JProf. Dr. Diyora Salimova
M.Sc. Ilkhom Mukhammadiev

Sheet 4

1. a) Specify explicitly measurable spaces $(\Omega, \mathcal{F})$ and $(S, \mathcal{S})$ and $\mathcal{F}/\mathcal{S}$-measurable functions $X, Y\colon \Omega \to S$ such that

$$\{X = Y\} = \{\omega \in \Omega\colon X(\omega) = Y(\omega)\} \notin \mathcal{F}. \tag{1}$$

Prove that your result is correct.

b) Give an example of stochastic processes X and Y which are modifications of each other but not indistinguishable from each other. Prove that your result is correct.

2. *Modifications with continuous sample paths:* Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $T \in [0, \infty)$, $m \in \mathbb{N}$, let $X, Y\colon [0,T] \times \Omega \to \mathbb{R}^m$ be stochastic processes with continuous sample paths which satisfy for all $t \in [0,T]$ that $P(X_t = Y_t) = 1$. Prove that X and Y are indistinguishable from each other.

Hint: Use the fact that the rational numbers are dense in $\mathbb{R}$.

3. *Itô's formula for time-dependent test functions:* Let $T \in (0, \infty)$, $d, m \in \mathbb{N}$, let $(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t\in[0,T]})$ be a filtered probability space, let

$$f\colon \mathbb{R}^{d+1} \ni (s, x) \mapsto f(s, x) \in \mathbb{R} \qquad \text{(test function)}$$

be a *twice* continuously differentiable function, and let $X\colon [0,T] \times \Omega \to \mathbb{R}^d$ be an Itô process with drift

$$Y\colon [0,T] \times \Omega \to \mathbb{R}^d, \qquad \text{(drift)}$$

diffusion

$$Z\colon [0,T] \times \Omega \to \mathbb{R}^{d\times m} \qquad \text{(diffusion)}$$

and standard Brownian motion $W\colon [0,T] \times \Omega \to \mathbb{R}^m$. Use the Itô's formula from the class to show that $(f(t, X_t))_{t\in[0,T]}$ is also an Itô process and derive the corresponding drift and diffusion processes.

Due: Friday, 15.11.2024.
Webpage: `https://aam.uni-freiburg.de/agsa/lehre/ws24/numsde/index.html`