University of Freiburg Numerical Approximation of SDEs Winter 2024/25
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Sheet 5

1. Let $\xi \in \mathcal{L}^p(P|_{\mathbb{F}_0}; |\cdot|)$ for some $p \geq 2$. Show that the SDE

$$dX_t = \log(1 + X_t^2)dt + \mathbf{1}_{\{X_t > 0\}} X_t dW_t, \quad t \in [0, T], \quad X_0 = \xi,$$

has a unique solution process $X \colon [0, T] \times \Omega \to \mathbb{R}$.

2. Show that the SDE

$$dX_t = 3X_t^{1/3} dt + 3X_t^{2/3} dW_t, \quad t \in [0, T], \quad X_0 = 0.$$

has infinitely many solution processes $X \colon [0, T] \times \Omega \to \mathbb{R}$. Explain which condition of the existence-and-uniqueness theorem from the lecture is violated.
Hint: Use the function $\theta_a := (x - a)^3 \mathbf{1}_{\{x \geq a\}}$ for some constant $a > 0$.

3. **a)** Let $\alpha \in \mathbb{R}$, $\beta \in (0, \infty)$, let $Y \colon \Omega \to \mathbb{R}$ be an $\mathcal{N}_{\alpha,\beta^2}$-distributed random variable, and let $\Phi \colon \mathbb{R} \to \mathbb{R}$ be the function which satisfies for all $x \in \mathbb{R}$ that

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \, dy. \tag{1}$$

Prove for all $K \in \mathbb{R}$ that

$$\mathbb{E}\big[\max\{e^Y - K, 0\}\big] = \begin{cases} e^{\alpha + \frac{1}{2}\beta^2} \, \Phi\Big(\frac{\alpha - \ln(K)}{\beta} + \beta\Big) - K \, \Phi\Big(\frac{\alpha - \ln(K)}{\beta}\Big) & : K > 0 \\ e^{\alpha + \frac{1}{2}\beta^2} - K & : K \leq 0 \end{cases}. \tag{2}$$

Hint: First show for an $\mathcal{N}_{0,1}$-distributed random variable X and $c \in (0, \infty)$ that $\mathbb{E}\big[e^{cX}\big] = \exp\big(\frac{1}{2}c^2\big)$.

b) Use the above formula to derive the price of a *European call option.*

Due: Friday, 22.11.2024.
Webpage: `https://aam.uni-freiburg.de/agsa/lehre/ws24/numsde/index.html`