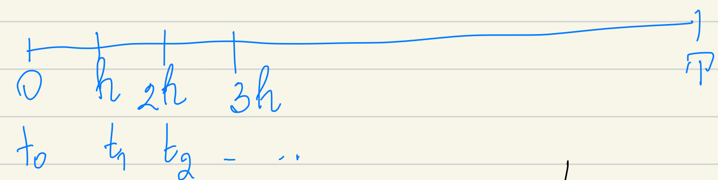
$Y_0 = \xi$

$$Y_{n+1} = Y_n + \mu(Y_n)\cdot\underbrace{\frac{T}{N}}_{h} + \sigma(Y_n)\Big(W_{\underbrace{\frac{(n+1)T}{N}}_{t_{n+1}}} - W_{\underbrace{\frac{nT}{N}}_{t_n}}\Big)$$

$$\forall n\in\{0,\dots,N-1\}$$

$$= Y_n + h\cdot\mu(Y_n) + \sigma(Y_n)\cdot\left(W_{t_{n+1}} - W_{t_n}\right)$$

Intuition:



$$X_{t_{n+1}} = \xi + \int_0^{t_{n+1}} \mu(X_s)\,ds + \int_0^{t_{n+1}} \sigma(X_s)\,dW_s$$

$$= \xi + \int_0^{t_n} \mu(X_s)\,ds + \int_{t_n}^{t_{n+1}} \mu(X_s)\,ds +$$

⑤

$$+\int_0^{t_n} \sigma(X_s)\,dW_s + \int_{t_n}^{t_{n+1}} \sigma(X_s)\,dW_s$$

$$= X_{t_n} + \int_{t_n}^{t_{n+1}} \mu(X_s)\,ds + \int_{t_n}^{t_{n+1}} \sigma(X_s)\,dW_s$$

$$X_{t_n} \sim Y_n$$

$$\int_{t_n}^{t_{n+1}} \mu(X_s)\,ds \approx \int_{t_n}^{t_{n+1}} \mu(Y_n)\,ds =$$

$$(t_{n+1} - t_n)\cdot\mu(Y_n) = h\cdot\mu(Y_n)$$

$$\int_{t_n}^{t_{n+1}} \sigma(X_s)\,dW_s = \int_{t_n}^{t_{n+1}} \sigma(Y_n)\,dW_s$$

$$= \sigma(Y_n)\,[W_{t_{n+1}} - W_{t_n}]$$

⑥

$$Y_{n+1} = Y_n + h\cdot\mu(Y_n) + \sigma(Y_n)\cdot[W_{t_{n+1}} - W_{t_n}].$$

We define linearly interpolated EM approximation by:

$$\bar{Y}: [0,T]\times\Omega \to \mathbb{R}^d, \quad \bar{Y}_0 = \xi$$

$$\forall n \in \{0,1,\dots,N-1\},\ t\in[t_n, t_{n+1}] = [n\cdot h, (n+1)h]$$

$$\bar{Y}_t = \bar{Y}_{n\cdot h} + (t - nh)\cdot\mu(\bar{Y}_{nh}) + \left(\frac{t}{h} - n\right)\sigma(\bar{Y}_{n\cdot h})(W_{t_{n+1}} - W_{t_n})$$

$$\bar{Y}_t = Y_n + \left(\frac{t}{h} - n\right)\left[\mu(Y_n)\cdot h + \sigma(Y_n)\cdot(W_{t_{n+1}} - W_{t_n})\right]$$

Q: How good are these app-ns? (7)

We have to differentiate several notions of convergence:

Def-ns: Let $Y^N : [0,T] \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$ be stochastic processes. We say that Y^N, $N\in\mathbb{N}$ converges to X at time T

(1) In the strong L^p sense ($p\in(0,\infty)$) if

$$\left(\mathbb{E}\left[\| Y^N_T - X_T\|^p\right]\right)^{1/p} \xrightarrow[N\to\infty]{} 0$$

L^p error $\xrightarrow[N\to\infty]{} 0$

with order $\alpha\in(0,\infty)$ if $\exists\, C\in(0,\infty)$:

$$\| Y^N_T - X_T \|_{L^p(\mathbb{P};\mathbb{R}^d)} \le C\cdot N^{-\alpha} \quad \forall N\in\mathbb{N}$$

(2) $\mathbb{P}$-a.s. if (8)

$$\mathbb{P}\left(\limsup_{N\to\infty} \| Y^N_T - X_T\|_{\mathbb{R}^d} = 0\right) = 1.$$

with order $\alpha\in(0,\infty)$ if $\exists\, C:\Omega\to\mathbb{R}$ (random variable) s.t.

$$\mathbb{P}\left(\| Y^N_T - X_N\| \le C\cdot N^{-\alpha}\right) = 1.$$

(3) in probability if $\forall \varepsilon\in(0,\infty)$

$$\limsup_{N\to\infty} \mathbb{P}\left(\| Y^N_T - X_T\| \ge \varepsilon\right) = 0$$

(4) in numerically weak sense if $\forall \varphi \in C^\infty(\mathbb{R}^d,\mathbb{R})$ [test f-ns] with at most polynomially growing derivatives $\mathbb{E}[|\varphi(Y^N_T)|]<\infty$, $\mathbb{E}[|\varphi(X_T)|]<\infty$

(9)

$$\left|\mathbb{E}[\varphi(Y_T^N)] - \mathbb{E}[\varphi(X_T)]\right| \xrightarrow[N\to\infty]{} 0$$

with order $\alpha \in (0,\infty)$ if:

$$\exists C \in (0,\infty):\ \underbrace{\left|\mathbb{E}[\varphi(Y_T^N)] - \mathbb{E}[\varphi(X_T)]\right|}_{\text{weak error wrt } \varphi.} \le C \cdot N^{-\alpha}$$

Recall: $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ normed spaces; $f: E \to F$ grows at most polynomially if $\exists C \in (0,\infty)$

$$\|f(x)\|_F \le C \cdot (1 + \|x\|_E)^C$$

Theorem (Strong / uniform strong convergence of EM scheme).

(10)

Let $p \in [2,\infty)$, $\xi \in L^p(\mathbb{P}|_{\mathcal{F}_0}; \mathbb{R}^d)$ and μ and σ be globally Lipschitz cont., i.e.

$$\exists L_\mu:\ \|\mu(x) - \mu(y)\| \le L_\mu \cdot \|x-y\|$$

$$\exists L_\sigma:\ \|\sigma(x) - \sigma(y)\|_{HS} \le L_\sigma \|x-y\|,\quad \forall x,y \in \mathbb{R}^d.$$

[The same assumptions as for existence & uniqueness theorem]

Then we have L^p convergence with order $1/2$:

$$(1)\ \sup_{t\in[0,T]} \|X_t - \bar{Y}_t\|_{L^p(\mathbb{P};\mathbb{R}^d)} = \sup_{t\in[0,T]} \left(\mathbb{E}\left[\|X_t - \bar{Y}_t\|^p_{\mathbb{R}^d}\right]\right)^{1/p} \le$$

$$\leq C(L_\mu, L_\sigma, T, p) \cdot \|X\|_{C^{1/2}([0,T], L^p)} \cdot \frac{1}{\sqrt{N}}$$ (11)

$$\sup_{t,s} \left(\frac{\|X_t - X_s\|_{L^p(P;\mathbb{R}^d)}}{(t-s)^{1/2}} \right)$$

(ii) $\exists \tilde{C}$ s.t.

uniform

$$\left\| \sup_{t\in[0,T]} \|X_t - \bar{Y}_t\|_{\mathbb{R}^d} \right\|_{L^p(P;\mathbb{R})}$$

$$= \left(\mathbb{E}\left[\left(\sup_{t\in[0,T]} \|X_t - \bar{Y}_t\|_{\mathbb{R}^d} \right)^p \right] \right)^{1/p}$$

$$\leq \tilde{C} \left(\sqrt{1 + \ln(N)} \right) \cdot \frac{1}{\sqrt{N}}$$

Smaller conv. order, $\frac{1}{2} - \varepsilon$ (12)

$\forall \varepsilon \in (0,\infty)$.

$$\frac{\ln(N)}{N^\alpha} \to 0 \quad \forall \alpha \in (0,\infty)$$

It has essentially order $\frac{1}{2}$.

Proof: (of) (i) uses discrete Gronwall lemma + Burkholder Davis-Gundy inequality (used for stochastic integral).

We will only prove (i) next time

increment tamed EM, higher order (Milstein)