University of Freiburg — Numerical Approximation of SDEs — Winter 2024/25
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Practical Sheet 3

Note that we do not distinguish between pseudo random numbers and actual random numbers.

1. Let $T, x_0 \in (0,\infty)$, $\alpha, \beta \in \mathbb{R}$, let $(\Omega, \mathcal{F}, P)$ be a probability space, let $W\colon [0,T]\times\Omega \to \mathbb{R}$ be a standard Brownian motion, let $X\colon [0,T]\times\Omega\to\mathbb{R}$ be the stochastic process which satisfies for all $t\in[0,T]$ that

$$X_t = e^{(\alpha t + \beta W_t)}\, x_0. \tag{1}$$

The stochastic process X is known as *geometric Brownian motion* and used in the *Black-Scholes model* for the valuation of financial derivatives.

a) Write a Matlab function `MonteCarloGBM(`T`,`α`,`β`,`x_0`,`f`,`N`)` with input $T\in(0,\infty)$, $\alpha,\beta\in\mathbb{R}$, $x_0\in(0,\infty)$, $f\colon\mathbb{R}\to\mathbb{R}$, $N\in\mathbb{N}$ and output a Monte Carlo approximation of

$$\mathbb{E}\big[f(X_T)\big] \tag{2}$$

based on $N\in\mathbb{N}$ samples. Call your function `MonteCarloGBM(`T`, `α`, `β`, `x_0`,`f`,`N`)` with the parameters $T=1$, $\beta=\frac{1}{10}$, $\alpha=\ln(1.06)-\frac{\beta^2}{2}$, $x_0=92$, $f=\mathbb{R}\ni x\mapsto [x-100]^+\in\mathbb{R}$, $N=10^4$.

b) Let $\Phi\colon\mathbb{R}\to\mathbb{R}$ be the $\mathcal{N}(0,1)$-distribution function, i.e. $\Phi(y):=\int_{-\infty}^{y}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2}dx$ for all $y\in\mathbb{R}$. It can be shown that for all $K\in\mathbb{R}$ it holds that

$$\begin{aligned}&\mathbb{E}\big[\max\{X_T-K,0\}\big]\\ &=\begin{cases} e^{\left(\alpha+\frac{1}{2}\beta^2\right)T}\,x_0-K & : K\le 0\\ e^{\left(\alpha+\frac{1}{2}\beta^2\right)T}\,x_0\,\Phi\Big(\frac{\alpha T+\ln(\frac{x_0}{K})}{\beta\sqrt{T}}+\beta\sqrt{T}\Big)-K\,\Phi\Big(\frac{\alpha T+\ln(\frac{x_0}{K})}{\beta\sqrt{T}}\Big) & : K>0\end{cases}.\end{aligned}\tag{3}$$

Use equation (3) and the built-in Matlab function `normcdf(...)` to calculate $\mathbb{E}\big[\max\{X_T-K,0\}\big]$ in the case $T=1$, $\beta=\frac{1}{10}$, $\alpha=\ln(1.06)-\frac{\beta^2}{2}$, $x_0=92$, $K=100$. Compare this result with the result of item a).

2. Let $T\in(0,\infty)$, $d,m,N\in\mathbb{N}$, $\xi\in\mathbb{R}^d$, $\mu\in\mathcal{M}\big(\mathcal{B}(\mathbb{R}^d),\mathcal{B}(\mathbb{R}^d)\big)$, $\sigma\in\mathcal{M}\big(\mathcal{B}(\mathbb{R}^d),\mathcal{B}(\mathbb{R}^{d\times m})\big)$, let $(\Omega,\mathcal{F},P,(\mathbb{F}_t)_{t\in[0,T]})$ be a filtered probability space, let $W\colon[0,T]\times\Omega\to\mathbb{R}^m$ be an m-dimensional standard $(\Omega,\mathcal{F},P,(\mathbb{F}_t)_{t\in[0,T]})$-Brownian motion, and consider the SDE

$$dX_t=\mu(X_t)\,dt+\sigma(X_t)\,dW_t,\qquad t\in[0,T],\qquad X_0=\xi. \tag{4}$$

(i) Write a MATLAB function EulerMaruyama(T,m,ξ,μ,σ,W) which performs the Euler–Maruyama scheme with time step size $h = \frac{T}{N}$ for the SDE above, where (besides the input parameters T, m, ξ named above) the input functions $\mu\colon \mathbb{R}^{d\times M} \to \mathbb{R}^{d\times M}$ and $\sigma\colon \mathbb{R}^{d\times M} \to \mathbb{R}^{d\times Mm}$ can be thought of as extended versions of above μ and σ, and the input parameter $W \in \mathbb{R}^{(N+1)\times Mm}$ is a realization of M independent m-dimensional Brownian motions at the nodes $\left\{\frac{nT}{N} : n = 0, \ldots, N\right\}$, i.e. $W^{:,i+(k-1)M} = \left(W_0^k, W_{\frac{T}{N}}^k, W_{\frac{2T}{N}}^k, \ldots, W_{\frac{(N-1)T}{N}}^k, W_T^k\right)(\omega_i)$ for $i = 1, 2, \ldots, M$ and $k = 1, \ldots, m$. The function should return the realizations of the Euler-Maruyama approximation $Y_N(\omega_i) \in \mathbb{R}^d$ at the endpoint T for $i = 1, 2, \ldots, M$. You can use the template `EulerMaruyama_template.m` for this.

(ii) Investigate the strong error of the Euler–Maruyama scheme by fixing the parameters $d = 2$, $m = 2$, $T = 1$,

$$\mu(x_1, x_2) = \begin{pmatrix} 0.5x_1 \\ 2x_2 \end{pmatrix}, \qquad \sigma(x_1, x_2) = \begin{pmatrix} x_1 & 0 \\ 0 & 2x_2 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \tag{5}$$

and using $N_\ell = 10 \cdot 2^\ell$, $h_\ell = T/N_\ell$ for $\ell \in \{0, 1, \ldots, 4\}$. To do so: generate $M = 10^5$ sample paths of the Brownian motion at the finest grid points $\{jh_4 \colon j = 0, 1, \ldots, N_4\}$. Based on these, generate for each $\ell \in \{0, 1, \ldots, 4\}$ the M corresponding approximations to X_T with values $Y_{N_\ell}^{h_\ell}$. Then use the M simulations of $\mathcal{E}^{h_\ell} := \|Y_{N_\ell}^{h_\ell} - X_T\|_{\mathbb{R}^2}$, where $\|\cdot\|_{\mathbb{R}^2}$ is the Euclidean norm on $\mathbb{R}^2$, for $\ell = 0, 1, \ldots, 4$ to determine the “experimental strong convergence rate” of $E_M(\mathcal{E}^{h_\ell}) := \frac{1}{M}\sum_{j=1}^{M} \mathcal{E}_{(j)}^{h_\ell}$ with respect to h_ℓ. Hints:

- Using the SDE representations of two suitable geometric Brownian motions, you can derive the exact solution X of (4).
- Estimate the convergence rate by a linear regression of $\log(E_M(\mathcal{E}^{h_\ell}))$ on the log-stepsizes $\log(h_\ell)$. For this you may use the MATLAB function `polyfit`.

3. Let $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $W\colon [0, T] \times \Omega \to \mathbb{R}$ be a one-dimensional standard $(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t\in[0,T]})$-Brownian motion, let $\xi \in \mathbb{R}$, let $\mu : \mathbb{R} \to \mathbb{R}$ be globally Lipschitz continuous and let $\sigma \in C^1(\mathbb{R}; \mathbb{R})$. Consider the (general) one-dimensional SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad t \in [0, T], \quad X_0 = \xi. \tag{6}$$

(i) Write a Matlab function Milstein1D($T, \xi, \mu, \sigma, \sigma', W$) that applies the Milstein scheme to discretize the one-dimensional SDE (6). The input parameters μ, σ, and σ' are function handles and $W \in \mathbb{R}^{(N+1)\times M}$ is a realization of M independent one-dimensional Brownian motions at the nodes $\left\{\frac{nT}{N} : n = 0, \ldots, N\right\}$, i.e. $(W^{:,i}) = \left(W_0, W_{\frac{T}{N}}, W_{\frac{2T}{N}}, \ldots, W_{\frac{(N-1)T}{N}}, W_T\right)(\omega_i)$ for $i = 1, 2, \ldots, M$.

Hint: The easiest way is to modify the solution `EulerMaruyama.m` from the above Exercise 2 for a one-dimensional SDE.

(ii) Test your implementation for the SDE

$$dX_t = X_t dt + \log\left(1 + X_t^2\right) dW_t, \quad t \in [0,T], \quad X_0 = 1. \tag{7}$$

Find experimental convergence rates. For this use $M = 10^5$ and $N_\ell = 10 \cdot 2^\ell$, $h_\ell = T/N_\ell$ for $\ell = 0, 1, ..., 4$ and report on the experimental rates of strong convergence in L^1 and L^2, i.e., of $\frac{1}{M}\sum_{j=1}^{M} \mathcal{E}_{(j)}^{h_\ell}$ and $\left(\frac{1}{M}\sum_{j=1}^{M}(\mathcal{E}_{(j)}^{h_\ell})^2\right)^{\frac{1}{2}}$. Use as an approximation of the exact solution a numerical solution of the SDE on the level $\underline{L = 7}$ of refinement.

For this task, you may use the template `Milstein_SDE_template.m`

(iii) Repeat item (ii) for the SDE

$$dX_t = X_t dt + \sin\left(1 + X_t^2\right) dW_t, \quad t \in [0,T], \quad X_0 = 1. \tag{8}$$

Comment on the results for the SDEs (7) and (8).

Due: Friday, 29.11.2024.
Webpage: `https://aam.uni-freiburg.de/agsa/lehre/ws24/numsde/index.html`