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Sheet 3

1. Let (E, d_E) and (F, d_F) be metric spaces and let $f\colon E \to F$ be a function.

a) Prove that f is uniformly continuous if and only if

$$\lim_{h \searrow 0} w_f(h) = w_f(0). \tag{1}$$

b) Next let $\alpha \in (0,1]$. Prove that

$$|f|_{\mathcal{C}^\alpha(E,F)} = \sup_{h\in(0,\infty)} \left[\frac{w_f(h)}{h^\alpha}\right]. \tag{2}$$

2. Let $n \in \mathbb{N}$, $\alpha \in (0,1]$, $a, b \in \mathbb{R}$ with $a < b$. Show that

a) for $f \in \mathcal{L}^1(B_{[a,b]}; |\cdot|_{\mathbb{R}})$:

$$\left|T^n_{[a,b]}[f] - \int_a^b f(x)\,dx\right|_{\mathbb{R}} \leq (b-a) \cdot w_f(\tfrac{(b-a)}{2n}) \leq \frac{(b-a)^{(1+\alpha)}\,|f|_{\mathcal{C}^\alpha([a,b],\mathbb{R})}}{(2n)^\alpha} \tag{3}$$

b) for $f \in C^1([a,b], \mathbb{R})$:

$$\left|T^n_{[a,b]}[f] - \int_a^b f(x)\,dx\right|_{\mathbb{R}} \leq \tfrac{(b-a)^2}{n} \cdot w_{f'}(\tfrac{(b-a)}{2n}) \leq \frac{(b-a)^{(2+\alpha)}\,|f'|_{\mathcal{C}^\alpha([a,b],\mathbb{R})}}{2^\alpha\, n^{(1+\alpha)}}. \tag{4}$$

3. Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $f \in \mathcal{M}(\mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$ be a measurable and bounded function, and let $U_n \in \mathcal{M}(\mathcal{F}, \mathcal{B}(\mathbb{R}))$, $n \in \mathbb{N}$, be independent $\mathcal{U}_{(-1,1)}$-distributed random variables. Prove that

$$\left(\mathbb{E}_P\left[\left|\frac{f(U_1) + \ldots + f(U_{5000})}{2500} - \int_{-1}^1 f(x)\,dx\right|_{\mathbb{R}}^2\right]\right)^{\frac{1}{2}} \leq \frac{\sup_{x\in\mathbb{R}} |f(x)|_{\mathbb{R}}}{30}. \tag{5}$$

Due: Friday, 08.11.2024.
Webpage: `https://aam.uni-freiburg.de/agsa/lehre/ws24/numsde/index.html`