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Sheet 1

1. Let $d \in \mathbb{N}$ and let $\mathcal{N}_{0,I_{\mathbb{R}^d}} \colon \mathcal{B}(\mathbb{R}^d) \to [0,1]$ be the d-dimensional standard normal distribution given by

$$\mathcal{N}_{0,I_{\mathbb{R}^d}}(B) \colon = \frac{1}{(2\pi)^{d/2}} \int_B e^{-\frac{1}{2}\|x\|^2_{\mathbb{R}^d}}\, dx.$$

Show that for all $i, j \in \{1, \ldots, d\}$ it holds that

$$\int_{\mathbb{R}^d} x_i \mathcal{N}_{0,I_{\mathbb{R}^d}}(dx_1, \ldots, dx_d) = 0, \quad \int_{\mathbb{R}^d} x_i x_j \mathcal{N}_{0,I_{\mathbb{R}^d}}(dx_1, \ldots, dx_d) = \begin{cases} 1, & \text{if } \; i = j \\ 0, & \text{if } \; i \neq j \end{cases}.$$

Hint: You may use without proof that $\int_{\mathbb{R}} e^{-\frac{1}{2}x^2}\, dx = \sqrt{2\pi}$. This can be shown by considering a standard normal density in two dimensions ($d = 2$) and by using polar coordinates.

2. Let $F \colon \mathbb{R} \to [0,1]$ be a distribution function and let the *generalized inverse* of F be given as in in the class by

$$I_F \colon (0,1) \to \mathbb{R}, \quad y \mapsto \inf\{x \in \mathbb{R} \colon F(x) \geq y\} = \inf(F^{-1}[y,1]).$$

a) Prove or disprove the following statement: For every distribution function $F \colon \mathbb{R} \to [0,1]$ and every $y \in (0,1)$ it holds that $F(x) > y$ if and only if $x > I_F(y)$.

b) Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $c \in \mathbb{R}$, let $X \colon \Omega \to \mathbb{R}$ be a $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable function which satisfies for all $\omega \in \Omega$ that

$$X(\omega) = c,$$

and let $F \colon \mathbb{R} \to [0,1]$ be the distribution function of X. What is $I_F(y),\ y \in (0,1)$?

c) Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $U \colon \Omega \to \mathbb{R}$ a $\mathcal{U}_{(0,1)}$-distributed random variable, let $X \colon \Omega \to \mathbb{R}$ be a function which satisfies for all $\omega \in \Omega$ that

$$X(\omega) = \sin(U(\omega)),$$

and let $F \colon \mathbb{R} \to [0,1]$ be the distribution function of X. What is $I_F(y),\ y \in (0,1)$?

3. Let $\lambda \in (0, \infty)$. Then, a Laplace distributed random variable X with parameter λ has the density function

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|}, \qquad x \in \mathbb{R}, \tag{1}$$

and we write $X \sim \text{Laplace}_\lambda$. Let $F\colon \mathbb{R} \to [0, 1]$ be the distribution function of X.

a) Show for all $x \in \mathbb{R}$ that

$$F(x) = \text{Laplace}_\lambda((-\infty, x]) = \begin{cases} \frac{1}{2} e^{\lambda x} & \text{if} \quad x < 0, \\ 1 - \frac{1}{2} e^{-\lambda x} & \text{if} \quad x \geq 0. \end{cases} \tag{2}$$

b) Show for all $y \in (0, 1)$ that

$$I_F(y) = \begin{cases} \frac{1}{\lambda} \ln(2y) & \text{if} \quad 0 < y < \frac{1}{2}, \\ -\frac{1}{\lambda} \ln(2 - 2y) & \text{if} \quad \frac{1}{2} \leq y < 1. \end{cases} \tag{3}$$

Due: Friday, 25.10.2024.
Webpage: `https://aam.uni-freiburg.de/agsa/lehre/ws24/numsde/index.html`