Error estimates for the Willmore flow of graphs

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Abstract: The evolution of two–dimensional graphs under Willmore flow is approximated by a finite element method. The highly nonlinear fourth order problem is split into two coupled second order problems using height and a weighted mean curvature as variables. We prove a–priori error estimates for the resulting time–continuous scheme and present results of test calculations.

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1 Introduction

The purpose of this paper is to analyze a finite element scheme for the approximation of two–dimensional surfaces in $\mathbb{R}^3$ which evolve according to Willmore flow. This flow can be interpreted as the L^2–gradient flow for the Willmore functional

$$W(f) := \frac{1}{2}\int_\Gamma H^2 dA, \qquad \Gamma = f(M),$$

where $f : M \to \mathbb{R}^3$ is a smooth immersion and $H = \kappa_1 + \kappa_2$ denotes the mean curvature of Γ. Considering normal variations $f_\epsilon(x) := f(x) + \epsilon\phi(x)\nu(x), x \in M$, where ν is a unit normal field to Γ and $\phi : M \to \mathbb{R}$ is smooth and vanishes near ∂M, one obtains the formula

$$\langle W'(f), \phi\rangle := \frac{d}{d\epsilon}W(f_\epsilon)_{|\epsilon=0} = -\int_\Gamma \phi\big(\Delta_\Gamma H + \frac{1}{2}H^3 - 2HK\big)dA.$$

Here, $K = \kappa_1\kappa_2$ is the Gauss curvature of Γ and Δ_Γ denotes the Laplace–Beltrami operator. The sign of H is chosen in such a way that $H > 0$ for a sphere with outward pointing normal. For details on this calculation and more information on the Willmore functional see [17]. Given a smooth, oriented surface $\Gamma_0 \subset \mathbb{R}^3$, the Willmore flow problem now consists in finding a family of smooth, oriented surfaces $(\Gamma(t))_{t\in[0,T]}$, which satisfy

$$\begin{aligned} V &= \Delta_\Gamma H + \frac{1}{2}H^3 - 2HK \qquad \text{on } \Gamma(t), && (1.1)\\ \Gamma(0) &= \Gamma_0, && (1.2) \end{aligned}$$

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where V denotes the normal velocity.
In this paper we shall assume that the surfaces $\Gamma(t)$ are graphs over some base domain $\Omega \subset \mathbb{R}^2$, i.e.

$$\Gamma(t) = \{(x, u(x,t)) \,|\, x \in \Omega\},$$

oriented by the unit normal field

$$\nu = \frac{(\nabla u, -1)}{\sqrt{1+|\nabla u|^2}}. \tag{1.3}$$

It will be convenient to use the abbreviation

$$Q := \sqrt{1+|\nabla u|^2} \tag{1.4}$$

for the area element. The evolution law (1.1) gives rise to a partial differential equation for the height function u. In order to write down this equation we note that the quantities V, H, K and $\Delta_\Gamma H$ appearing in (1.1) are expressed in terms of u as follows:

$$\begin{aligned} V &= -\frac{u_t}{Q}, \qquad H = \nabla\cdot(\frac{\nabla u}{Q}), \qquad K = \frac{\det D^2 u}{Q^4}, && (1.5)\\ \Delta_\Gamma H &= \frac{1}{Q}\nabla\cdot\Big((QI - \frac{\nabla u\otimes\nabla u}{Q})\nabla H\Big). && (1.6)\end{aligned}$$

The last relation can be rewritten as

$$\Delta_\Gamma H = \nabla\cdot\Big(\frac{1}{Q}(I - \frac{\nabla u\otimes\nabla u}{Q^2})\nabla(QH)\Big) - H\,\nabla\cdot\Big(\frac{1}{Q}(I - \frac{\nabla u\otimes\nabla u}{Q^2})\nabla Q\Big). \tag{1.7}$$

Using the expression for H we coclude

$$\frac{1}{Q}(I - \frac{\nabla u\otimes\nabla u}{Q^2})\nabla Q = \frac{1}{Q}(\nabla Q - \frac{\Delta u}{Q}\nabla u) + H\,\frac{\nabla u}{Q} \tag{1.8}$$

and a calculation shows that

$$\nabla\cdot\Big(\frac{1}{Q}(\nabla Q - \frac{\Delta u}{Q}\nabla u)\Big) = -2K. \tag{1.9}$$

Inserting (1.8) and (1.9) into (1.7) we obtain

$$\begin{aligned}\Delta_\Gamma H &= \nabla\cdot\Big(\frac{1}{Q}(I - \frac{\nabla u\otimes\nabla u}{Q^2})\nabla(QH)\Big) + 2HK - H\nabla\cdot(H\,\frac{\nabla u}{Q})\\ &= \nabla\cdot\Big(\frac{1}{Q}(I - \frac{\nabla u\otimes\nabla u}{Q^2})\nabla(QH)\Big) + 2HK - \frac{1}{2}\nabla\cdot\Big(\frac{H^2}{Q}\nabla u\Big) - \frac{1}{2}H^3.\end{aligned}$$

Comparing this expression with (1.1) and recalling that $V = -\dfrac{u_t}{Q}$ we obtain the following fourth order parabolic PDE for u,

$$u_t + Q\,\nabla\cdot\Big(\frac{1}{Q}(I - \frac{\nabla u\otimes\nabla u}{Q^2})\nabla(QH)\Big) - \frac{1}{2}Q\,\nabla\cdot\Big(\frac{H^2}{Q}\nabla u\Big) = 0 \qquad \text{in } \Omega\times(0,T). \tag{1.10}$$

Note that the above equation has (after division by Q) a nice divergence structure in which the Gauss curvature K no longer appears. This structure was exploited by Droske and Rumpf in [8] for a level set approach to Willmore flow. We shall use the boundary condtions

$$u = g, \quad H = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.11}$$

for a given smooth function $g : \bar{\Omega} \to \mathbb{R}$. The condition (1.11) is motivated by calculations in [14], where it is shown that if $\Gamma = f(M)$ is stationary for E with respect to variations which keep the boundary of Γ fixed, then necessarily $H = 0$ on $\partial\Gamma$. Finally, we prescribe the initial condition

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega. \tag{1.12}$$

In what follows we shall assume that the initial–boundary value problem (1.10)–(1.12) has a unique solution u, which satisfies

$$u \in L^\infty((0,T); H^{4,\infty}(\Omega)) \cap L^2((0,T); H^5(\Omega)), \tag{1.13}$$
$$u_t \in L^\infty((0,T); H^{2,\infty}(\Omega)) \cap L^2((0,T); H^3(\Omega)), \tag{1.14}$$
$$u_{tt} \in L^\infty((0,T); L^\infty(\Omega)) \cap L^2((0,T); H^1(\Omega)). \tag{1.15}$$

Since the focus of our work is on the analysis of a finite element scheme approximating solutions of (1.10)–(1.12), we do not address the question of whether a function u verifying the regularity assumptions (1.13)–(1.15) exists. We expect a positive answer at least for small T, but at the moment little seems to be known about boundary value problems for Willmore flow. The situation is different for the evolution of closed surfaces. In [16] it is shown that a unique local solution of (1.1), (1.2) exists provided that Γ_0 is a compact closed immersed and orientable $C^{2,\alpha}$–surface in $\mathbb{R}^3$. The solution exists globally in time if Γ_0 is sufficiently close to a sphere in $C^{2,\alpha}$. Using different methods, Kuwert & Schätzle ([11]) obtain global existence of solutions provided that $\int_{\Gamma_0} |A^\circ|^2$ is sufficiently small, where A° denotes the trace–free part of the second fundamental form. They were subsequently able to remove the smallness assumption and to prove the existence of a global smooth solution provided that $W(f_0) \leq 16\pi$, where $\Gamma_0 = f_0(S^2)$ (see [12] and note that our definition differs from the one used in [12] by a factor of 2). There is numerical evidence ([13]) that the above condition is optimal in the sense that the flow develops a singularity if the initial surface has energy greater that 16π.

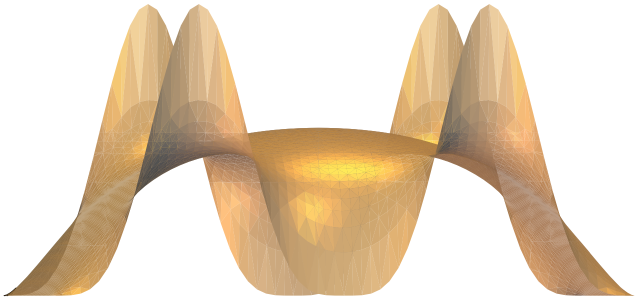

Figure 1: A stationary solution of (1.10)–(1.11): Willmore surface with prescribed boundary and with mean curvature equal to zero on the boundary. The solution was computed with the time dependent algorithm (4.1)–(4.2).

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2 Variational formulation and discretization

In order to approximate solutions of (1.10)–(1.12) by a finite element scheme, we need to derive a variational formulation of (1.10). To begin, let us introduce

$$E(p)_{ij} := \frac{1}{\sqrt{1+|p|^2}}\left(\delta_{ij} - \frac{p_i p_j}{1+|p|^2}\right), \qquad i,j = 1,2,\, p \in \mathbb{R}^2. \tag{2.1}$$

Furthermore (1.10) suggests to use $w = -QH$ rather than H as the second variable in a splitting method (cf. [8]). If we divide (1.10) by Q, multiply by a test function $\varphi \in H_0^1(\Omega)$ and integrate by parts we are led to

$$\begin{aligned}\int_\Omega \frac{u_t\varphi}{Q} + \int_\Omega E(\nabla u)\nabla w \cdot \nabla\varphi + \frac{1}{2}\int_\Omega \frac{w^2}{Q^3}\nabla u \cdot \nabla\varphi &= 0 \quad \forall \varphi \in H_0^1(\Omega) &(2.2)\\ \int_\Omega \frac{w\zeta}{Q} - \int_\Omega \frac{\nabla u \cdot \nabla\zeta}{Q} &= 0 \quad \forall \zeta \in H_0^1(\Omega), &(2.3)\end{aligned}$$

where the second relation stems from the definition of w and (1.5).

Before we proceed and base the discretization of our problem on (2.2), (2.3), we want to deduce the decrease in time of the Willmore energy $\frac{1}{2}\int_{\Gamma(t)} H^2 dS = \frac{1}{2}\int_\Omega H^2 Q$ from these relations. To begin, differentiating (2.3) with respect to time gives

$$\int_\Omega \frac{w_t\,\zeta}{Q} - \int_\Omega \frac{w\,\zeta\,Q_t}{Q^2} - \int_\Omega E(\nabla u)\nabla u_t \cdot \nabla\zeta = 0 \qquad \forall \zeta \in H_0^1(\Omega). \tag{2.4}$$

If we insert $\zeta = w$ into (2.4), $\varphi = u_t$ into (2.2) and add the resulting equations we obtain, keeping in mind that $\nabla u \cdot \nabla u_t = Q_t Q$,

$$0 = \int_\Omega \frac{u_t^2}{Q} - \frac{1}{2}\int_\Omega \frac{w^2\,Q_t}{Q^2} + \int_\Omega \frac{w\,w_t}{Q} = \int_\Omega \frac{u_t^2}{Q} + \frac{1}{2}\frac{d}{dt}\int_\Omega \frac{w^2}{Q}. \tag{2.5}$$

Observing that $\int_\Omega \frac{w^2}{Q} = \int_\Omega H^2\,Q$ we see that the energy decreases in time.

Let us now turn to the discretization in space. The variational formulation (2.2), (2.3) suggests to use a second order splitting scheme with linear finite elements in order to approximate the pair (u,w). Let $\mathcal{T}_h$ be a family of triangulations of Ω with maximum mesh size $h := \max_{S\in\mathcal{T}_h} \operatorname{diam}(S)$. We suppose that $\bar\Omega$ is the union of the elements of $\mathcal{T}_h$ so that element edges lying on the boundary are allowed to be curved. Furthermore we suppose that the triangulation is quasiuniform in the sense that there exists a constant $\kappa > 0$ (independent of h) such that each $S \in \mathcal{T}_h$ is contained in a ball of radius $\kappa^{-1}h$ and contains a ball of radius κh. The discrete space is defined by

$$X_h := \{v_h \in C^0(\bar\Omega) \,|\, v_h \text{ is a linear polynomial on each } S \in \mathcal{T}_h\},$$

with suitable modifications for boundary elements. There exists an interpolation operator $I_h : H^2(\Omega) \to X_h$ such that

$$\|v - I_h v\| + h\|\nabla(v - I_h v)\| \le ch^2 \|v\|_{H^2(\Omega)} \qquad \text{for all } v \in H^2(\Omega). \tag{2.6}$$

Furthermore, let $X_{h0} := X_h \cap H^1_0(\Omega)$ and suppose that $I_h v \in X_{h0}$ for $v \in H^2(\Omega) \cap H^1_0(\Omega)$. Here and throughout the paper we will denote the $L^2(\Omega)$–norm by $\| \cdot \|$.

Our discrete problem now reads: find $(u_h(t), w_h(t)), 0 \le t \le T$, such that $u_h(t) - I_h g \in X_{h0}$, $w_h(t) \in X_{h0}$, $u_h(0) = u_{0h} \in X_{h0}$ and

$$\begin{aligned} \int_\Omega \frac{u_{ht}\varphi_h}{Q_h} + \int_\Omega E(\nabla u_h)\nabla w_h \cdot \nabla\varphi_h + \frac{1}{2}\int_\Omega \frac{w_h^2}{Q_h^3}\nabla u_h \cdot \nabla\varphi_h &= 0 \quad \forall \varphi_h \in X_{h0} &(2.7)\\ \int_\Omega \frac{w_h \zeta_h}{Q_h} - \int_\Omega \frac{\nabla u_h \cdot \nabla\zeta_h}{Q_h} &= 0 \quad \forall \zeta_h \in X_{h0}. &(2.8)\end{aligned}$$

Lemma 2.1. *The system (2.7), (2.8) has a unique solution* (u_h, w_h) *on* $[0,T]$ *for all* $T < \infty$.

Proof. Local existence on an interval $[0, t_h)$ follows from the theory of ordinary differential equations. Since $u_h(t), w_h(t)$ have values in a finite–dimensional space, it is sufficient to bound some norm of (u_h, w_h) in order to obtain existence on $[0,T]$. If we repeat the argument which led to (2.5) in the discrete setting, the result is

$$\int_0^t \int_\Omega \frac{u_{ht}^2}{Q_h} + \frac{1}{2}\int_\Omega \frac{w_h^2}{Q_h} \le C(h, u_{0h}), \qquad 0 \le t < t_h. \tag{2.9}$$

Using $\zeta_h = u_{ht} \in X_{h0}$ in (2.8) we obtain $\frac{d}{dt}\int_\Omega Q_h = \int_\Omega \frac{w_h u_{ht}}{Q_h}$, which combined with (2.9) yields

$$\int_\Omega Q_h(t) \le c + \left(\int_0^t \int_\Omega \frac{w_h^2}{Q_h}\right)^{\frac{1}{2}} \left(\int_0^t \int_\Omega \frac{u_{ht}^2}{Q_h}\right)^{\frac{1}{2}} \le c(h, T, u_{0h}), \qquad 0 \le t < t_h.$$

This implies that $Q_h(x,t) \le c(h,T)$ uniformly in $(x,t) \in \Omega \times [0, t_h)$, and (2.9) then shows that $\|u_h(t)\|$, $\|w_h(t)\|$ remain bounded on $[0, t_h)$. We leave the proof of uniqueness to the reader. ∎

Our main result is the following error estimate:

Theorem 2.2. *Let us assume that (1.10)–(1.12) has a unique solution* u *on the interval* $[0,T]$*, which satisfies (1.13)–(1.15). Also suppose that* $u_{0h} = \widehat{u}_{0h}$*, where* $\widehat{u}_{0h}$ *is defined as the projection (2.14) of* u_0*. Then*

$$\begin{aligned} \sup_{0\le t\le T} \|(u - u_h)(t)\| + \sup_{0\le t\le T} \|(w - w_h)(t)\| &\le ch^2|\log h|^2 &(2.10)\\ \sup_{0\le t\le T} \|\nabla(u - u_h)(t)\| &\le ch &(2.11)\\ \int_0^T \|u_t - u_{ht}\|^2 dt &\le ch^4|\log h|^4 &(2.12)\\ \int_0^T \|\nabla(w - w_h)\|^2 dt &\le ch^2. &(2.13)\end{aligned}$$

A detailed proof of this theorem will be given in Section 3. One of the crucial points of our error analysis is the use of suitable projections of u and w which we introduce next. To begin, let $\widehat{u}_h$ be defined by: $\widehat{u}_h - I_h g \in X_{h0}$ and

$$\int_\Omega \frac{\nabla \widehat{u}_h \cdot \nabla \zeta_h}{\widehat{Q}_h} = \int_\Omega \frac{\nabla u \cdot \nabla \zeta_h}{Q} \qquad \forall \zeta_h \in X_{h0}, \tag{2.14}$$

where $\widehat{Q}_h = \sqrt{1 + |\nabla \widehat{u}_h|^2}$. Note that t is just a parameter. Estimates for the error between u and $\widehat{u}_h$ in the time–independent case were first carried out in [10], L^∞–estimates are due to [9]. For functions u which depend on time it was proved in [4], [5] that

$$\begin{aligned}
\sup_{0\le t\le T} \|(u-\widehat{u}_h)(t)\| + h \sup_{0\le t\le T} \|\nabla(u-\widehat{u}_h)(t)\| &\le ch^2, && (2.15)\\
\sup_{0\le t\le T} \|(u-\widehat{u}_h)(t)\|_{L^\infty} + h \sup_{0\le t\le T} \|\nabla(u-\widehat{u}_h)(t)\|_{L^\infty} &\le ch^2|\log h|, && (2.16)\\
\sup_{0\le t\le T} \|(u_t-\widehat{u}_{h,t})(t)\| &\le ch^2|\log h|^2, && (2.17)\\
\sup_{0\le t\le T} \|\nabla(u_t-\widehat{u}_{h,t})(t)\| &\le ch. && (2.18)
\end{aligned}$$

With the help of $\widehat{u}_h$, we next define a projection $\widehat{w}_h \in X_{h0}$ of w as follows:

$$\int_\Omega E(\nabla \widehat{u}_h)\nabla \widehat{w}_h \cdot \nabla \varphi_h = \int_\Omega E(\nabla u)\nabla w \cdot \nabla \varphi_h + \frac{1}{2}\int_\Omega w^2\Big(\frac{\nabla u}{Q^3} - \frac{\nabla \widehat{u}_h}{\widehat{Q}_h^3}\Big)\cdot \nabla \varphi_h \qquad \forall \varphi_h \in X_{h0}. \tag{2.19}$$

The proof of the following error bounds will be given in Lemma A.1 of the Appendix:

$$\begin{aligned}
\sup_{0\le t\le T} \|\nabla(w-\widehat{w}_h)(t)\| &\le ch && (2.20)\\
\sup_{0\le t\le T} \|(w-\widehat{w}_h)(t)\| &\le ch^2|\log h| && (2.21)\\
\sup_{0\le t\le T} \|\nabla(w_t-\widehat{w}_{ht})(t)\| &\le ch && (2.22)\\
\sup_{0\le t\le T} \|(w_t-\widehat{w}_{ht})(t)\| &\le ch^2|\log h|^2. && (2.23)
\end{aligned}$$

Note that (1.13), (1.14), (2.16)–(2.18), (2.20)–(2.23) together with interpolation and inverse estimates imply that

$$\|\widehat{u}_h\|_{W^{1,\infty}}, \|\widehat{u}_{ht}\|_{W^{1,\infty}}, \|\widehat{w}_h\|_{W^{1,\infty}}, \|\widehat{w}_{ht}\|_{W^{1,\infty}} \le c \tag{2.24}$$

uniformly in h.

Let us briefly sketch the main ideas in the proof of Theorem 2.2. Combining the variational identities (2.2), (2.3) and (2.7), (2.8) with (2.14) and (2.19) we shall derive corresponding relations for $e_u := \widehat{u}_h - u_h, e_w := \widehat{w}_h - w_h$. A careful analysis, which essentially mimicks the a–priori estimate (2.5), leads to

$$\sup_{0\le t\le T}\big(\|e_w(t)\|^2 + \|e_u(t)\|^2 + \|\nabla e_u(t)\|^2\big) \le ch^4|\log h|^4.$$

Superconvergence for $\|\nabla e_u\|$ is then exploited via a bootstrap argument in order to argue that Q_h remains bounded uniformly, a property which is crucial for the analysis to ensure

ellipticity of $E(\nabla u_h)$ and control on $\|e_{ut}\|$.
We end this Section with a few remarks on numerical approaches to Willmore flow and related problems. In [13], a finite difference scheme is derived to approximate axisymmetric solutions of evolution laws $V = f(\kappa_1, \kappa_2)$. The Willmore flow is studied in detail and numerical evidence is provided that the flow may develop singularities in finite time. [8] derives a level set formulation using the level set function and a weighted mean curvature as variables. The formal similarity to the graph approach motivated our choice of variables. For the parametric approach to Willmore flow, [15] derives a variational form which employs position and mean curvature vector as variables and allows the use of linear finite elements to discretize in space. This approach is subsequently extended in [3] to surfaces with boundaries and applied to problems in surface restoration. We also refer to [2], where numerical simulations of anisotropic surface diffusion are carried out for a surface energy which consists of a strongly anisotropic nonconvex part and the Willmore functional (weighted by a small factor).
Let us finally mention evolution by surface diffusion,

$$V = \Delta_\Gamma H \qquad \text{on } \Gamma(t),$$

which coincides with Willmore flow in the highest order term, but which is simpler in that the nonlinear terms in the principal curvatures are absent. In the graph case, it is possible to write down a splitting method using height and mean curvature as variables. Finite element error bounds both for time–continuous and fully discrete schemes have been derived in [6], [1], [7].

3 Proof of the error estimate

Let (u_h, w_h) be the discrete solution and denote by $Q_h := \sqrt{1 + |\nabla u_h|^2}$ the discrete area element. Furthermore, define

$$C_0 := \sup_{x \in \Omega, 0 \le t \le T} Q(x,t), \qquad \text{and } C_1 := \sup_{x \in \Omega, 0 \le t \le T} |w(x,t)|.$$

The initial condition together with a continuity argument then yields for small h

$$\sup_{x \in \Omega} Q_h(x,t) \le 2C_0, \quad \sup_{x \in \Omega} |w_h(x,t)| \le 2C_1 \tag{3.1}$$

for small t. Let us define

$$T_h := \sup\{t \in [0,T] \,|\, (3.1) \text{ holds on } [0,t]\}. \tag{3.2}$$

Our strategy is to first prove the error estimates on the interval $[0, T_h)$ and subsequently use these bounds to show that $T_h = T$. Thus, in what follows we shall assume (3.1). Let us decompose the errors $u - u_h$ and $w - w_h$ according to

$$\begin{aligned} u - u_h &= (u - \widehat{u}_h) + (\widehat{u}_h - u_h) =: \rho_u + e_u, \\ w - w_h &= (w - \widehat{w}_h) + (\widehat{w}_h - w_h) =: \rho_w + e_w. \end{aligned}$$

Furthermore, it will be convenient to work with the following discrete normals

$$\widehat{\nu}_h := \frac{(\nabla \widehat{u}_h, -1)}{\widehat{Q}_h}, \quad \nu_h := \frac{(\nabla u_h, -1)}{Q_h}.$$

We infer from Proposition 2 in [5] that

$$|\widehat{\nu}_h - \nu_h| \leq |\nabla(\widehat{u}_h - u_h)| \leq (1 + \sup_{\Omega} |\nabla \widehat{u}_h|) Q_h \, |\widehat{\nu}_h - \nu_h|,$$

which, combined with (3.1) and (2.24), yields

$$|\widehat{\nu}_h - \nu_h| \leq |\nabla e_u| \leq (1 + c) 2 C_0 |\widehat{\nu}_h - \nu_h|. \tag{3.3}$$

In order to make the error analysis more transparent we split it up into a series of auxiliary results.

Lemma 3.1. *Suppose that $F : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable and that $f \in H^1(\Omega)$. Then*

$$\int_\Omega \big(F(\nabla u) - F(\nabla \widehat{u}_h)\big) f = - \int_\Omega \nabla \cdot \big(f \, F'(\nabla u)\big) \rho_u + R,$$

where R satisfies

$$|R| \leq C h^2 |\log h| \|f\|.$$

Proof. Clearly,

$$\begin{aligned}
\int_\Omega \big(F(\nabla u) - F(\nabla \widehat{u}_h)\big) f &= \int_\Omega F'(\nabla u) \cdot \nabla \rho_u \, f + \int_0^1 \int_\Omega \big(F'(\nabla u - s \nabla \rho_u) - F'(\nabla u)\big) \cdot \nabla \rho_u \, f \\
&= - \int_\Omega \nabla \cdot \big(f \, F'(\nabla u)\big) \rho_u + R,
\end{aligned}$$

where

$$|R| \leq c \|\nabla \rho_u\|_{L^\infty} \|\nabla \rho_u\| \, \|f\| \leq C h^2 |\log h| \|f\|$$

by (2.15), (2.16) and (2.24). ∎

Lemma 3.2. *For every $\epsilon > 0$ there exists c_ϵ such that*

$$||\nabla e_u(t)||^2 \leq \epsilon \, ||e_w(t)||^2 + c_\epsilon ||e_u(t)||^2 + c h^4 |\log h|^2, \qquad 0 \leq t < T_h.$$

Proof. In view (2.3), (2.8) and (2.14) we have

$$\int_\Omega \left(\frac{\nabla \widehat{u}_h}{\widehat{Q}_h} - \frac{\nabla u_h}{Q_h} \right) \cdot \nabla \varphi_h = \int_\Omega \left(\frac{w}{Q} - \frac{w_h}{Q_h} \right) \varphi_h \qquad \forall \varphi_h \in X_{h0}.$$

Using $\varphi_h = \widehat{u}_h - u_h = e_u \in X_{h0}$ and applying Lemma 3.1 with $F(p) = \dfrac{1}{\sqrt{1 + |p|^2}}$, $f = w \, e_u$ we derive

$$\begin{aligned}
\int_\Omega \left(\frac{\nabla \widehat{u}_h}{\widehat{Q}_h} - \frac{\nabla u_h}{Q_h} \right) \cdot \nabla e_u &= \int_\Omega \left(\frac{1}{Q} - \frac{1}{\widehat{Q}_h} \right) w \, e_u + \int_\Omega \left(\frac{1}{\widehat{Q}_h} - \frac{1}{Q_h} \right) w \, e_u + \int_\Omega \frac{(w - w_h) e_u}{Q_h} \\
&\leq \int_\Omega \nabla \cdot \left(w \, e_u \frac{\nabla u}{Q^3} \right) \rho_u + c h^2 |\log h| \, \|w \, e_u\| + c \|\nabla e_u\| \, \|e_u\| + \big(\|e_w\| + \|\rho_w\|\big) \|e_u\| \\
&\leq c h^2 \|\nabla e_u\| + c\big(\|\nabla e_u\| + \|e_w\| + h^2 |\log h|\big) \|e_u\|,
\end{aligned} \tag{3.4}$$

in view of (2.15) and (2.21). Observing that $(\nabla e_u, 0)^t = \widehat{Q}_h \widehat{\nu}_h - Q_h \nu_h$ we may write

$$\left(\frac{\nabla \widehat{u}_h}{\widehat{Q}_h} - \frac{\nabla u_h}{Q_h} \right) \cdot \nabla e_u = \big(\widehat{\nu}_h - \nu_h\big) \cdot \big(\widehat{Q}_h \widehat{\nu}_h - Q_h \nu_h\big) = \frac{1}{2} |\widehat{\nu}_h - \nu_h|^2 (\widehat{Q}_h + Q_h) \geq c_0 \|\nabla e_u\|^2$$

by (2.24) and (3.3). Combining this inequality with (3.4) implies the result. ∎

Lemma 3.3.

$$||\nabla e_w(t)||^2 \leq c\big(\|\nabla e_u(t)\|^2 + \|e_{ut}(t)\|^2 + \|e_w(t)\|^2 + h^4|\log h|^4\big), \qquad 0 \leq t < T_h.$$

Proof. First note that in view of the definition of $\widehat{w}_h$ we have for all $\varphi_h \in X_{h0}$

$$\begin{aligned} &\int_\Omega \frac{\widehat{u}_{ht}\,\varphi_h}{Q_h} + \int_\Omega E(\nabla\widehat{u}_h)\nabla\widehat{w}_h\cdot\nabla\varphi_h + \frac{1}{2}\int_\Omega \frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla\widehat{u}_h\cdot\nabla\varphi_h \\ &\quad= \int_\Omega \frac{(\widehat{u}_{ht}-u_t)\varphi_h}{Q_h} + \int_\Omega u_t\left(\frac{1}{Q_h}-\frac{1}{Q}\right)\varphi_h + \frac{1}{2}\int_\Omega(\widehat{w}_h^2-w^2)\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla\varphi_h, \end{aligned}$$

from which we infer

$$\begin{aligned} &\int_\Omega \frac{e_{ut}\,\varphi_h}{Q_h} + \int_\Omega\big(E(\nabla\widehat{u}_h)\nabla\widehat{w}_h - E(\nabla u_h)\nabla w_h\big)\cdot\nabla\varphi_h + \frac{1}{2}\int_\Omega\left(\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla\widehat{u}_h - \frac{w_h^2}{Q_h^3}\nabla u_h\right)\cdot\nabla\varphi_h \\ &\quad= -\int_\Omega \frac{\rho_{ut}\,\varphi_h}{Q_h} + \int_\Omega u_t\left(\frac{1}{Q_h}-\frac{1}{Q}\right)\varphi_h + \frac{1}{2}\int_\Omega(\widehat{w}_h^2-w^2)\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla\varphi_h. \end{aligned} \tag{3.5}$$

Inserting $\varphi_h = e_w$ into (3.5) we derive

$$\begin{aligned} \int_\Omega\big(E(\nabla\widehat{u}_h)\nabla\widehat{w}_h - E(\nabla u_h)\nabla w_h\big)\cdot\nabla e_w &= -\int_\Omega \frac{e_{ut}\,e_w}{Q_h} - \frac{1}{2}\int_\Omega\left(\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla\widehat{u}_h - \frac{w_h^2}{Q_h^3}\nabla u_h\right)\cdot\nabla e_w \\ &\quad -\int_\Omega \frac{\rho_{ut}\,e_w}{Q_h} + \int_\Omega u_t\Big(\frac{1}{Q_h}-\frac{1}{Q}\Big)e_w + \frac{1}{2}\int_\Omega(\widehat{w}_h^2-w^2)\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla e_w. \end{aligned}$$

Since $Q_h \leq 2C_0$ we have

$$E(\nabla u_h)\zeta\cdot\zeta \geq \frac{1}{\sqrt{1+4C_0^2}^3}|\zeta|^2 \qquad \forall\zeta\in\mathbb{R}^2, \tag{3.6}$$

so that

$$\begin{aligned} &\int_\Omega\big(E(\nabla\widehat{u}_h)\nabla\widehat{w}_h - E(\nabla u_h)\nabla w_h\big)\cdot\nabla e_w \\ &\quad= \int_\Omega E(\nabla u_h)\nabla e_w\cdot\nabla e_w + \int_\Omega\big(E(\nabla\widehat{u}_h) - E(\nabla u_h)\big)\nabla\widehat{w}_h\cdot\nabla e_w \\ &\quad\geq \frac{1}{\sqrt{1+4C_0^2}^3}\|\nabla e_w\|^2 - c\|\nabla e_u\|\,\|\nabla e_w\| \geq \frac{1}{2\sqrt{1+4C_0^2}^3}\|\nabla e_w\|^2 - c\|\nabla e_u\|^2. \end{aligned}$$

by (2.24). Furthermore, we infer from Lemma 3.1

$$\begin{aligned} \int_\Omega u_t\Big(\frac{1}{Q_h}-\frac{1}{Q}\Big)e_w &= \int_\Omega\Big(\frac{1}{Q_h}-\frac{1}{\widehat{Q}_h}\Big)u_t\,e_w - \int_\Omega\Big(\frac{1}{Q}-\frac{1}{\widehat{Q}_h}\Big)u_t\,e_w \\ &\leq c\|\nabla e_u\|\,\|e_w\| - \int_\Omega\nabla\cdot\Big(u_t\,e_w\frac{\nabla u}{Q^3}\Big)\rho_u + ch^2|\log h|\,\|u_te_w\| \\ &\leq c\|\nabla e_u\|\,\|e_w\| + ch^2\|\nabla e_w\| + ch^2|\log h|\|e_w\|, \end{aligned}$$

by (2.15). The remaining terms are estimated in a straightforward manner so that we finally obtain

$$\begin{aligned}\frac{1}{2\sqrt{1+4C_0^2}^3}\|\nabla e_w\|^2 &\le c\|\nabla e_u\|^2 + \|e_{ut}\|\,\|e_w\| + c\big(\|e_w\| + \|\nabla e_u\|\big)\|\nabla e_w\| + \|\rho_{ut}\|\,\|e_w\| \\ &\quad + c\|\nabla e_u\|\,\|e_w\| + ch^2\|\nabla e_w\| + ch^2|\log h|\,\|e_w\| + c\|\rho_w\|\,\|\nabla e_w\|.\end{aligned}$$

The result now follows from Young's inequality, (2.17) and (2.21). ∎

Lemma 3.4. *We have for* $0 \le t < T_h$

$$\begin{aligned}&\frac{1}{2C_0}\|e_{ut}\|^2 + \int_\Omega (E(\nabla\widehat{u}_h)\nabla\widehat{w}_h - E(\nabla u_h)\nabla w_h)\cdot\nabla e_{ut} + \frac{1}{2}\int_\Omega\left(\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla\widehat{u}_h - \frac{w_h^2}{Q_h^3}\nabla u_h\right)\cdot\nabla e_{ut} \\ &\le -\frac{d}{dt}\int_\Omega u_t\frac{\nabla u}{Q^3}\cdot\nabla e_u\,\rho_u + \frac{1}{2}\frac{d}{dt}\int_\Omega(\widehat{w}_h^2 - w^2)\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla e_u + c\|\nabla e_u\|^2 + ch^4|\log h|^4.\end{aligned}$$

Proof. Inserting $\varphi_h = e_{ut}$ into (3.5) and recalling (3.1) we obtain

$$\begin{aligned}&\frac{1}{2C_0}\|e_{ut}\|^2 + \int_\Omega\big(E(\nabla\widehat{u}_h)\nabla\widehat{w}_h - E(\nabla u_h)\nabla w_h\big)\cdot\nabla e_{ut} + \frac{1}{2}\int_\Omega\left(\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla\widehat{u}_h - \frac{w_h^2}{Q_h^3}\nabla u_h\right)\cdot\nabla e_{ut} \\ &\le -\int_\Omega\frac{\rho_{ut}\,e_{ut}}{Q_h} + \int_\Omega u_t\Big(\frac{1}{Q_h} - \frac{1}{\widehat{Q}_h}\Big)e_{ut} + \int_\Omega u_t\Big(\frac{1}{\widehat{Q}_h} - \frac{1}{Q}\Big)e_{ut} + \frac{1}{2}\int_\Omega(\widehat{w}_h^2 - w^2)\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla e_{ut} \\ &\equiv I + II + III + IV.\end{aligned}$$

We infer from (2.17)

$$|I| + |II| \le \|\rho_{ut}\|\,\|e_{ut}\| + c\|\nabla e_u\|\,\|e_{ut}\| \le \epsilon\|e_{ut}\|^2 + c_\epsilon(h^4|\log h|^4 + \|\nabla e_u\|^2), \tag{3.7}$$

while Lemma 3.1 yields with $F(p) = \dfrac{1}{\sqrt{1+|p|^2}}$ and $f = u_t e_{ut}$

$$\begin{aligned}III &= -\int_\Omega\nabla\cdot\Big(u_t e_{ut}\frac{\nabla u}{Q^3}\Big)\rho_u + R = -\int_\Omega\nabla\cdot\Big(u_t\frac{\nabla u}{Q^3}\Big)e_{ut}\rho_u - \frac{d}{dt}\int_\Omega u_t\frac{\nabla u}{Q^3}\cdot\nabla e_u\,\rho_u \\ &\quad + \int_\Omega\Big(u_t\frac{\nabla u}{Q^3}\Big)_t\cdot\nabla e_u\,\rho_u + \int_\Omega u_t\frac{\nabla u}{Q^3}\cdot\nabla e_u\,\rho_{ut} + R,\end{aligned}$$

where $|R| \le Ch^2|\log h|\|u_t e_{ut}\|$. Thus,

$$\begin{aligned}III &\le -\frac{d}{dt}\int_\Omega u_t\frac{\nabla u}{Q^3}\cdot\nabla e_u\,\rho_u + c\|e_{ut}\|\big(\|\rho_u\| + h^2|\log h|\big) + c\|\nabla e_u\|\big(\|\rho_u\| + \|\rho_{ut}\|\big) \\ &\le -\frac{d}{dt}\int_\Omega u_t\frac{\nabla u}{Q^3}\cdot\nabla e_u\,\rho_u + \epsilon\|e_{ut}\|^2 + c\|\nabla e_u\|^2 + c_\epsilon h^4|\log h|^4.\end{aligned}\tag{3.8}$$

by (2.15), (2.17). Finally,

$$\begin{aligned}IV &= \frac{1}{2}\frac{d}{dt}\int_\Omega(\widehat{w}_h^2 - w^2)\,\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla e_u - \int_\Omega(\widehat{w}_h\widehat{w}_{ht} - ww_t)\,\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla e_u \\ &\quad -\frac{1}{2}\int_\Omega(\widehat{w}_h^2 - w^2)\left(\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\right)_t\cdot\nabla e_u.\end{aligned}$$

We infer from (2.18), (2.21) and (2.24) that

$$\begin{aligned} IV &\leq \frac{1}{2}\frac{d}{dt}\int_\Omega (\widehat{w}_h^2 - w^2)\,\frac{\nabla \widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla e_u + c||\nabla e_u||\big(||\rho_w|| + ||\rho_{wt}||\big) \\ &\leq \frac{1}{2}\frac{d}{dt}\int_\Omega (\widehat{w}_h^2 - w^2)\,\frac{\nabla \widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla e_u + c||\nabla e_u||^2 + ch^4|\log h|^4 \end{aligned} \tag{3.9}$$

by (2.21), (2.23). Summing (3.7)–(3.9) and choosing ϵ sufficiently small yields the result. ∎

Lemma 3.5. *We have for* $0 \leq t < T_h$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{e_w^2}{Q_h} - \frac{1}{2}\int_\Omega \frac{e_w^2}{Q_h^2}Q_{ht} - \int_\Omega \widehat{w}_h\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{Q_{ht}}{Q_h^2}\right)e_w \\ &\qquad - \int_\Omega (E(\nabla\widehat{u}_h)\nabla\widehat{u}_{ht} - E(\nabla u_h)\nabla u_{ht})\cdot\nabla e_w \\ &\leq \epsilon||\nabla e_w||^2 + c_\epsilon\big(||\nabla e_u||^2 + ||e_w||^2\big) + c_\epsilon h^4|\log h|^4. \end{aligned}$$

Proof. Starting from (2.3) and recalling the definition of $\widehat{u}_h$ we infer

$$\int_\Omega \frac{w\xi_h}{Q} = \int_\Omega \frac{\nabla u\cdot\nabla\xi_h}{Q} = \int_\Omega \frac{\nabla\widehat{u}_h\cdot\nabla\xi_h}{\widehat{Q}_h} \qquad \forall \xi_h \in X_{h0},$$

from which we obtain after differentiation with respect to time

$$\int_\Omega \frac{w_t\xi_h}{Q} - \int_\Omega \frac{w\xi_h}{Q^2}Q_t - \int_\Omega E(\nabla\widehat{u}_h)\nabla\widehat{u}_{ht}\cdot\nabla\xi_h = 0 \qquad \forall\xi_h \in X_{h0}.$$

Similarly

$$\int_\Omega \frac{w_{ht}\xi_h}{Q_h} - \int_\Omega \frac{w_h\xi_h}{Q_h^2}Q_{ht} - \int_\Omega E(\nabla u_h)\nabla u_{ht}\cdot\nabla\xi_h = 0 \qquad \forall\xi_h \in X_{h0}.$$

Taking the difference of the above relations we obtain

$$\begin{aligned} &\int_\Omega \frac{e_{wt}\xi_h}{Q_h} - \int_\Omega \frac{e_w\xi_h}{Q_h^2}Q_{ht} - \int_\Omega \widehat{w}_h\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{Q_{ht}}{Q_h^2}\right)\xi_h \\ &\qquad - \int_\Omega (E(\nabla\widehat{u}_h)\nabla\widehat{u}_{ht} - E(\nabla u_h)\nabla u_{ht})\cdot\nabla\xi_h \\ &= \int_\Omega \widehat{w}_{ht}\left(\frac{1}{Q_h} - \frac{1}{Q}\right)\xi_h - \int_\Omega \rho_{wt}\frac{1}{Q}\xi_h + \int_\Omega \rho_w\frac{Q_t}{Q^2}\xi_h - \int_\Omega \widehat{w}_h\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{Q_t}{Q^2}\right)\xi_h. \end{aligned}$$

If we use $\xi_h = e_w$ the result is

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{e_w^2}{Q_h} - \frac{1}{2}\int_\Omega \frac{e_w^2 Q_{ht}}{Q_h^2} - \int_\Omega \widehat{w}_h\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{Q_{ht}}{Q_h^2}\right)e_w \\ &\qquad - \int_\Omega (E(\nabla\widehat{u}_h)\nabla\widehat{u}_{ht} - E(\nabla u_h)\nabla u_{ht})\cdot\nabla e_w \\ &= \int_\Omega \widehat{w}_{ht}\left(\frac{1}{Q_h} - \frac{1}{Q}\right)e_w - \int_\Omega \rho_{wt}\frac{1}{Q}e_w + \int_\Omega \rho_w\frac{Q_t}{Q^2}e_w - \int_\Omega \widehat{w}_h\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{Q_t}{Q^2}\right)e_w \\ &\equiv I + \ldots + IV. \end{aligned}$$

In order to deal with the first term we apply again Lemma 3.1 with $F(p) := \frac{1}{\sqrt{1+|p|^2}}$, $f = \widehat{w}_{ht} e_w$:

$$\begin{aligned} I &= \int_\Omega \Big(\frac{1}{Q_h} - \frac{1}{\widehat{Q}_h} \Big) \widehat{w}_{ht} e_w + \int_\Omega \Big(\big(\frac{1}{\widehat{Q}_h} - \frac{1}{Q} \big) \Big) \widehat{w}_{ht} e_w \\ &= \int_\Omega \Big(\frac{1}{Q_h} - \frac{1}{\widehat{Q}_h} \Big) \widehat{w}_{ht} e_w - \int_\Omega \nabla \cdot \Big(\widehat{w}_{ht} e_w \frac{\nabla u}{Q^3} \Big) \, \rho_u + R, \end{aligned}$$

where $|R| \leq ch^2 |\log h| \|\widehat{w}_{ht} e_w\|$. As a consequence,

$$\begin{aligned} |I| &\leq c\|\widehat{w}_{ht}\|_{L^\infty} \big(\|\nabla e_u\| \, \|e_w\| + \|\nabla e_w\| \, \|\rho_u\| + h^2 |\log h| \, \|e_w\| \big) + c\|\nabla \widehat{w}_{ht}\|_{L^\infty} \|e_w\| \, \|\rho_u\| \\ &\leq \epsilon \|\nabla e_w\|^2 + c_\epsilon h^4 |\log h|^2 + c\|\nabla e_u\|^2 + c\|e_w\|^2 \end{aligned} \tag{3.10}$$

by (2.24). Next, (2.21) and (2.23) imply

$$|II + III| \leq c\big(||\rho_{wt}|| + \|\rho_w\| \big) ||e_w|| \leq c\|e_w\|^2 + ch^4 |\log h|^4. \tag{3.11}$$

Abbreviating $G_i(p) := \frac{p_i}{\sqrt{1+|p|^2}^3}, i = 1, 2$ we obtain

$$\begin{aligned} IV &= \int_\Omega \left(\frac{\nabla u \cdot \nabla u_t}{Q^3} - \frac{\nabla \widehat{u}_h \cdot \nabla \widehat{u}_{ht}}{\widehat{Q}_h^3} \right) \widehat{w}_h e_w \\ &= \int_\Omega \big(G_i(\nabla u) - G_i(\nabla \widehat{u}_h) \big) u_{tx_i} \widehat{w}_h e_w - \int_\Omega \big(G_i(\nabla u) - G_i(\nabla \widehat{u}_h) \big) \rho_{utx_i} \widehat{w}_h e_w \\ &\quad + \int_\Omega G_i(\nabla u) \rho_{utx_i} \widehat{w}_h e_w \\ &= -\int_\Omega \nabla \cdot \big(G_i'(\nabla u) \widehat{w}_h e_w u_{tx_i} \big) \, \rho_u + R - \int_\Omega \big(G_i(\nabla u) - G_i(\nabla \widehat{u}_h) \big) \rho_{utx_i} \widehat{w}_h e_w \\ &\quad - \int_\Omega \frac{\partial}{\partial x_i} G_i(\nabla u) \, \rho_{ut} \widehat{w}_h e_w - \int_\Omega G_i(\nabla u) \widehat{w}_{hx_i} \rho_{ut} e_w - \int_\Omega G_i(\nabla u) e_{wx_i} \widehat{w}_h \rho_{ut}, \end{aligned}$$

where $|R| \leq ch^2 |\log h| \|\widehat{w}_h e_w \nabla u_t\|$. Using (2.24) we can estimate

$$\begin{aligned} |IV| &\leq c\|\widehat{w}_h\|_{W^{1,\infty}} \|e_w\|_{H^1} \|\rho_u\| + ch^2 |\log h| \, \|e_w\| + c\|\nabla \rho_u\|_{L^\infty} \|\nabla \rho_{ut}\| \, \|e_w\| \\ &\quad + c\|\rho_{ut}\| \, \|e_w\| + c\|\nabla \widehat{w}_h\|_{L^\infty} \|\rho_{ut}\| \, \|e_w\| + c\|\nabla e_w\| \, \|\widehat{w}_h\|_{L^\infty} \|\rho_{ut}\| \\ &\leq \epsilon \|\nabla e_w\|^2 + c_\epsilon h^4 |\log h|^4 + c\|e_w\|^2. \end{aligned} \tag{3.12}$$

Combining (3.10)–(3.12) we finally obtain the result. ∎

It follows from Lemma 3.4 and 3.5 that

$$\begin{aligned}
&\frac{1}{2C_0}\|e_{ut}\|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega \frac{e_w^2}{Q_h} \\
&\quad + \frac{1}{2}\int_\Omega \left(\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla\widehat{u}_h - \frac{w_h^2}{Q_h^3}\nabla u_h\right)\cdot\nabla e_{ut} - \frac{1}{2}\int_\Omega \frac{e_w^2}{Q_h^2}Q_{ht} - \int_\Omega \widehat{w}_h\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{Q_{ht}}{Q_h^2}\right)e_w \\
&\quad + \int_\Omega (E(\nabla\widehat{u}_h)\nabla\widehat{w}_h - E(\nabla u_h)\nabla w_h)\cdot\nabla e_{ut} - \int_\Omega (E(\nabla\widehat{u}_h)\nabla\widehat{u}_{ht} - E(\nabla u_h)\nabla u_{ht})\cdot\nabla e_w \\
\leq\ & -\frac{d}{dt}\int_\Omega u_t\frac{\nabla u}{Q^3}\cdot\nabla e_u\,\rho_u + \frac{1}{2}\frac{d}{dt}\int_\Omega (\widehat{w}_h^2 - w^2)\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla e_u \\
&\qquad + c_\epsilon h^4|\log h|^4 + \epsilon||\nabla e_w||^2 + c_\epsilon\big(||e_u||_{H^1}^2 + ||e_w||^2\big).
\end{aligned} \tag{3.13}$$

A short calculation shows

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla\widehat{u}_h - \frac{w_h^2}{Q_h^3}\nabla u_h\right)\cdot\nabla e_{ut} - \frac{1}{2}\frac{e_w^2}{Q_h^2}Q_{ht} - \widehat{w}_h\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{Q_{ht}}{Q_h^2}\right)e_w \\
&\qquad = -\frac{1}{2}\widehat{w}_h^2\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{1}{2}\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla\widehat{u}_h\cdot\nabla u_{ht} - \frac{1}{2}\frac{w_h^2}{Q_h^3}\nabla u_h\cdot\nabla\widehat{u}_{ht} + \frac{1}{2}\widehat{w}_h^2\frac{Q_{ht}}{Q_h^2} + \widehat{w}_h w_h\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} \\
&\qquad \equiv S_1 + \ldots + S_5.
\end{aligned}$$

Clearly,

$$S_1 = \frac{1}{2}\widehat{w}_h^2\frac{\partial}{\partial t}\Big(\frac{1}{\widehat{Q}_h}\Big), \quad S_4 = -\frac{1}{2}\widehat{w}_h^2\frac{\partial}{\partial t}\Big(\frac{1}{Q_h}\Big).$$

Observing that $|\widehat{\nu}_h - \nu_h|^2 = 2 - 2\frac{1 + \nabla\widehat{u}_h\cdot\nabla u_h}{\widehat{Q}_h Q_h}$ we obtain

$$\begin{aligned}
S_2 &= -\frac{1}{2}\widehat{w}_h^2\frac{\partial}{\partial t}\left(\frac{\nabla\widehat{u}_h\cdot\nabla u_h}{\widehat{Q}_h^3}\right) + \frac{1}{2}\widehat{w}_h^2\frac{\nabla\widehat{u}_{ht}\cdot\nabla u_h}{\widehat{Q}_h^3} - \frac{3}{2}\widehat{w}_h^2\frac{\nabla\widehat{u}_h\cdot\nabla u_h}{\widehat{Q}_h^4}\widehat{Q}_{ht} \\
&= -\frac{1}{2}\widehat{w}_h^2\frac{\partial}{\partial t}\left(\frac{\nabla\widehat{u}_h\cdot\nabla u_h}{\widehat{Q}_h^3}\right) + \frac{1}{2}\widehat{w}_h^2\frac{\nabla\widehat{u}_{ht}\cdot\nabla u_h}{\widehat{Q}_h^3} + \frac{3}{4}\widehat{w}_h^2\frac{Q_h\widehat{Q}_{ht}}{\widehat{Q}_h^3}|\widehat{\nu}_h - \nu_h|^2 \\
&\quad -\frac{3}{2}\widehat{w}_h^2\frac{Q_h\widehat{Q}_{ht}}{\widehat{Q}_h^3} - \frac{1}{2}\widehat{w}_h^2\frac{\partial}{\partial t}\left(\frac{1}{\widehat{Q}_h^3}\right),
\end{aligned}$$

while

$$S_3 = -\frac{1}{2}(\widehat{w}_h - w_h)^2\frac{\nabla u_h\cdot\nabla\widehat{u}_{ht}}{Q_h^3} - \widehat{w}_h w_h\frac{\nabla u_h\cdot\nabla\widehat{u}_{ht}}{Q_h^3} + \frac{1}{2}\widehat{w}_h^2\frac{\nabla u_h\cdot\nabla\widehat{u}_{ht}}{Q_h^3}.$$

Summation of $S_1, \ldots, S_5$ yields

$$\begin{aligned}
S_1 + \ldots + S_5 \quad = \quad & \frac{1}{2}\widehat{w}_h^2 \frac{\partial}{\partial t}\left(\frac{1}{\widehat{Q}_h} - \frac{1}{Q_h} - \frac{\nabla\widehat{u}_h \cdot \nabla u_h}{\widehat{Q}_h^3} - \frac{1}{\widehat{Q}_h^3}\right) \\
& + \frac{3}{4}\widehat{w}_h^2 \frac{Q_h \widehat{Q}_{ht}}{\widehat{Q}_h^3} |\widehat{\nu}_h - \nu_h|^2 - \frac{1}{2} e_w^2 \frac{\nabla u_h \cdot \nabla \widehat{u}_{ht}}{Q_h^3} \\
& + \frac{1}{2}\widehat{w}_h^2\, \nabla(u_h - \widehat{u}_h) \cdot \nabla\widehat{u}_{ht}\left(\frac{1}{\widehat{Q}_h^3} - \frac{1}{Q_h^3}\right) + \widehat{w}_h^2 \frac{\nabla u_h \cdot \nabla\widehat{u}_{ht}}{Q_h^3} + \frac{1}{2}\widehat{w}_h^2 \frac{\nabla\widehat{u}_h \cdot \nabla\widehat{u}_{ht}}{\widehat{Q}_h^3} \\
& - \frac{1}{2}\widehat{w}_h^2 \frac{\nabla\widehat{u}_h \cdot \nabla\widehat{u}_{ht}}{Q_h^3} + w_h \widehat{w}_h \left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{\nabla u_h \cdot \nabla\widehat{u}_{ht}}{Q_h^3}\right) - \frac{3}{2}\widehat{w}_h^2 \frac{Q_h \widehat{Q}_{ht}}{\widehat{Q}_h^3} \\
= \quad & \frac{1}{2}\widehat{w}_h^2 \frac{\partial}{\partial t}\left(\frac{2}{\widehat{Q}_h} - \frac{1}{Q_h} - \frac{\nabla\widehat{u}_h \cdot \nabla u_h}{\widehat{Q}_h^3} - \frac{1}{\widehat{Q}_h^3}\right) + \frac{3}{4}\widehat{w}_h^2 \frac{Q_h \widehat{Q}_{ht}}{\widehat{Q}_h^3} |\widehat{\nu}_h - \nu_h|^2 \\
& - \frac{1}{2} e_w^2 \frac{\nabla u_h \cdot \nabla\widehat{u}_{ht}}{Q_h^3} + \frac{1}{2}\widehat{w}_h^2\, \nabla(u_h - \widehat{u}_h) \cdot \nabla\widehat{u}_{ht}\left(\frac{1}{\widehat{Q}_h^3} - \frac{1}{Q_h^3}\right) \\
& + \widehat{w}_h (w_h - \widehat{w}_h)\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{\nabla u_h \cdot \nabla\widehat{u}_{ht}}{Q_h^3}\right) - \frac{1}{2}\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^3}\big(-4\widehat{Q}_h + 3Q_h + \frac{\widehat{Q}_h^4}{Q_h^3}\big).
\end{aligned}$$

A short calculation shows that $D := \left(\frac{2}{\widehat{Q}_h} - \frac{1}{Q_h} - \frac{\nabla\widehat{u}_h \cdot \nabla u_h}{\widehat{Q}_h^3} - \frac{1}{\widehat{Q}_h^3}\right)$ can be written as

$$D = \frac{1}{2}\frac{Q_h}{\widehat{Q}_h^2}|\widehat{\nu}_h - \nu_h|^2 - \frac{1}{Q_h \widehat{Q}_h^2}(\widehat{Q}_h - Q_h)^2. \tag{3.14}$$

Integration over Ω together with (3.3) gives

$$\begin{aligned}
& \frac{1}{2}\int_\Omega \left(\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla\widehat{u}_h - \frac{w_h^2}{Q_h^3}\nabla u_h\right) \cdot \nabla e_{ut} - \frac{1}{2}\int_\Omega \frac{e_w^2}{Q_h^2} Q_{ht} - \int_\Omega \widehat{w}_h \left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{Q_{ht}}{Q_h^2}\right) e_w \\
\geq \quad & \frac{1}{2}\frac{d}{dt}\int_\Omega \widehat{w}_h^2 D - \int_\Omega \widehat{w}_h \widehat{w}_{ht} D - c\|\nabla e_u\|^2 - c\|e_w\|^2 \\
& - c\int_\Omega |e_w|\,|\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{\nabla u_h \cdot \nabla\widehat{u}_{ht}}{Q_h^3}| - c\int_\Omega | - 4\widehat{Q}_h + 3Q_h + \frac{\widehat{Q}_h^4}{Q_h^3}|.
\end{aligned}$$

Since

$$\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{\nabla\widehat{u}_{ht} \cdot \nabla u_h}{Q_h^3} = \frac{\nabla\widehat{u}_{ht}}{\widehat{Q}_h^2}\left(\frac{\nabla\widehat{u}_h}{\widehat{Q}_h} - \frac{\nabla u_h}{Q_h}\right) + \frac{\nabla u_h}{Q_h} \cdot \nabla\widehat{u}_{ht}\frac{Q_h + \widehat{Q}_h}{\widehat{Q}_h^2 Q_h}\,\frac{Q_h - \widehat{Q}_h}{Q_h}$$

we obtain

$$\left|\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{\nabla\widehat{u}_{ht} \cdot \nabla u_h}{Q_h^3}\right| \leq c\,|\widehat{\nu}_h - \nu_h| \leq c\|\nabla e_u\|.$$

It is easily shown that

$$|-4\widehat{Q}_h + 3Q_h + \frac{\widehat{Q}_h^4}{Q_h^3}| \le c(Q_h - \widehat{Q}_h)^2 \le c|\nabla e_u|^2.$$

Recalling (3.14) we have in conclusion

$$\begin{aligned} &\frac{1}{2}\int_\Omega \left(\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla \widehat{u}_h - \frac{w_h^2}{Q_h^3}\nabla u_h\right)\cdot \nabla e_{ut} - \frac{1}{2}\int_\Omega \frac{e_w^2}{Q_h^2}Q_{ht} - \int_\Omega \widehat{w}_h\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{Q_{ht}}{Q_h^2}\right)e_w \qquad (3.15)\\ \ge\ & \frac{1}{2}\frac{d}{dt}\int_\Omega \widehat{w}_h^2\left\{\frac{1}{2}\frac{Q_h}{\widehat{Q}_h^2}|\widehat{\nu}_h - \nu_h|^2 - \frac{1}{Q_h\widehat{Q}_h^2}(\widehat{Q}_h - Q_h)^2\right\} - c\big(||e_w||^2 + \|\nabla e_u\|^2\big). \end{aligned}$$

Next,

$$\begin{aligned} &(E(\nabla\widehat{u}_h)\nabla\widehat{w}_h - E(\nabla u_h)\nabla w_h)\cdot\nabla e_{ut} - (E(\nabla\widehat{u}_h)\nabla\widehat{u}_{ht} - E(\nabla u_h)\nabla u_{ht})\cdot\nabla e_w\\ &\quad = (E(\nabla\widehat{u}_h) - E(\nabla u_h))\nabla(\widehat{u}_{ht} - u_{ht})\cdot\nabla\widehat{w}_h - (E(\nabla\widehat{u}_h) - E(\nabla u_h))\nabla\widehat{u}_{ht}\cdot\nabla e_w. \end{aligned}$$

In order to deal with the first term we introduce $\widehat{P}_h, P_h \in M(3,3)$ by

$$\widehat{P}_{h,ij} = \delta_{ij} - \widehat{\nu}_{hi}\widehat{\nu}_{hj}, \quad P_{h,ij} = \delta_{ij} - \nu_{hi}\nu_{hj}, \qquad i,j = 1,2,3.$$

Since $\big(\nabla(\widehat{u}_h - u_h), 0\big)^t = \widehat{Q}_h\widehat{\nu}_h - Q_h\nu_h$ we may calculate

$$\begin{aligned} \left(\frac{1}{\widehat{Q}_h}\widehat{P}_h - \frac{1}{Q_h}P_h\right)\big(\nabla(\widehat{u}_{ht} - u_{ht}), 0\big)^t &= \left(\frac{1}{\widehat{Q}_h}\widehat{P}_h - \frac{1}{Q_h}P_h\right)\Big(\widehat{Q}_{ht}\widehat{\nu}_h + \widehat{Q}_h\widehat{\nu}_{ht} - Q_{ht}\nu_h - Q_h\nu_{ht}\Big)\\ &= \widehat{\nu}_{ht} - \frac{Q_{ht}}{\widehat{Q}_h}\widehat{P}_h\nu_h - \frac{Q_h}{\widehat{Q}_h}\widehat{P}_h\nu_{ht} - \frac{\widehat{Q}_{ht}}{Q_h}P_h\widehat{\nu}_h - \frac{\widehat{Q}_h}{Q_h}P_h\widehat{\nu}_{ht} + \nu_{ht} \qquad (3.16)\\ &= \frac{\partial}{\partial t}\left\{\nu_h - \widehat{\nu}_h - \frac{Q_h}{\widehat{Q}_h}\widehat{P}_h\nu_h\right\} + S, \end{aligned}$$

where

$$S = -\frac{\widehat{Q}_{ht}Q_h}{\widehat{Q}_h^2}\widehat{P}_h\nu_h + \frac{Q_h}{\widehat{Q}_h}\widehat{P}_{ht}\nu_h + 2\widehat{\nu}_{ht} - \frac{\widehat{Q}_{ht}}{Q_h}P_h\widehat{\nu}_h - \frac{\widehat{Q}_h}{Q_h}P_h\widehat{\nu}_{ht}.$$

Observe that

$$\widehat{P}_h\nu_h = \nu_h - \big(\nu_h\cdot\widehat{\nu}_h\big)\widehat{\nu}_h = \nu_h - \widehat{\nu}_h + \frac{1}{2}|\widehat{\nu}_h - \nu_h|^2\widehat{\nu}_h$$

and similarly

$$P_h\widehat{\nu}_h = \widehat{\nu}_h - \nu_h + \frac{1}{2}|\widehat{\nu}_h - \nu_h|^2\nu_h.$$

This gives

$$\begin{aligned} S \ =\ & -\frac{\widehat{Q}_{ht}Q_h}{\widehat{Q}_h^2}\left(\widehat{\nu}_h - \nu_h + \frac{1}{2}|\widehat{\nu}_h - \nu_h|^2\widehat{\nu}_h\right) + \frac{Q_h}{\widehat{Q}_h}\big(-(\nu_h\cdot\widehat{\nu}_h)\widehat{\nu}_{ht} - (\widehat{\nu}_{ht}\cdot\nu_h)\widehat{\nu}_h\big)\\ & +2\widehat{\nu}_{ht} - \frac{\widehat{Q}_{ht}}{Q_h}\left(\nu_h - \widehat{\nu}_h + \frac{1}{2}|\widehat{\nu}_h - \nu_h|^2\nu_h\right) - \frac{\widehat{Q}_h}{Q_h}\big(\widehat{\nu}_{ht} - (\widehat{\nu}_{ht}\cdot\nu_h)\nu_h\big)\\ =\ & -\frac{1}{2}\widehat{Q}_{ht}|\widehat{\nu}_h - \nu_h|^2\left(\frac{Q_h}{\widehat{Q}_h^2}\widehat{\nu}_h + \frac{1}{Q_h}\nu_h\right) + \frac{\widehat{Q}_{ht}}{\widehat{Q}_h}\left(\frac{\widehat{Q}_h}{Q_h} - \frac{Q_h}{\widehat{Q}_h}\right)\big(\widehat{\nu}_h - \nu_h\big)\\ & + \left(2 - \frac{Q_h}{\widehat{Q}_h}\left(\widehat{\nu}_h\cdot\nu_h\right) - \frac{\widehat{Q}_h}{Q_h}\right)\widehat{\nu}_{ht} + \big(\widehat{\nu}_{ht}\cdot(\nu_h - \widehat{\nu}_h)\big)\left(\frac{\widehat{Q}_h}{Q_h}\nu_h - \frac{Q_h}{\widehat{Q}_h}\widehat{\nu}_h\right) \end{aligned}$$

since $\widehat{\nu}_h \cdot \widehat{\nu}_{ht} = 0$. We infer from (3.1) and (3.3) that

$$\left|\frac{\widehat{Q}_h}{Q_h} - \frac{Q_h}{\widehat{Q}_h}\right|, \quad \left|\frac{\widehat{Q}_h}{Q_h}\nu_h - \frac{Q_h}{\widehat{Q}_h}\widehat{\nu}_h\right| \le c|\nabla e_u|,$$

as well as

$$\left|2 - \frac{Q_h}{\widehat{Q}_h}(\widehat{\nu}_h \cdot \nu_h) - \frac{\widehat{Q}_h}{Q_h}\right| = \left|\frac{1}{2}\frac{Q_h}{\widehat{Q}_h}|\widehat{\nu}_h - \nu_h|^2 - \frac{(\widehat{Q}_h - Q_h)^2}{\widehat{Q}_h Q_h}\right| \quad \le \quad c|\nabla e_u|^2,$$

which implies

$$|S| \le c|\nabla e_u|^2. \tag{3.17}$$

In conclusion

$$\begin{aligned}
&(E(\nabla\widehat{u}_h) - E(\nabla u_h))\nabla(\widehat{u}_{ht} - u_{ht}) \cdot \nabla\widehat{w}_h \\
&= \left(\frac{1}{\widehat{Q}_h}\widehat{P}_h - \frac{1}{Q_h}P_h\right)\big(\nabla(\widehat{u}_{ht} - u_{ht}), 0\big)^t \cdot (\nabla\widehat{w}_h, 0)^t \\
&= \left\{\frac{\partial}{\partial t}\left(\big(\frac{Q_h}{\widehat{Q}_h} - 1\big)\big(\widehat{\nu}_h - \nu_h\big) - \frac{1}{2}\frac{Q_h}{\widehat{Q}_h}|\widehat{\nu}_h - \nu_h|^2\widehat{\nu}_h\right) + S\right\} \cdot (\nabla\widehat{w}_h, 0)^t.
\end{aligned}$$

If we integrate this relation over Ω and observe (3.17) and (2.24) we obtain

$$\begin{aligned}
&\int_\Omega (E(\nabla\widehat{u}_h) - E(\nabla u_h))\nabla(\widehat{u}_{ht} - u_{ht}) \cdot \nabla\widehat{w}_h \\
&= \frac{d}{dt}\int_\Omega \left(\left(\frac{Q_h}{\widehat{Q}_h} - 1\right)(\widehat{\nu}_h - \nu_h) - \frac{1}{2}\frac{Q_h}{\widehat{Q}_h}|\widehat{\nu}_h - \nu_h|^2\widehat{\nu}_h\right) \cdot (\nabla\widehat{w}_h, 0)^t \\
&\quad - \int_\Omega \left(\left(\frac{Q_h}{\widehat{Q}_h} - 1\right)(\widehat{\nu}_h - \nu_h) - \frac{1}{2}\frac{Q_h}{\widehat{Q}_h}|\widehat{\nu}_h - \nu_h|^2\widehat{\nu}_h\right) \cdot (\nabla\widehat{w}_{ht}, 0)^t + \int_\Omega S \cdot (\nabla\widehat{w}_h, 0)^t \\
&\ge \frac{d}{dt}\int_\Omega \left(\left(\frac{Q_h}{\widehat{Q}_h} - 1\right)(\widehat{\nu}_h - \nu_h) - \frac{1}{2}\frac{Q_h}{\widehat{Q}_h}|\widehat{\nu}_h - \nu_h|^2\widehat{\nu}_h\right) \cdot (\nabla\widehat{w}_h, 0)^t - c\|\nabla e_u\|^2, \qquad (3.18)
\end{aligned}$$

Recalling the definitions of $\hat{P}_h$, P_h and observing (3.1) we may estimate

$$\begin{aligned}
|\int_\Omega (E(\nabla\widehat{u}_h) - E(\nabla u_h))\nabla\widehat{u}_{ht} \cdot \nabla e_w| &= |\int_\Omega \left(\frac{1}{\widehat{Q}_h}\hat{P}_h - \frac{1}{Q_h}P_h\right)(\nabla\widehat{u}_{ht}, 0)^t \cdot (\nabla e_w, 0)^t| \\
&\le c\int_\Omega |\nabla e_u|\,|\nabla e_w| \le \epsilon||\nabla e_w||^2 + c_\epsilon\|\nabla e_u\|^2. \qquad (3.19)
\end{aligned}$$

If we insert (3.15), (3.18) and (3.19) into (3.13) we finally obtain

$$\begin{aligned}
\frac{1}{2C_0}\|e_{ut}\|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega \frac{e_w^2}{Q_h} \quad &\le \quad -\frac{d}{dt}\int_\Omega u_t\frac{\nabla u}{Q^3}\nabla e_u\,\rho_u + \frac{1}{2}\frac{d}{dt}\int_\Omega (\widehat{w}_h^2 - w^2)\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3} \cdot \nabla e_u \\
&\quad -\frac{1}{2}\frac{d}{dt}\int_\Omega \widehat{w}_h^2\left\{\frac{1}{2}\frac{Q_h}{\widehat{Q}_h^2}|\widehat{\nu}_h - \nu_h|^2 - \frac{1}{Q_h\widehat{Q}_h^2}(\widehat{Q}_h - Q_h)^2\right\} \\
&\quad -\frac{d}{dt}\int_\Omega \left(\left(\frac{Q_h}{\widehat{Q}_h} - 1\right)(\widehat{\nu}_h - \nu_h) - \frac{1}{2}\frac{Q_h}{\widehat{Q}_h}|\widehat{\nu}_h - \nu_h|^2\widehat{\nu}_h\right) \cdot (\nabla\widehat{w}_h, 0)^t \\
&\quad +\epsilon||\nabla e_w||^2 + c_\epsilon\big(\|e_w\|^2 + \|\nabla e_u\|^2 + h^4|\log h|^4\big).
\end{aligned}$$

Integrating with respect to time between 0 and t $(0 \le t < T_h)$ and taking into account Lemma 3.2 and Lemma 3.3 yields

$$\begin{aligned}
\int_0^t ||e_{ut}||^2 ds + ||e_w(t)||^2 \quad &\le \quad c\|\nabla e_u(t)\| \big(\|\rho_u(t)\| + \|e_w(t)\| + \|\nabla e_u(t)\|\big) \\
&\quad + \epsilon \int_0^t \|\nabla e_w\|^2 ds + c_\epsilon \int_0^t \big(\|e_w\|^2 + \|\nabla e_u\|^2\big) ds + c_\epsilon h^4 |\log h|^4 \\
&\le \quad \epsilon\big(\|e_w(t)\|^2 + \int_0^t \|e_{ut}\|^2 ds\big) + c_\epsilon\Big(\|e_u(t)\|^2 + h^4|\log h|^4 + \int_0^t \big(\|e_u\|^2 + \|e_w\|^2\big) ds\Big).
\end{aligned}$$

After choosing ϵ sufficiently small we obtain

$$\int_0^t ||e_{ut}||^2 ds + ||e_w(t)||^2 \le c_1 \|e_u(t)\|^2 + c \int_0^t \big(||e_u||^2 + ||e_w||^2\big) ds + ch^4 |\log h|^4. \tag{3.20}$$

On the other hand we have in view of $e_u(0) = 0$

$$\|e_u(t)\|^2 \le 2 \int_0^t \|e_u\| \, \|e_{ut}\| ds \le \frac{1}{2} \int_0^t ||e_{ut}||^2 ds + c \int_0^t ||e_u||^2 ds.$$

If we use this inequality in (3.20) we finally obtain

$$\int_0^t ||e_{ut}||^2 ds + ||e_w(t)||^2 + \|e_u(t)\|^2 \le c \int_0^t \big(||e_u||^2 + ||e_w||^2\big) ds + ch^4 |\log h|^4, \qquad 0 \le t < T_h$$

and Gronwall's inequality yields

$$||e_w(t)||^2 + \|e_u(t)\|^2 \le ch^4 |\log h|^4, \qquad 0 \le t \le T_h. \tag{3.21}$$

Lemma 3.2 implies that $\|\nabla e_u(t)\| \le ch^2 |\log h|^2$ for $0 \le t \le T_h$ which together with (2.16) and an inverse estimate gives

$$\begin{aligned}
\|Q_h(t)\|_{L^\infty} \quad &\le \quad \|Q(t)\|_{L^\infty} + c\|\nabla(u - \widehat{u}_h)(t)\|_{L^\infty} + c\|\nabla e_u(t)\|_{L^\infty} \\
&\le \quad C_0 + ch|\log h| + ch|\log h|^2 \quad \le \quad \frac{3}{2} C_0,
\end{aligned}$$

and similarly $\|w_h(t)\|_{L^\infty} \le \frac{3}{2} C_1$, $0 \le t \le T_h$, provided that $h \le h_0$. Now we are in position to prove that $T_h = T$. If not, the above argument would imply that $Q_h(x, T_h) \le \frac{3}{2} C_0$, $|w_h(x, T_h)| \le \frac{3}{2} C_1$ for $x \in \bar{\Omega}$, and we could establish (3.1) on $[0, T_h + \delta]$ for some $\delta > 0$ contradicting the definition of T_h. The estimates (2.10)–(2.13) now follow from (3.21), Lemma 3.2, Lemma 3.3 and the interpolation results for $\widehat{u}_h, \widehat{w}_h$.

4 Numerical results

For our numerical tests we have to use a time discretization. We have chosen a semi–implicit discretization with respect to the time variable in the spatially discrete scheme (2.7), (2.8). For a generic function v we denote its evaluation at the m–th time level $t_m = m\tau$ by $v^m = v(\cdot, t_m)$. The time discretization then is given by: $u_h^{m+1} \in X_h$, $u_h^{m+1} - I_h g \in X_{h0}$, $w_h^{m+1} \in X_{h0}$,

$$\frac{1}{\tau}\int_\Omega \frac{(u_h^{m+1} - u_h^m)\varphi_h}{Q_h^m} + \int_\Omega E(\nabla u_h^m)\nabla w_h^{m+1} \cdot \nabla\varphi_h + \frac{1}{2}\int_\Omega \frac{(w_h^m)^2}{(Q_h^m)^3}\nabla u_h^{m+1} \cdot \nabla\varphi_h = \int_\Omega f^m \varphi_h \quad \forall \varphi_h \in X_{h0}, \quad (4.1)$$

$$\int_\Omega \frac{w_h^{m+1}\zeta_h}{Q_h^m} - \int_\Omega \frac{\nabla u_h^{m+1} \cdot \nabla\zeta_h}{Q_h^m} = 0 \quad \forall \zeta_h \in X_{h0}. \quad (4.2)$$

for $m = 0, 1, \ldots, m(T)$ with $\tau m(T) = T$ and $Q_h^m = \sqrt{1 + |\nabla u_h^m|^2}$. As initial data we use $u_h^0 = I_h u_0$ and w_h^0, which is computed from

$$\int_\Omega \frac{w_h^0 \zeta_h}{Q_h^0} - \int_\Omega \frac{\nabla u_h^0 \cdot \nabla\zeta_h}{Q_h^0} = 0 \quad \forall \zeta_h \in X_{h0}.$$

We have introduced an additional given right hand side f, which we will need for our numerical tests. Let us denote by $\{\varphi_j\}_{j=1,\ldots,N}$ the usual nodal basis of X_{h0}. Then, (4.1), (4.2) represents a linear system for the coefficients of u_h^{m+1} and w_h^{m+1} in the expansions

$$u_h^m = \sum_{j=1}^N U_j^m \varphi_j, \qquad w_h^m = \sum_{j=1}^N W_j^m \varphi_j.$$

Denote by $U^m = (U_1^m, \ldots, U_N^m)$, $W^m = (W_1^m, \ldots, W_N^m)$ the coefficient vectors and set

$$M_{i,j}^m = \int_\Omega \frac{\varphi_i \varphi_j}{Q_h^m}, \quad E_{i,j}^m = \int_\Omega E(\nabla u_h^m)\nabla\varphi_i \cdot \nabla\varphi_j, \quad A_{i,j}^m = \int_\Omega \frac{\nabla\varphi_i \cdot \nabla\varphi_j}{Q_h^m},$$

as well as

$$B_{i,j}^m = \frac{1}{2}\int_\Omega \frac{(w_h^m)^2}{(Q_h^m)^3}\nabla\varphi_i \cdot \nabla\varphi_j, \quad F_i^m = \int_\Omega f^m \varphi_i$$

for $i, j = 1, \ldots, N$. With these settings we can write the linear system (4.1), (4.2) in the form

$$\begin{aligned} \frac{1}{\tau}M^m U^{m+1} + E^m W^{m+1} + B^m U^{m+1} &= \frac{1}{\tau}M^m U^m + F^m \quad (4.3) \\ M^m W^{m+1} - A^m U^{m+1} &= 0. \quad (4.4) \end{aligned}$$

Eliminating W^{m+1} from the first equation by inverting the (weighted) mass matrix in the second equation leads to the linear system

$$\left(\frac{1}{\tau}M^m + E^m \left(M^m\right)^{-1} A^m + B^m\right) U^{m+1} = \frac{1}{\tau}M^m U^m + F^m. \quad (4.5)$$

We solve this non-symmetric system by a conjugate gradient method. In the practical computations we use mass lumping, so that M^m becomes a diagonal matrix.

We have tested our algorithm with the help of the following problem: let $\Omega = \{x \in \mathbb{R}^2 \mid |x| < 1\}$, $(0, T) = (0, 0.5)$ and

$$u(x, t) = 2.5\cos(2\pi t)(|x| - 1)^3 |x|^5.$$

h	$E_{\infty,2,u}$	eoc	$E_{\infty,2,\nabla u}$	eoc	$E_{2,2,u_t}$	eoc	$E_{\infty,\infty,u}$	eoc
1.0	0.59486	-	1.41194	-	1.90956	-	1.09159	-
0.73681	0.30771	2.158	0.90635	1.451	0.97860	2.188	0.30057	4.222
0.42033	0.11159	1.807	0.53259	0.947	0.50503	1.178	0.11449	1.719
0.22192	0.047902	1.324	0.34495	0.680	0.23983	1.165	0.046528	1.409
0.11373	0.013300	1.916	0.18205	0.956	0.070064	1.840	0.012886	1.920
0.057535	0.0034522	1.979	0.094278	0.965	0.018898	1.922	0.0033490	1.977

Table 1: u-errors for the test problem, $\tau = 0.1h^2$

The function f is calculated in such a way that u is a solution of the PDE

$$\frac{u_t}{Q} + \nabla \cdot \left(\frac{1}{Q} \big(I - \frac{\nabla u \otimes \nabla u}{Q^2} \big) \nabla (QH) \right) - \frac{1}{2} \nabla \cdot \left(\frac{H^2}{Q} \nabla u \right) = f \qquad \text{in } \Omega \times (0,T).$$

In order not to destroy the second order convergence properties in some norms we choose the time step as $\tau = 0.1h^2$. In Tables 1 and 4 we show the absolute errors for u in the norms

$$E_{\infty,2,u} = \max_{m=0,\dots,m(T)} \|u^m - u_h^m\|, \quad E_{\infty,2,\nabla u} = \max_{m=0,\dots,m(T)} \|\nabla u^m - \nabla u_h^m\|,$$

$$E_{2,2,u_t} = \left(\tau \sum_{m=1}^{m(T)} \|u_t^m - \frac{u_h^m - u_h^{m-1}}{\tau}\|^2 \right)^{\frac{1}{2}}, \quad E_{\infty,\infty,u} = \max_{m=0,\dots,m(T)} \sup_{\Omega} |u^m - u_h^m|,$$

as well as for w in the norms

$$E_{\infty,2,w} = \max_{m=0,\dots,m(T)} \|w^m - w_h^m\|, \quad E_{2,2,\nabla w} = \left(\tau \sum_{m=1}^{m(T)} \|\nabla w^m - \nabla w_h^m\|^2 \right)^{\frac{1}{2}},$$

$$E_{\infty,\infty,w} = \max_{m=0,\dots,m(T)} \sup_{\Omega} |w^m - w_h^m|.$$

The computational results confirm the theoretical results of Theorem 2.2. Between two spatial grid levels with maximal grid size h_1 and h_2 and errors $E(h_1)$, $E(h_2)$ we computed the experimental order of convergence according to

$$\operatorname{eoc}(h_1, h_2) = \log \frac{E(h_1)}{E(h_2)} \left(\log \frac{h_1}{h_2} \right)^{-1}.$$

We demonstrate the possible effect that the gradient of the continuous solution may blow up at some time. Figure 2 shows the norm gradient$(t) = \|\nabla u_h(\cdot,t)\|_{L^\infty(\Omega)}$. From Lemma 2.1 we know that the discrete solution exists for all times. The continuous one may exist for finite time only. A blow up of the continuous gradient may be deduced practically from the

h	$E_{\infty,2,w}$	eoc	$E_{2,2,\nabla w}$	eoc	$E_{\infty,\infty,w}$	eoc
1.0	1.58839	-	4.58182	-	3.83664	-
0.73681	1.44454	0.310	2.59686	1.859	1.79998	2.477
0.42033	0.74736	1.174	2.99178	-0.252	0.86798	1.299
0.22192	0.40324	0.965	1.76851	0.823	0.41747	1.146
0.11373	0.11953	1.818	0.90402	1.003	0.11833	1.885
0.057535	0.031087	1.976	0.44252	1.048	0.028734	2.077

Table 2: w-errors for the test problem, $\tau = 0.1h^2$

behaviour of the discrete gradient. As initial function for the computations in Figure 2 we have chosen

$$u_0(x_1, x_2) = \delta \sin^2(\pi(1 + x_1)) \sin^2(\pi x_1) \tag{4.6}$$

on the domain $\Omega = (-1, 1) \times (-1, 1)$. For the choice $\delta = 1$ (left in Fig. 2) a blow up of the gradient is indicated, which is not the case when $\delta = 0.5$ (right in Fig. 2).