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Finite elements on evolving surfaces

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In this article we define a new evolving surface finite element method (ESFEM) for numerically approximating partial differential equations on hypersurfaces $\Gamma(t)$ in $\mathbb{R}^{n+1}$ which evolve with time. The key idea is based on approximating $\Gamma(t)$ by an evolving interpolated polyhedral (polygonal if $n = 1$) surface $\Gamma_h(t)$ consisting of a union of simplices (triangles for $n = 2$) whose vertices lie on $\Gamma(t)$. A finite element space of functions is then defined by taking the set of all continuous functions on $\Gamma_h(t)$ which are linear affine on each simplex. The finite element nodal basis functions enjoy a transport property which simplifies the computation. We formulate a conservation law for a scalar quantity on $\Gamma(t)$ and, in the case of a diffusive flux, derive a transport and diffusion equation which takes into account the tangential velocity and the local stretching of the surface. Using surface gradients to define weak forms of elliptic operators naturally generates weak formulations of elliptic and parabolic equations on $\Gamma(t)$. Our finite element method is applied to the weak form of the conservation equation. The computation of the mass and element stiffness matrices are simple and straightforward. Error bounds are derived in the case of semi-discretization in space. Numerical experiments are described which indicate the order of convergence and also the power of the method. We describe how this framework may be employed in applications.

Keywords: finite elements, evolving surfaces, conservation, diffusion, existence, error estimates, computations.

1. Introduction

Partial differential equations on evolving surfaces occur in many applications. For example, traditionally they arise naturally in fluid dynamics and materials science and more recently in the mathematics of images. In this paper we propose a mathematical approach to the formulation and approximation of transport and diffusion of a material quantity on an evolving surface in $\mathbb{R}^{n+1}$ (n=1,2). We have in mind a surface which not only evolves in the normal direction so as to define the surface evolution but also has a tangential velocity associated with the motion of material points in the surface which advects material quantities such as heat or mass. For our purposes here we assume that the surface evolution is prescribed.

1.1 *The advection diffusion equation*

Conservation of a scalar with a diffusive flux on an evolving hypersurface $\Gamma(t)$ leads to the diffusion equation

$$\dot{u} + u\nabla_\Gamma \cdot v - \nabla_\Gamma \cdot (\mathscr{D}_0 \nabla_\Gamma u) = 0 \tag{1.1}$$

on $\Gamma(t)$. Here $\dot{u}$ denotes the covariant or advective surface material derivative, v is the velocity of the surface and ∇_Γ is the tangential surface gradient. If $\partial\Gamma(t)$ is empty then the equation does not need a boundary condition. Otherwise we can impose Dirichlet or Neumann boundary conditions on $\partial\Gamma(t)$.

1.2 *The finite element method*

In this paper we propose a finite element approximation based on the variational form

$$\frac{d}{dt}\int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \mathscr{D}_0 \nabla_\Gamma u \cdot \nabla_\Gamma \varphi = \int_{\Gamma(t)} u\dot{\varphi} \tag{1.2}$$

where φ is an arbitrary test function defined on the surface $\Gamma(t)$ for all t. This provides the basis of our evolving surface finite element method (ESFEM) which is applicable to arbitrary evolving n–dimensional hypersurfaces in $\mathbb{R}^{n+1}$ (curves in $\mathbb{R}^2$) with or without boundary. Indeed this is the extension of the method of Dziuk (6) for the Laplace-Beltrami equation on a stationary surface. The principal idea is to use a polyhedral approximation of Γ based on a triangulated surface. It follows that a quite natural local piecewise linear parameterisation of the surface is employed rather than a global one. The finite element space is then the space of continuous piecewise linear functions on the triangulated surface whose nodal basis functions enjoy the remarkable property

$$\dot{\phi}_j = 0.$$

The implementation is thus rather similar to that for solving the diffusion equation on flat stationary domains. For example, the backward Euler time discretization leads to the ESFEM scheme

$$\frac{1}{\tau}\left(\mathscr{M}(t^{m+1})\alpha^{m+1} - \mathscr{M}(t^m)\alpha^m\right) + \mathscr{S}(t^{m+1})\alpha^{m+1} = 0$$

where $\mathscr{M}(t)$ and $\mathscr{S}(t)$ are the time dependent surface mass and stiffness matrices and α^m is the vector of nodal values at time t^m. Here, τ denotes the time step size.

1.3 *Level set or implicit surface approach*

An alternative approach to our method based on the use of (1.2) is to embed the surface in a family of level set surfaces (1; 15; 9). This Eulerian approach can be discretized on a Cartesian grid in $\mathbb{R}^{n+1}$ and has the usual advantages and disadvantages of level set methods.

1.4 *Applications*

Such a problem arises, for example, when modeling the transport of an insoluble surfactant on the interface between two fluids, (14; 10). Here one views the velocity of the surface as being the fluid velocity and hence the surfactant is transported by advection via the tangential fluid velocity (and hence the tangential surface velocity) as well by diffusion within the surface. The evolution of the surface itself in the normal direction is then given by the normal component of the fluid velocity.

Diffusion induced grain boundary motion, (4; 7; 13; 5), has the feature of coupling forced mean curvature flow for the motion of a grain boundary with a diffusion equation for a concentration of mass in the grain boundary. In this case there is no material tangential velocity of the grain boundary so it is sufficient to consider the surface velocity as being in the normal direction.

Another example is pattern formation on the surfaces of growing organisms modelled by reaction diffusion equations, (12). Possible applications in image processing are suggested by the article (11).

1.5 *Outline of paper*

The layout of the paper is as follows. We begin in section 2 by defining notation and essential concepts from elementary differential geometry necessary to describe the problem and numerical method. The equations presented above are justified in section 3. The weak form of the equations is derived in section 4 and the well posedness of the initial boundary value problem is established. In section 5 the finite element method is defined and some preliminary approximations results are shown. Error bounds for the semi-discretization in space are proved in section 6. Implementation issues are discussed in section 7 and the results of numerical experiments are presented. Finally in section 8 we make some concluding remarks.

2. Basic notation and surface derivatives

2.1 *Notation*

For each $t \in [0,T_0]$, $T_0 > 0$, let $\Gamma(t)$ be a compact smooth connected and oriented hypersurface in $\mathbb{R}^{n+1}$ (n=1,2) and $\Gamma_0 = \Gamma(0)$. In order to formulate the model it is convenient to use two descriptions of $\Gamma(t)$, one using a diffeomorphic parametrization and the other a level set function.

Note that to define an evolving surface $\Gamma(t)$ it is sufficient to prescribe the normal velocity. However we wish to consider time dependent material surfaces $\Gamma = \Gamma(t)$ for which a material particle P located at $X_P(t)$ on $\Gamma(t)$ has a velocity $\dot{X}_P(t)$ not necessarily only in the normal direction. Thus we assume that there is a velocity field v so that points P on $\Gamma(t)$ evolve with velocity $\dot{X}_P(t) = v(X_P(t),t)$.

Hence for our first description, we assume that there exists a map

$$\Phi(\cdot,t) : \Gamma_0 \to \Gamma(t)\ ,\ \Phi \in C^1([0,T_0],C^1(\Gamma_0)) \cap C^0([0,T_0],C^3(\Gamma_0)),$$

so that $\Phi(\cdot,t)$ is a diffeomorphism from Γ_0 to $\Gamma(t)$ for every $t \in [0,T_0]$ and that it solves the equation

$$\Phi_t(\cdot,t) = v(\Phi(\cdot,t),t), \Phi(\cdot,0) = \mathrm{Id}.$$

Thus for $X_P(0) = P \in \Gamma_0$ we have $X_P(t) = \Phi(P,t) \in \Gamma(t)$.

It follows that $\Gamma(t)$ has a second representation defined by a smooth level set function $d = d(x,t)$, $x \in \mathbb{R}^{n+1}, t \in [0,T_0]$ so that

$$\Gamma(t) = \{x \in \mathcal{N}(t) | \, d(x,t) = 0\}$$

where $\mathcal{N}(t)$ is an open subset of $\mathbb{R}^{n+1}$ in which $\nabla d \neq 0$ and chosen so that

$$d, d_t, d_{x_i}, d_{x_i x_j} \in C^1(\mathcal{N}_{T_0})\,(i,j = 1,\dots,n)$$

for $\mathcal{N}_{T_0} = \ \cup_{t\in[0,T_0]}\ \mathcal{N}(t) \times \{t\}$.

The orientation of Γ is set by taking the normal ν to Γ to be in the direction of increasing d. Hence we define a normal vector field by

$$\nu(x,t) = \frac{\nabla d(x,t)}{|\nabla d(x,t)|}$$

so that the normal velocity V of Γ is given by

$$V(x,t) = -\frac{d_t(x,t)}{|\nabla d(x,t)|}.$$

We assume that the velocity field v is C^1 in $\mathcal{N}_{T_0}$. It has the decomposition $v = V\nu + T$ into normal velocity $V = -\frac{d_t}{|\nabla d|}|_{\Gamma(t)}$ and tangential velocity T.

Observe that a possible choice for d is a signed distance function and in that case $|\nabla d| = 1$ on $\mathcal{N}_{T_0}$. For later use we mention that $\mathcal{N}(t)$ can be chosen such that for every $x \in \mathcal{N}(t)$ and $t \in [0, T_0]$ there exists a unique $a(x,t) \in \Gamma(t)$ such that

$$x = a(x,t) + d(x,t)\nu(a(x,t),t), \tag{2.1}$$

where here d denotes the signed distance function to $\Gamma(t)$.

For any function η defined on an open subset $\mathcal{N}(t)$ of $\mathbb{R}^{n+1}$ containing $\Gamma(t)$ we define its tangential gradient on Γ by

$$\nabla_\Gamma \eta = \nabla\eta - \nabla\eta \cdot \nu\, \nu$$

where, for x and y in $\mathbb{R}^{n+1}$, $x \cdot y$ denotes the usual scalar product and $\nabla\eta$ denotes the usual gradient on $\mathbb{R}^{n+1}$. The tangential gradient $\nabla_\Gamma \eta$ only depends on the values of η restricted to $\Gamma(t)$ and $\nabla_\Gamma \eta \cdot \nu = 0$. The components of the tangential gradient will be denoted by

$$\nabla_\Gamma \eta = \left(\underline{D}_1 \eta, \ldots, \underline{D}_{n+1} \eta\right).$$

The Laplace-Beltrami operator on $\Gamma(t)$ is defined as the tangential divergence of the tangential gradient:

$$\Delta_\Gamma \eta = \nabla_\Gamma \cdot \nabla_\Gamma \eta = \sum_{i=1}^{n+1} \underline{D}_i \underline{D}_i \eta.$$

Let $\Gamma(t)$ have a boundary $\partial\Gamma(t)$ whose intrinsic unit outer normal, tangential to $\Gamma(t)$, is denoted by μ. Then the formula for integration by parts on $\Gamma(t)$ is

$$\int_\Gamma \nabla_\Gamma \eta = -\int_\Gamma \eta H \nu + \int_{\partial\Gamma} \eta \mu, \tag{2.2}$$

where H denotes the mean curvature of Γ with respect to ν, which is given by

$$H = -\nabla_\Gamma \cdot \nu. \tag{2.3}$$

The orientation is such that for a sphere $\Gamma = \{x \in \mathbb{R}^{n+1} \,|\, |x - x_0| = R\}$ and the choice $d(x) = R - |x - x_0|$ the normal is pointing into the ball $B_R(x_0) = \{x \in \mathbb{R}^{n+1} \,|\, |x - x_0| < R\}$ and the mean curvature of Γ is given by $H = \frac{n}{R}$. Note that H is the sum of the principle curvatures rather than the arithmetic mean and hence differs from the common definition by a factor n. The mean curvature vector $H\nu$ is invariant with respect to the choice of the sign of d.

Green's formula on the surface Γ is

$$\int_\Gamma \nabla_\Gamma \xi \cdot \nabla_\Gamma \eta = \int_{\partial\Gamma} \xi \nabla_\Gamma \eta \cdot \mu - \int_\Gamma \xi \Delta_\Gamma \eta. \tag{2.4}$$

If Γ is closed then $\partial\Gamma$ is empty and the boundary terms do not appear. For these facts about tangential derivatives we refer to (8), pp 389–391. Note that, in general, higher order tangential derivatives do not commute.

We shall use Sobolev spaces on surfaces Γ. For a given Lipschitz surface Γ we define

$$H^1(\Gamma) = \{\eta \in L^2(\Gamma) \mid \nabla_\Gamma \eta \in L^2(\Gamma)^{n+1}\}$$

and $H^1_0(\Gamma)$ in the obvious way, if $\partial\Gamma \neq \emptyset$. For smooth enough Γ we analoguously define the Sobolev spaces $H^k(\Gamma)$ for $k \in \mathbb{N}$.

2.2 *The material derivative and Leibniz formulae*

By a dot we denote the material derivative of a scalar function $f = f(x,t)$ defined on $\mathcal{N}_{T_0}$:

$$\dot{f} = \frac{\partial f}{\partial t} + v \cdot \nabla f. \tag{2.5}$$

In particular we note that

$$\dot{f}(\Phi(\cdot,t),t) = \frac{d}{dt} f(\Phi(\cdot,t),t)$$

and that the derivative depends only on the values of f on the evolving surface $\Gamma(t)$.

REMARK 2.1 The material derivative $\dot{g}$ of a function g defined on the $(n+1)$ dimensional hypersurface $\mathcal{G}_{T_0} = \cup_{t\in[0,T_0]} \Gamma(t) \times \{t\} \subset \mathbb{R}^{n+2}$ is related to the tangential gradient on this surface by the formula

$$\dot{g} = (1+V^2)\left(\nabla_{\mathcal{G}_{T_0}} g\right)_{n+2} + v \cdot \nabla_\Gamma g,$$

where $\left(\nabla_{\mathcal{G}_{T_0}} g\right)_{n+2}$ is the $n+2$-nd component of this tangential gradient. Note that

$$||g||^2_{L^2(\mathcal{G}_{T_0})} + ||\nabla_\Gamma g||^2_{L^2(\mathcal{G}_{T_0})} + ||\dot{g}||^2_{L^2(\mathcal{G}_{T_0})}$$

is equivalent to $||g||^2_{H^1(\mathcal{G}_{T_0})}$.

It is convenient to note that with (2.3) we obtain

$$\nabla_\Gamma \cdot v = \nabla_\Gamma \cdot (V\nu) + \nabla_\Gamma \cdot T = V\nabla_\Gamma \cdot \nu + \nabla_\Gamma \cdot T = -VH + \nabla_\Gamma \cdot T \tag{2.6}$$

and

$$\nabla_\Gamma \cdot v = \operatorname{trace}((\mathcal{I} - \nu \otimes \nu)\nabla v) \tag{2.7}$$

where $\mathscr{I} - \nu \otimes \nu$ denotes the matrix with entries $\delta_{ij} - \nu_i \nu_j$. For a scalar f we have

$$v \cdot \nabla f = V\nu \cdot \nabla f + T \cdot \nabla f = V\frac{\partial f}{\partial \nu} + T \cdot \nabla_\Gamma f. \tag{2.8}$$

The following formula for the differentiation of a parameter dependent surface integral will play a decisive role.

LEMMA 2.1 (LEIBNIZ FORMULA) Let Γ be a surface and f be a function defined in $\mathscr{N}_{T_0}$ such that all the following quantities exist. Then

$$\frac{d}{dt}\int_\Gamma f = \int_\Gamma \left(\dot{f} + f\nabla_\Gamma \cdot v\right) \tag{2.9}$$

and with the decomposition $v = V\nu + T$ of the velocity of Γ into normal and tangential velocity

$$\frac{d}{dt}\int_\Gamma f = \int_\Gamma \left(\frac{\partial f}{\partial t} + V\frac{\partial f}{\partial \nu} - fVH + \nabla_\Gamma \cdot (fT)\right). \tag{2.10}$$

Finally, with the deformation tensor $D(v)_{ij} = \frac{1}{2}\left(\underline{D}_i v_j + \underline{D}_j v_i\right)$ $(i, j = 1, \ldots, n)$,

$$\frac{1}{2}\frac{d}{dt}\int_\Gamma |\nabla_\Gamma f|^2 = \int_\Gamma \nabla_\Gamma f \cdot \nabla_\Gamma \dot{f} + \frac{1}{2}\int_\Gamma |\nabla_\Gamma f|^2 \nabla_\Gamma \cdot v - \int_\Gamma D(v)\nabla_\Gamma f \cdot \nabla_\Gamma f. \tag{2.11}$$

A proof of this Lemma is given in section 9.

3. Conservation and diffusion on $\Gamma(t)$

3.1 *Conservation law*

Let u be the density of a scalar quantity on $\Gamma(t)$ (for example mass per unit area $n = 2$ or mass per unit length $n = 1$). The basic conservation law we wish to consider can be formulated for an arbitrary portion $\mathscr{M}(t)$ of $\Gamma(t)$, which is the image of an arbitrary portion $\mathscr{M}(0)$ of $\Gamma(0)$ under the prescribed velocity flow. The law is that, for every $\mathscr{M}(t)$,

$$\frac{d}{dt}\int_{\mathscr{M}(t)} u = -\int_{\partial\mathscr{M}(t)} q \cdot \mu \tag{3.1}$$

where, $\partial\mathscr{M}(t)$ is the boundary of $\mathscr{M}(t)$ (a curve if $n = 2$ and the end points of a curve if $n = 1$) and μ is the conormal on $\partial\mathscr{M}(t)$. Thus μ is the unit normal to $\partial\mathscr{M}(t)$ pointing out of $\mathscr{M}(t)$ and tangential to $\Gamma(t)$. The surface flux is denoted by q. Observe that components of q normal to $\mathscr{M}$ do not contribute to the flux, so we may assume that q is a tangent vector.

With the use of integration by parts, (2.2), we obtain

$$\int_{\partial\mathscr{M}(t)} q \cdot \mu = \int_{\mathscr{M}(t)} \nabla_\Gamma \cdot q + \int_{\mathscr{M}(t)} q \cdot \nu H = \int_{\mathscr{M}(t)} \nabla_\Gamma \cdot q.$$

On the other hand by the Leibniz formula (2.9) we have

$$\frac{d}{dt}\int_{\mathscr{M}(t)} u = \int_{\mathscr{M}(t)} \left(\dot{u} + u\nabla_\Gamma \cdot v\right),$$

so that

$$\int_{\mathcal{M}(t)} \left(\dot{u} + u\nabla_\Gamma \cdot v + \nabla_\Gamma \cdot q\right) = 0,$$

which implies the pointwise conservation law

$$\dot{u} + u\nabla_\Gamma \cdot v + \nabla_\Gamma \cdot q = 0. \tag{3.2}$$

Now using a representation of u off the surface, so that u has usual spatial derivatives in $\mathbb{R}^{n+1}$, we can write (3.2) as

$$u_t + v \cdot \nabla u + u\nabla_\Gamma \cdot v + \nabla_\Gamma \cdot q = 0. \tag{3.3}$$

Observing (2.10), an alternative form is

$$u_t + V\frac{\partial u}{\partial \nu} - uVH + \nabla_\Gamma \cdot (uT) + \nabla_\Gamma \cdot q = 0. \tag{3.4}$$

Thus we arrive at some special cases:

1. *Divergence free velocity*,
$$u_t + v \cdot \nabla u - u\nu \cdot \nabla v \nu + \nabla_\Gamma \cdot q = 0. \tag{3.5}$$

2. *Zero tangential velocity*,
$$u_t + V\frac{\partial u}{\partial \nu} - uVH + \nabla_\Gamma \cdot q = 0. \tag{3.6}$$

3. *Zero normal velocity*,
$$u_t + \nabla_\Gamma \cdot (uv) + \nabla_\Gamma \cdot q = 0. \tag{3.7}$$

4. *Zero normal velocity and divergence free tangential velocity*,
$$u_t + v \cdot \nabla_\Gamma u + \nabla_\Gamma \cdot q = 0. \tag{3.8}$$

5. *Stationary surface*,
$$u_t + \nabla_\Gamma \cdot q = 0. \tag{3.9}$$

REMARK 3.1 Our approach does not require values of the scalar u away from the surface and so does not need to consider $\frac{\partial u}{\partial \nu}$. In some approaches this can be handled by assuming an extension of u away from the surface which is constant in the normal direction, (15), so $\frac{\partial u}{\partial \nu} = 0$. Furthermore there is no explicit need to compute the curvature or normal of the surface in (3.2).

REMARK 3.2 Our computational approach is based on (3.2) and depends only on explicit knowledge of the surface location and does not require explicit evaluations of the normal ν or the mean curvature H.

3.2 *Diffusion equation and variational form*

Taking q to be the diffusive flux

$$q = -\mathscr{D}_0 \nabla_\Gamma u \tag{3.10}$$

where the symmetric diffusion tensor is $\mathscr{D}_0 \geqslant d_0 \mathscr{I} > 0$ on the tangent space and $\mathscr{D}_0 \nu = 0$. This leads to the diffusion equation

$$\dot{u} + u \nabla_\Gamma \cdot v - \nabla_\Gamma \cdot (\mathscr{D}_0 \nabla_\Gamma u) = 0 \tag{3.11}$$

on $\Gamma(t)$.

If $\partial\Gamma = \emptyset$, i.e. the surface has no boundary, then there is no need for boundary conditions. For example, this would be the case if $\Gamma(t)$ is the bounding surface of a domain.

If $\partial\Gamma(t)$ is non-empty then we impose the homogeneous Dirichlet boundary condition

$$u = 0 \ \text{ on } \ \partial\Gamma(t). \tag{3.12}$$

Again if $\partial\Gamma(t)$ is non-empty then we could impose the Neumann flux condtion

$$\mathscr{D}_0 \nabla_\Gamma u \cdot \mu = 0. \tag{3.13}$$

The variational form (1.2) then is an easy consequence. We multiply equation (1.1) by an adequate test function φ and integrate over $\Gamma(t)$. We then obtain using integration by parts (2.2) and the Leibniz formula (2.9):

$$\begin{aligned}
0 &= \int_{\Gamma(t)} \left(\dot{u}\varphi + u\varphi \nabla_\Gamma \cdot v\right) + \int_{\Gamma(t)} \mathscr{D}_0 \nabla_\Gamma u \cdot \nabla_\Gamma \varphi \\
&= \int_{\Gamma(t)} \left((u\varphi)^{\cdot} - u\dot{\varphi} + u\varphi \nabla_\Gamma \cdot v\right) + \int_{\Gamma(t)} \mathscr{D}_0 \nabla_\Gamma u \cdot \nabla_\Gamma \varphi \\
&= \frac{d}{dt} \int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \mathscr{D}_0 \nabla_\Gamma u \cdot \nabla_\Gamma \varphi - \int_{\Gamma(t)} u\dot{\varphi}.
\end{aligned}$$

4. Weak form and wellposedness

We introduce the notion of a weak solution of the surface PDE (1.1), for which we derived a variational form in (1.2). Just as in the cartesian case one could integrate (1.2) with respect to time and then define a weak solution without using a time derivative of u. But since the purpose of this work is the approximation of stronger solutions we use a somewhat stronger notion of solution. We treat the case of a compact surface without boundary.

Definition 4.1 (Weak solution) Let $\mathscr{G}_{T_0} = \cup_{t\in[0,T_0]} \Gamma(t) \times \{t\}$ and $\mathscr{D}_0 \in L^\infty(\mathscr{G}_{T_0})$. A function $u \in H^1(\mathscr{G}_{T_0})$ is a weak solution of (1.1), if for almost every $t \in (0, T_0)$

$$\int_{\Gamma(t)} \dot{u}\varphi + \int_{\Gamma(t)} u\varphi \nabla_\Gamma \cdot v + \int_{\Gamma(t)} \mathscr{D}_0 \nabla_\Gamma u \cdot \nabla_\Gamma \varphi = 0 \tag{4.1}$$

for every $\varphi(\cdot,t) \in H^1(\Gamma(t))$.

In order to simplify the presentation we set

$$\mathscr{D}_0 = \mathscr{I}$$

in this section. With suitable assumptions on $\mathscr{D}_0$ the results can easily be extended to the general case. We first prove the basic energy equations for the problem. They will lead to existence and will be the basis for error estimates later.

LEMMA 4.1 Let u be a weak solution of (1.1). Then

$$\frac{1}{2}\frac{d}{dt}\int_{\Gamma(t)} u^2 + \int_{\Gamma(t)} |\nabla_\Gamma u|^2 + \frac{1}{2}\int_{\Gamma(t)} u^2 \nabla_\Gamma \cdot v = 0. \tag{4.2}$$

Proof. We choose $\varphi = u$ in

$$\frac{d}{dt}\int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \nabla_\Gamma u \cdot \nabla_\Gamma \varphi = \int_{\Gamma(t)} u\dot{\varphi}$$

and get

$$\frac{d}{dt}\int_\Gamma u^2 + \int_\Gamma |\nabla_\Gamma u|^2 = \int_\Gamma u\dot{u} = \frac{1}{2}\int_\Gamma (u^2)^{\cdot} = \frac{1}{2}\frac{d}{dt}\int_\Gamma u^2 - \frac{1}{2}\int_\Gamma u^2 \nabla_\Gamma \cdot v,$$

and this was the claim. □

LEMMA 4.2 Let u be a weak solution of (1.1), for which the following quantities exist. Then

$$\int_\Gamma \dot{u}^2 + \frac{1}{2}\frac{d}{dt}\int_\Gamma |\nabla_\Gamma u|^2 = \frac{1}{2}\int_\Gamma |\nabla_\Gamma u|^2 \nabla_\Gamma \cdot v - \int_\Gamma D(v)\nabla_\Gamma u \cdot \nabla_\Gamma u - \int_\Gamma u\dot{u}\nabla_\Gamma \cdot v. \tag{4.3}$$

Proof. We choose $\varphi = \dot{u}$ in (4.1) and get with the use of (2.11)

$$\begin{aligned} 0 &= \int_\Gamma \dot{u}^2 + \int_\Gamma u\dot{u}\nabla_\Gamma \cdot v + \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma \dot{u} \\ &= \int_\Gamma \dot{u}^2 + \frac{1}{2}\frac{d}{dt}\int_\Gamma |\nabla_\Gamma u|^2 - \frac{1}{2}\int_\Gamma |\nabla_\Gamma u|^2 \nabla_\Gamma \cdot v + \int_\Gamma D(v)\nabla_\Gamma u \cdot \nabla_\Gamma u + \int_\Gamma u\dot{u}\nabla_\Gamma \cdot v. \end{aligned}$$

□

Theorem 4.2 (Existence) Let $u_0 \in H^1(\Gamma_0)$. Then there exists a unique weak solution of (1.1) and the following energy estimates hold:

$$\sup_{(0,T_0)} \|u\|^2_{L^2(\Gamma)} + \int_0^{T_0} \|\nabla_\Gamma u\|^2_{L^2(\Gamma)} \leqslant c\|u_0\|^2_{L^2(\Gamma_0)}, \tag{4.4}$$

$$\int_0^{T_0} \|\dot{u}\|^2_{L^2(\Gamma)} + \sup_{(0,T_0)} \|\nabla_\Gamma u\|^2_{L^2(\Gamma)} \leqslant c\|u_0\|^2_{H^1(\Gamma_0)}. \tag{4.5}$$

Proof. That there can be no more than one weak solution is a consequence of the estimate (4.2) which applies to the difference of two weak solutions by linearity and a standard Gronwall argument. Let φ_j^0, $j \in \mathbb{N}$, denote the eigenfunctions of the Laplace-Beltrami operator on Γ_0, (see (2)). Let $\Phi = \Phi(y,t), y \in \Gamma_0, 0 \leqslant t \leqslant T_0$ denote the diffeomorphism (see section 2) between Γ_0 and $\Gamma(t)$. Set

$$\varphi_j(\Phi(\cdot,t),t) = \varphi_j^0.$$

This then gives a countable dense subset $\{\varphi_j(\cdot,t) \,|\, j \in \mathbb{N}\}$ of $H^1(\Gamma(t))$. For $j = 1,\dots,N$ one has the transport property

$$\dot{\varphi}_j = 0 \quad \text{on} \quad \Gamma. \tag{4.6}$$

Our ansatz for a Galerkin solution of (4.1) from $X_N = \operatorname{span}\{\varphi_1(\cdot,t),\dots,\varphi_N(\cdot,t)\}$ is

$$u_N(x,t) = \sum_{j=1}^{N} u_j(t)\varphi_j(x,t)$$

where $u_j(0) = (u_0,\varphi_j^0)_{L^2(\Gamma_0)}$. Because of the property (4.6) we have that

$$\dot{u}_N = \sum_{j=1}^{N} \dot{u}_j \varphi_j$$

is in the same finite dimensional space X_N as u_N. By (linear) ODE theory we have existence and uniqueness of u_N satisfying

$$\frac{d}{dt}\int_{\Gamma(t)} u_N\varphi + \int_{\Gamma(t)} \nabla_\Gamma u_N \cdot \nabla_\Gamma \varphi = \int_{\Gamma(t)} u_N \dot{\varphi} \tag{4.7}$$

for all $\varphi(\cdot,t) \in \operatorname{span}\{\varphi_1(\cdot,t),\dots,\varphi_N(\cdot,t)\}$. Lemma 4.1 then implies the energy equation

$$\frac{1}{2}\frac{d}{dt}\int_{\Gamma(t)} u_N^2 + \int_{\Gamma(t)} |\nabla_\Gamma u_N|^2 + \frac{1}{2}\int_{\Gamma(t)} u_N^2 \nabla_\Gamma \cdot v = 0. \tag{4.8}$$

and a Gronwall argument gives the estimate

$$\sup_{t\in(0,T_0)} \int_{\Gamma(t)} u_N(\cdot,t)^2\, dA + \int_0^{T_0}\int_{\Gamma(t)} |\nabla_\Gamma u_N(\cdot,t)|^2\, dA dt \leqslant C \tag{4.9}$$

where the constant depends on the geometry of $\Gamma(t)$, $t \in [0,T_0]$ and on the inital data u_0 but not on N. Similarly with Lemma 4.2 we get

$$\int_{\Gamma(t)} \dot{u}_N^2 + \frac{1}{2}\frac{d}{dt}\int_{\Gamma(t)} |\nabla_\Gamma u_N|^2 \leqslant c\int_{\Gamma(t)} |\nabla_\Gamma u_N|^2 + c\int_{\Gamma(t)} |u_N||\dot{u}_N|$$

so that with (4.9) and a Gronwall argument we arrive at the estimate

$$\int_0^{T_0}\int_{\Gamma(t)} \dot{u}_N(\cdot,t)^2\, dA dt + \sup_{t\in(0,T_0)}\int_{\Gamma(t)} |\nabla_\Gamma u_N(\cdot,t)|^2\, dA \leqslant C. \tag{4.10}$$

When we combine the estimates (4.9) and (4.10), then we obtain the boundedness of the sequence $(u_N)_{N\in\mathbb{N}}$ in $H^1(\mathscr{G}_{T_0})$: Thus there exists a $u = u(x,t)$, $u \in H^1(\mathscr{G}_{T_0})$ such that for a subsequence (which we again call u_N),

$$u_N \rightharpoonup u \quad (N \to \infty) \quad \text{in} \quad H^1(\mathscr{G}_{T_0}).$$

This, (4.7), the density of the sequence φ_j and Fubini's theorem imply that u is a weak solution as in Definition 4.1. □

For our error estimates we shall need regularity properties of the solution u for smoothly evolving smooth Γ.

Theorem 4.3 Let Γ be sufficiently smooth. Then $u(\cdot,t) \in H^2(\Gamma(t))$ and

$$\int_0^{T_0} \|u\|^2_{H^2(\Gamma)} \leqslant c\|u_0\|^2_{H^1(\Gamma_0)}. \tag{4.11}$$

Proof. Because of the smoothness of Γ we have from (2) and the elliptic PDE

$$\int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma \varphi = -\int_\Gamma \left(\dot{u} + u\nabla_\Gamma \cdot v\right) \varphi$$

for all φ that $u \in H^2(\Gamma)$ and $\|u\|_{H^2(\Gamma)} \leqslant c(\|\dot{u}\|_{L^2(\Gamma)} + \|u\|_{L^2(\Gamma)})$. The energy estimates (4.4) and (4.5) then prove the result. □

REMARK 4.1 The results of existence and uniqueness are easily extended to the case where $\partial\Gamma(t)$ is non-empty and either Dirichlet or Neumann conditions are prescribed. Then, for the regularity result of Theorem 4.3 we need regularity of the boundary $\partial\Gamma$.

5. Finite element approximation

5.1 *Finite elements on surfaces*

The smooth evolving surface $\Gamma(t)$ ($\partial\Gamma(t) = \emptyset$) is approximated by an evolving surface

$$\Gamma_h(t) \subset \mathcal{N}(t)\ ,\ (\partial\Gamma_h(t) = \emptyset),$$

which for each t is of class $C^{0,1}$ and in time is smooth. In particular for $n = 2$, $\Gamma_h(t)$ is a triangulated (and hence polyhedral) surface consisting of triangles e in $\mathscr{T}_h(t)$ with maximum diameter, uniformly in time, being denoted by h and inner radius bounded below by $\sigma_h \geqslant ch$ with some $c > 0$. The vertices $\{X_j(t)\}_{j=1}^N$ of the triangles are taken to sit on $\Gamma(t)$ so that $\Gamma_h(t)$ is an interpolation. Note that by (2.1) for every triangle $e(t) \subset \Gamma_h(t)$ there is a unique curved triangle $T(t) = a(e(t),t) \subset \Gamma(t)$. In order to avoid a global double covering (see Figure 1) we assume that,

$$\text{for each point } a \in \Gamma \text{ there is at most one point } x \in \Gamma_h \text{ with } a = a(x,\cdot). \tag{5.1}$$

This implies that there is a bijective correspondence between the triangles on Γ_h and the induced curvilinear triangles on Γ.

For any continuous function η defined on the discrete surface $\Gamma_h(t)$ we may define an extension or lift onto $\Gamma(t)$ by

$$\eta^l(a) = \eta(x(a)) \quad a \in \Gamma(t) \tag{5.2}$$

where by (2.1) and our assumptions, $x(a)$ is defined as the unique solution of

$$x = a + d(x,t)\nu(a,t). \tag{5.3}$$

Furthermore we understand by $\eta^l(x)$ the constant extension from $\Gamma(t)$ in the normal direction $\nu(a,t)$.

For each t we have a finite element space

$$S_h(t) = \left\{\phi \in C^0(\Gamma_h(t)) |\, \phi|_e \text{ is linear affine for each } e \in \mathscr{T}_h(t)\right\}.$$

It is convenient to introduce

$$S_h^l(t) = \left\{\eta^l \in C^0(\Gamma(t)) |\, \eta^l(a) = \eta(x(a)), \eta \in S_h(t) \text{ and } x(a) \text{ given by (5.3)}\right\}.$$

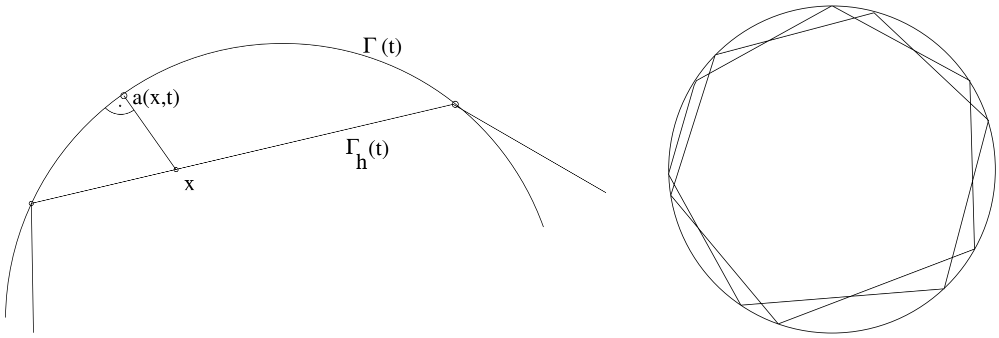

FIG. 1. Left: Approximation of a curve $\Gamma(t)$ by a polygon $\Gamma_h(t)$, a point $x \in \Gamma_h(t)$ and its orthogonal projection $a(x,t)$ onto $\Gamma(t)$. Right: A polygonal approximation to a circle, violating the condition (5.1).

Similarly each $e(t)$ defines a curvilinear triangle $T(t)$ on $\Gamma(t)$ by

$$T(t) = \{a(x,t) | x \in e(t)\}.$$

In the error analysis of the finite element scheme we shall need the following technical Lemma, which gives more detailed information about the order of approximation of the geometry. It will become clear in the proof of Theorem 6.1 how we shall exploit the following estimates.

LEMMA 5.1 Assume Γ and Γ_h are as above. Then

$$\sup_{t\in(0,T_0)} \|d(\cdot,t)\|_{L^\infty(\Gamma_h(t))} \leqslant ch^2. \tag{5.4}$$

The quotient, δ_h, between the smooth and discrete surface measures dA and dA_h, defined by $\delta_h dA_h = dA$, satisfies

$$\sup_{t\in(0,T_0)} \sup_{\Gamma_h(t)} |1-\delta_h| \leqslant ch^2. \tag{5.5}$$

Let P and P_h be the projections onto the tangent planes, $P_{ij} = \delta_{ij} - \nu_i\nu_j$, $P_{h,ij} = \delta_{ij} - \nu_{h,i}\nu_{h,j}$, and let $R_h = \frac{1}{\delta_h} P(I - d\mathcal{H})P_h(I - d\mathcal{H})$, $\mathcal{H}_{ij} = d_{x_i x_j} = (\nu_i)_{x_j}$. Then

$$\sup_{t\in(0,T_0)} \sup_{\Gamma_h(t)} |(I-R_h)P| \leqslant ch^2 \tag{5.6}$$

Proof. For ease of exposition and without loss of generality, we treat two dimensional surfaces and omit the time dependence of all quantities. Let $e \subset \Gamma_h$ be a triangle of the discrete surface. The corresponding curved triangle $T = a(e)$ thus is parametrized over e. Again without loss of generality, we may assume that $e \subset \mathbb{R}^2 \times \{0\}$. By I_h we denote the Lagrange interpolation on e.

Since the vertices of e lie on Γ we have that the interpolate $I_h d$ vanishes identically on e and

$$\|d\|_{L^\infty(e)} = \|d - I_h d\|_{L^\infty(T_h)} \leqslant ch^2 |d|_{H^{2,\infty}(T_h)} \leqslant ch^2 \|d\|_{C^{1,1}(\mathcal{N}_{T_0})}$$

and similarly

$$\|\nu_j\|_{L^\infty(e)} = \|d_{x_j}\|_{L^\infty(e)} \leqslant ch \quad (j = 1,2). \tag{5.7}$$

For $x=(x_1,x_2,0)\in e$ we have by (2.1) that the map $a(x)$ satisfies

$$a_{i,x_j}=\delta_{ji}-\nu_j\nu_i-d\mathscr{H}_{ji}.$$

Furthermore since $dA_h=dx_1dx_2$ and $dA=|a_{x_1}\wedge a_{x_2}|\,dx$ we have

$$\delta_h=|a_{x_1}\wedge a_{x_2}|.$$

To derive the estimate of the surface elements (5.5) we observe that from (5.4)

$$a_{i,x_j}=\delta_{ji}-\nu_j\nu_i-d\mathscr{H}_{ji}=P_{ji}+O(h^2).$$

This implies for $n=2$

$$\begin{aligned} a_{x_1}\wedge a_{x_2} &= \left(e_1-\nu_1\nu-d\nu_{x_1}\right)\wedge\left(e_2-\nu_2\nu-d\nu_{x_2}\right)=\left(e_1-\nu_1\nu\right)\wedge\left(e_2-\nu_2\nu\right)+O(h^2)\\ &= e_3-\nu_2e_1\wedge\nu-\nu_1\nu\wedge e_2+O(h^2)=\nu_3\nu+O(h^2)\end{aligned}$$

together with

$$|a_{x_1}\wedge a_{x_2}|^2\geqslant 1-O(h^2)\geqslant c_0>0$$

for $h\leqslant h_0$. Hence we have from (5.7)

$$|1-\delta_h|=|1-|a_{x_1}\wedge a_{x_2}||=\frac{|1-|a_{x_1}\wedge a_{x_2}|^2|}{1+|a_{x_1}\wedge a_{x_2}|}=\frac{|\nu_1^2+\nu_2^2+O(h^2)|}{1+|a_{x_1}\wedge a_{x_2}|}\leqslant ch^2,$$

and we have proved (5.5).

The proof of (5.6) follows from the previous estimates when we keep in mind that in our situation $\nu_h=e_3$. Note that by ν_h we mean the piece-wise constant vector defined by the normals to the triangles on $\Gamma_h(t)$. We find that

$$(R_h-I)P=PP_hP-P+O(h^2)=O(h^2)$$

since for a unit vector z we have

$$|(PP_hP-P)z|=|z\cdot\left(\nu_h-(\nu_h\cdot\nu)\nu\right)\left(\nu_h-(\nu_h\cdot\nu)\nu\right)|\leqslant ch^2,$$

because from (5.7),

$$|\nu_h-(\nu_h\cdot\nu)\nu|=|e_3-\nu_3\nu|=\sqrt{1-\nu_3^2}=\sqrt{\nu_1^2+\nu_2^2}=O(h).$$

This proves (5.6). □

In order to compare the norms between functions and their lift we need the following Lemma.

LEMMA 5.2 Let $\eta:\Gamma_h\to\mathbb{R}$ with lift $\eta^l:\Gamma\to\mathbb{R}$. Then for the corresponding plane, $e\subset\Gamma_h$, and smooth, $T\subset\Gamma$, triangles the following estimates hold if the norms exist. There is a constant $c>0$ independent of h such that

$$\frac{1}{c}\|\eta\|_{L^2(e)}\leqslant\|\eta^l\|_{L^2(T)}\leqslant c\|\eta\|_{L^2(e)} \tag{5.8}$$

$$\frac{1}{c}\|\nabla_{\Gamma_h}\eta\|_{L^2(e)}\leqslant\|\nabla_\Gamma\eta^l\|_{L^2(T)}\leqslant c\|\nabla_{\Gamma_h}\eta\|_{L^2(e)} \tag{5.9}$$

$$\|\nabla_{\Gamma_h}^2\eta\|_{L^2(e)}\leqslant c\|\nabla_\Gamma^2\eta^l\|_{L^2(T)}+ch\|\nabla_\Gamma\eta^l\|_{L^2(T)} \tag{5.10}$$

Proof. The proof is contained in (6). Here we only give the main ideas. In the following let d be the distance function with respect to the smooth surface Γ. By definition (see (5.2))

$$\eta(x) = \eta^l(x - d(x)\nu(x)), \quad x \in \Gamma_h.$$

The chain rule together with the definition of the tangential gradients on smooth and discrete surface gives

$$\nabla_{\Gamma_h}\eta(x) = P_h(x)\,(I - d(x)\mathscr{H}(x))\,\nabla_\Gamma \eta^l(a(x)), \quad x \in \Gamma_h,$$

where P_h and $\mathscr{H}$ are as in Lemma 5.1. The results then easily follow from the estimates of that Lemma and in particular the estimate $0 < \frac{1}{c} \leqslant \delta_h \leqslant c < \infty$. □

For later use we list interpolation inequalities which now are available. The Lemma was proved in (6) for the gradient. It is easily extended to the L^2-estimate.

LEMMA 5.3 (INTERPOLATION) For given $\eta \in H^2(\Gamma)$ there exists a unique $I_h\eta \in S_h^l$ such that

$$\|\eta - I_h\eta\|_{L^2(\Gamma)} + h\|\nabla_\Gamma(\eta - I_h\eta)\|_{L^2(\Gamma)} \leqslant ch^2\left(\|\nabla_\Gamma^2\eta\|_{L^2(\Gamma)} + h\|\nabla_\Gamma\eta\|_{L^2(\Gamma)}\right) \tag{5.11}$$

The interpolant is constructed in an obvious way. Since $\eta \in H^2(\Gamma)$, by Sobolev's embedding it is in $C^0(\Gamma)$ since the surface Γ is two dimensional. Thus the pointwise linear interpolation $\tilde{I}_h\eta \in S_h$ is well defined. The vertices of Γ_h lie on the smooth surface Γ and so the nodal values of η are well defined for this interpolation. We then lift $\tilde{I}_h\eta$ onto Γ by the process $I_h\eta = (\tilde{I}_h\eta)^l$ according to (5.2).

5.2 *Transport property of basis functions*

Each triangle $e(t)$ with vertices $X_k^e, k = 1,2,3,$ on the discrete surface can be parameterized using barycentric coordinates over the triangle $\hat{e} = \{0 \leqslant \lambda_k \leqslant 1,\ \sum_{j=1}^3 \lambda_k = 1\}$ by

$$x^e(\lambda_1, \lambda_2, \lambda_3, t) = \sum_{k=1}^3 \lambda_k X^e_{j(e,k)}(t). \tag{5.12}$$

For each $t \in [0, T_0]$ we define (moving) nodal basis functions $\{\phi_j(\cdot,t)\}_{j=1}^N$ defined on $\mathscr{N}(t)$ satisfying

$$\phi_j(\cdot,t) \in C^0(\Gamma_h(t)), \quad \phi_j(X_i(t),t) = \delta_{ij}, \quad \phi_j(\cdot,t)|_e \text{ is linear affine} \tag{5.13}$$

and on $e(t)$,

$$\phi_j|_e = \lambda_k,$$

where $k = k(e,j)$.

Clearly $\phi_j(\cdot,t) \in S_h(t)$ for each j and $\mathrm{span}\{\phi_j(\cdot,t)\} \equiv S_h(t)$. The linear independence of these nodal functions implies that for each t they form a basis of $S_h(t)$ so that for each $\phi(\cdot,t) \in S_h(t)$,

$$\phi(\cdot,t) = \sum_{j=1}^N \gamma_j(t)\phi_j(\cdot,t).$$

Observing the definition of material derivative we find that

$$\dot{\phi}_j|_e = \frac{d}{dt}\phi_j(x^e(\lambda_1,\lambda_2,\lambda_3,t),t) = \frac{d}{dt}\lambda_{k(e,j)} = 0$$

which yields the remarkable property

$$\dot{\phi}_j = 0 \ \text{ on } \ \Gamma_h(t).$$

Thus we have the following Proposition describing the transport property of the finite element functions:

PROPOSITION 5.1 (TRANSPORT PROPERTY) On $\Gamma_h(t)$, for each $j = 1,\dots,N$,

$$\dot{\phi}_j = 0$$

and for each $\phi = \sum_{j=1}^N \gamma_j(t)\phi_j \in S_h(t)$

$$\dot{\phi} = \sum_{j=1}^N \dot{\gamma}_j(t)\phi_j.$$

REMARK 5.1 Observe that the derivative $\dot{\phi}$ is defined with respect to the evolving surface on which ϕ takes its values. Noting that

$$\frac{dX_j}{dt}(t) = v(X_j(t),t) =: V_j(t)$$

and $v_h(\cdot,t)$ is the velocity of $\Gamma_h(t)$ where

$$v_h(\cdot,t) = \sum_{j=1}^N V_j(t)\phi_j(\cdot,t) \tag{5.14}$$

we may write the transport property of these finite element basis functions in the interior of $e(t)$ as

$$\phi_{j,t} + v_h \cdot \nabla\phi_j = 0.$$

Here we use the lift extension of a finite element function on the discrete surface which is constant in a direction normal to the underlying smooth surface.

5.3 *Semi-discrete approximation*

Our ESFEM is based on the evolving finite element spaces introduced in this section and the variational form (1.2) of the diffusion equation.

Definition 5.1 (Semi-discretization in Space) Find $U(\cdot,t) \in S_h(t)$ such that

$$\frac{d}{dt}\int_{\Gamma_h(t)} U\phi + \int_{\Gamma_h(t)} \mathscr{D}_0^{-l}\nabla_{\Gamma_h} U \cdot \nabla_{\Gamma_h}\phi = \int_{\Gamma_h(t)} U\dot{\phi} \quad \forall \phi \in S_h(t). \tag{5.15}$$

Here $\mathscr{D}_0^{-l}$ is such that its lift is $\mathscr{D}_0$ so that $\left(\mathscr{D}_0^{-l}\right)^l = \mathscr{D}_0$.

Using the Leibniz formula for the evolving triangulated surface $\Gamma_h(t)$, it is easily seen that an equivalent formulation is:

$$\int_{\Gamma_h(t)} \dot{U}\phi + \int_{\Gamma_h(t)} U\phi\nabla_{\Gamma_h}\cdot v_h + \int_{\Gamma_h(t)} \mathscr{D}_0^{-l}\nabla_{\Gamma_h} U \cdot \nabla_{\Gamma}\phi = 0 \quad \forall \phi \in S_h(t). \tag{5.16}$$

Setting

$$U(\cdot,t) = \sum_{j=1}^N \alpha_j(t)\phi_j(\cdot,t)$$

and using Proposition 5.1 we find that

$$\int_{\Gamma_h(t)} \sum_{j=1}^{N} \alpha_{j,t}\phi_j\phi + \int_{\Gamma_h(t)} \sum_{j=1}^{N} \alpha_j\phi_j\phi\nabla_{\Gamma_h}\cdot v_h + \int_{\Gamma_h(t)} \mathscr{D}_0^{-l} \sum_{j=1}^{N} \alpha_j(t)\nabla_{\Gamma_h}\phi_j\cdot\nabla_\Gamma\phi = 0 \quad \forall\phi\in S_h(t)$$

and taking $\phi = \phi_i, i = 1,\ldots,N$ we obtain

$$\mathscr{M}(t)\dot{\alpha} + \tilde{\mathscr{M}}(t)\alpha + \mathscr{S}(t)\alpha = 0 \tag{5.17}$$

where $\mathscr{M}(t)$ is the evolving mass matrix

$$\mathscr{M}(t)_{jk} = \int_{\Gamma_h(t)} \phi_j\phi_k,$$

$\tilde{\mathscr{M}}(t)$ is a mass matrix weighted by the surface divergence of the velocity,

$$\tilde{\mathscr{M}}(t)_{jk} = \int_{\Gamma_h(t)} \phi_j\phi_k\nabla_{\Gamma_h(t)}\cdot v_h$$

and $\mathscr{S}(t)$ is the evolving stiffness matrix

$$\mathscr{S}(t)_{jk} = \int_{\Gamma_h(t)} \mathscr{D}_0^{-l}\nabla_{\Gamma_h}\phi_j\nabla_{\Gamma_h}\phi_k.$$

A consequence of the fact that the covariant derivatives of the evolving basis functions vanish is:

PROPOSITION 5.2

$$\frac{d\mathscr{M}}{dt} = \tilde{\mathscr{M}} \tag{5.18}$$

Proof. A simple application of the Leibniz formula yields

$$\frac{d}{dt}\int_{\Gamma_h(t)} \phi_j\phi_k = \int_{\Gamma_h(t)} \left(\dot{\phi}_j\phi_k + \phi_j\dot{\phi}_k + \phi_j\phi_k\nabla_{\Gamma_h(t)}\cdot v_h\right)$$

and since $\dot{\phi}_j \equiv 0$ we have the result. □

Thus we arrive at a simpler version of the finite element approximation which does not explicitly involve the velocity of the surface. The system

$$\frac{d}{dt}(\mathscr{M}(t)\alpha) + \mathscr{S}(t)\alpha = 0. \tag{5.19}$$

is equivalent to equation (5.15). Since the mass matrix $\mathscr{M}(t)$ is uniformly positive definite on $[0,T_0]$ and the stiffness matrix $\mathscr{S}(t)$ is positive semi-definite, we get existence and uniqueness of the semi-discrete finite element solution.

REMARK 5.2 A significant feature of our approach is the fact that the matrices $\mathscr{M}(t)$ and $\mathscr{S}(t)$ depend only on the knowledge of the position of the nodes on the discrete surface. The computational method does not require a numerical evaluation of the normal or curvature.

REMARK 5.3 The numerical approximation can be directly applied to surfaces having a boundary.

6. Error bounds

In this section we will prove a convergence result. To start with we prove the basic stability results. They are similar to the energy estimates in Theorem 4.2.

LEMMA 6.1 (STABILITY) Let U be a solution of the semi-discrete scheme as in Definition 5.1 with initial value $U(\cdot,0) = U_0$ and U^l its lift according to (5.2). Then the following stability estimates hold:

$$\sup_{(0,T_0)} \|U^l\|^2_{L^2(\Gamma)} + \int_0^{T_0} \|\nabla_\Gamma U^l\|^2_{L^2(\Gamma)} \leqslant c\|U^l_0\|^2_{L^2(\Gamma_0)}, \tag{6.1}$$

$$\int_0^{T_0} \|\dot{U}^l\|^2_{L^2(\Gamma)} + \sup_{(0,T_0)} \|\nabla_\Gamma U^l\|^2_{L^2(\Gamma)} \leqslant c\|U^l_0\|^2_{H^1(\Gamma_0)}. \tag{6.2}$$

Proof. The estimates for U follow from the Leibniz formulas in Lemmas 4.1 and 4.2 in the same way as this was done for the continuous equation. We then lift U to the continuous surface and use the estimates of Lemma 5.2 together with (6.6). For the last argument we refer to the proof of the following Theorem. □

Theorem 6.1 (Convergence) Let u be a sufficiently smooth solution of (4.1) and let U be the discrete solution from Definition 5.1. With the lift U^l of U we then have the following error estimate:

$$\sup_{t\in(0,T_0)} \|u(\cdot,t) - U^l(\cdot,t)\|^2_{L^2(\Gamma(t))} + \int_0^{T_0} \|\nabla_\Gamma(u(\cdot,t) - U^l(\cdot,t))\|^2_{L^2(\Gamma(t))}dt$$

$$\leqslant ch^2\|u_0\|^2_{H^2(\Gamma_0)} + ch^4 \sup_{s\in(0,T_0)} \|u(\cdot,s)\|^2_{H^2(\Gamma(s))} + ch^6\int_0^{T_0} \|\dot{u}(\cdot,s)\|^2_{H^2(\Gamma(s))}ds \tag{6.3}$$

Proof. The error bounds rely on a suitable form of the error equation. In order to compare discrete and continuous solution both should be defined on the same surface which we take to be the continuous surface $\Gamma(t)$. The continuous equation reads

$$\frac{d}{dt}\int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \mathscr{D}_0\nabla_\Gamma u\cdot\nabla_\Gamma\varphi = \int_{\Gamma(t)} u\dot{\varphi} \quad \forall\varphi\in H^1(\Gamma(t)), \tag{6.4}$$

and the discrete equation is given by

$$\frac{d}{dt}\int_{\Gamma_h(t)} U\phi + \int_{\Gamma_h(t)} \mathscr{D}_0^{-l}\nabla_{\Gamma_h} U\cdot\nabla_{\Gamma_h}\phi = \int_{\Gamma_h(t)} U\dot{\phi} \quad \forall\phi\in S_h(t). \tag{6.5}$$

Here again $\mathscr{D}_0^{-l}$ is defined such that the lift of $\mathscr{D}_0^{-l}$ is $\mathscr{D}_0$. We lift the discrete equation to the continuous surface as it was described in the previous section. We define U^l and ϕ^l by

$$U(x,t) = U^l(a(x,t),t), \quad \phi(x,t) = \phi^l(a(x,t),t), \quad x\in\Gamma_h(t).$$

The transformation of the material derivative of ϕ with respect to the discrete surface Γ_h is done as follows. For $x\in\Gamma_h(t)$ we have

$$\begin{aligned}\dot{\phi}(x,t) &= \phi_t(x,t) + v_h(x,t)\cdot\nabla\phi(x,t)\\ &= \dot{\phi}^l(a(x,t),t) + \big((P(I-d\mathscr{H})v_h - d_t\nu - d\nu_t)(x,t) - v(a(x,t),t)\big)\cdot\nabla\phi^l(a(x,t),t).\end{aligned}$$

By definition $\nabla\phi^l(a(x,t),t)\cdot\nu(x,t) = \nabla\phi^l(a(x,t),t)\cdot\nu(a(x,t),t) = 0$ and so $\nabla\phi^l = \nabla_\Gamma\phi^l$. With the use of the estimate (5.4) this leads to

$$\dot\phi(x,t) = \dot\phi^l(a(x,t),t) + (v_h(x,t) - v(a(x,t),t))\cdot\nabla_\Gamma\phi^l(a(x,t),t) + O\left(h^2|\nabla_\Gamma\phi^l(a(x,t),t)|\right).$$

Here we also have used that v_h is bounded independently of h. Since v_h is the interpolant of v (see (5.14)), we have that $|v_h(x,t) - v(a(x,t),t)| \leqslant ch^2$, and we arrive at

$$\dot\phi(x,t) = \dot\phi^l(a(x,t),t) + O\left(h^2|\nabla_\Gamma\phi^l(a(x,t),t)|\right), \quad x\in\Gamma_h(t). \tag{6.6}$$

For better understanding of the following, we introduce the notation

$$u_h(x,t) = U^l(x,t), \quad x\in\Gamma(t)$$

and the abbreviation

$$R_h(x,t) = \frac{1}{\delta_h(x,t)}(\mathscr{D}_0^{-l}(x,t))^{-1}P(x,t)(I - d(x,t)\mathscr{H}(x,t))P_h(x,t)\mathscr{D}_0^{-l}(x,t)P_h(x,t)(I - d(x,t)\mathscr{H}(x,t))$$

($x\in\Gamma_h(t)$), and its lifted version for $R_h^l(a(x,t),t) = R_h(x,t)$, $x\in\Gamma_h(t)$. Lemma 5.1 holds with a minor modification in the proof for the case that $\mathscr{D}_0$ is not the identity on the tangent space by observing the identity

$$(\mathscr{D}_0^{-l})^{-1}PP_h\mathscr{D}_0^{-l}P_hP - P = (\mathscr{D}_0^{-l})^{-1}\left(PP_hP - P\right)\mathscr{D}_0^{-l}PP_hP + PP_hP - P.$$

Then (6.5) and

$$\begin{aligned}
\mathscr{D}_0^{-l}\nabla_{\Gamma_h}U\cdot\nabla_{\Gamma_h}\phi &= \mathscr{D}_0^{-l}P_h(P - d\mathscr{H})\nabla u_h(a,\cdot)\cdot P_h(P - d\mathscr{H})\nabla\varphi_h(a,\cdot)\\
&= \mathscr{D}_0^{-l}P_hP(I - d\mathscr{H})\nabla_\Gamma u_h(a,\cdot)\cdot P_hP(I - d\mathscr{H})\nabla_\Gamma\varphi_h(a,\cdot)\\
&= \mathscr{D}_0^{-l}R_h\nabla_\Gamma u_h(a,\cdot)\cdot\nabla_\Gamma\varphi_h(a,\cdot)
\end{aligned}$$

on $\Gamma_h(t)$, together with the estimate (6.6) lead to the inequality

$$\frac{d}{dt}\int_{\Gamma(t)} u_h\varphi_h\frac{1}{\delta_h^l} + \int_{\Gamma(t)}\mathscr{D}_0R_h^l\nabla_\Gamma u_h\cdot\nabla_\Gamma\varphi_h \geqslant \int_{\Gamma(t)} u_h\dot\varphi_h\frac{1}{\delta_h^l} - ch^2\int_{\Gamma(t)}|u_h|\,|\nabla_\Gamma\varphi_h|\frac{1}{\delta_h^l} \tag{6.7}$$

for all $\varphi_h\in S_h^l(t)$. We take the difference of equation (6.4) at φ_h and equation (6.7). The error relation between continuous and lifted discrete solution then is given by

$$\begin{aligned}
&\frac{d}{dt}\int_{\Gamma(t)}(u - \frac{1}{\delta_h^l}u_h)\varphi_h + \int_{\Gamma(t)}\mathscr{D}_0\left(\nabla_\Gamma u - R_h^l\nabla_\Gamma u_h\right)\cdot\nabla_\Gamma\varphi_h\\
&\leqslant \int_{\Gamma(t)}(u - \frac{1}{\delta_h^l}u_h)\dot\varphi_h + ch^2\int_{\Gamma(t)}|u_h|\,|\nabla_\Gamma\varphi_h|\frac{1}{\delta_h^l} \quad \forall\varphi_h\in S_h^l(t),
\end{aligned} \tag{6.8}$$

or, written in a more convenient form:

$$\begin{aligned}
&\frac{d}{dt}\int_{\Gamma(t)}(u - u_h)\varphi_h + \int_{\Gamma(t)}\mathscr{D}_0\nabla_\Gamma(u - u_h)\cdot\nabla_\Gamma\varphi_h\\
&\leqslant \int_{\Gamma(t)}\mathscr{D}_0\left(R_h^l - I\right)\nabla_\Gamma u_h\cdot\nabla_\Gamma\varphi_h + \frac{d}{dt}\int_{\Gamma(t)}(\frac{1}{\delta_h^l} - 1)u_h\varphi_h - \int_{\Gamma(t)}(\frac{1}{\delta_h^l} - 1)u_h\dot\varphi_h\\
&\quad + \int_{\Gamma(t)}(u - u_h)\dot\varphi_h + ch^2\int_{\Gamma(t)}|u_h|\,|\nabla_\Gamma\varphi_h|\frac{1}{\delta_h^l} \quad \forall\varphi_h\in S_h^l(t).
\end{aligned} \tag{6.9}$$

This implies

$$\begin{aligned}
&\frac{d}{dt}\int_{\Gamma(t)}(u-u_h)^2+d_0\int_{\Gamma(t)}|\nabla_\Gamma(u-u_h)|^2-\int_{\Gamma(t)}(u-u_h)(\dot u-\dot u_h) \qquad (6.10)\\
&\quad\leqslant \frac{d}{dt}\int_{\Gamma(t)}(u-u_h)(u-\varphi_h)+\int_{\Gamma(t)}\mathscr{D}_0\nabla_\Gamma(u-u_h)\cdot\nabla_\Gamma(u-\varphi_h)\\
&\quad -\int_{\Gamma(t)}(u-u_h)(\dot u-\dot\varphi_h)\\
&\quad +\int_{\Gamma(t)}\mathscr{D}_0\left(R_h^l-I\right)\nabla_\Gamma u_h\cdot\nabla_\Gamma(\varphi_h-u_h)+\frac{d}{dt}\int_{\Gamma(t)}(\frac{1}{\delta_h^l}-1)u_h(\varphi_h-u_h)\\
&\quad -\int_{\Gamma(t)}(\frac{1}{\delta_h^l}-1)u_h(\dot\varphi_h-\dot u_h)+ch^2\int_{\Gamma(t)}|u_h||\nabla_\Gamma(\varphi_h-u_h)|\frac{1}{\delta_h^l}\quad\forall\varphi_h\in S_h^l(t).
\end{aligned}$$

We now observe that from (2.9)

$$\begin{aligned}
&\frac{d}{dt}\int_{\Gamma(t)}(u-u_h)^2-\int_{\Gamma(t)}(u-u_h)(\dot u-\dot u_h)\\
&\quad = \quad \frac{1}{2}\frac{d}{dt}\int_{\Gamma(t)}(u-u_h)^2+\frac{1}{2}\int_{\Gamma(t)}(u-u_h)^2\nabla_\Gamma\cdot v\\
&\quad \geqslant \quad \frac{1}{2}\frac{d}{dt}\int_{\Gamma(t)}(u-u_h)^2-c\int_{\Gamma(t)}(u-u_h)^2,
\end{aligned}$$

and similarly

$$\frac{d}{dt}\int_{\Gamma(t)}(u-u_h)(u-\varphi_h)-\int_{\Gamma(t)}(u-u_h)(\dot u-\dot\varphi_h)\leqslant c\int_{\Gamma(t)}\left(|u-u_h|+|\dot u-\dot u_h|\right)|u-\varphi_h|.$$

Thus for any $\varphi_h\in S_h^l(t)$ we have from (6.10) using the geometry estimates from Lemma 5.1 together with the fact that $(R_h^l-I)\nabla_\Gamma=(R_h^l-I)P\nabla_\Gamma$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Gamma(t)}(u-u_h)^2+d_0\int_{\Gamma(t)}|\nabla_\Gamma(u-u_h)|^2\\
&\quad\leqslant c\int_{\Gamma(t)}(u-u_h)^2+c\int_{\Gamma(t)}\left(|u-u_h|+|\dot u-\dot u_h|\right)|u-\varphi_h|\\
&\quad +c\int_{\Gamma(t)}|\nabla_\Gamma(u-u_h)|\,|\nabla_\Gamma(u-\varphi_h)|\\
&\quad +ch^2\int_{\Gamma(t)}\left(|u_h|+|\nabla_\Gamma u_h|\right)|\nabla_\Gamma(u-u_h)|+ch^2\int_{\Gamma(t)}\left(|u_h|+|\nabla_\Gamma u_h|\right)|\nabla_\Gamma(u-\varphi_h)|\\
&\quad +\frac{d}{dt}\int_{\Gamma(t)}(\frac{1}{\delta_h^l}-1)u_h(u-u_h)-\frac{d}{dt}\int_{\Gamma(t)}(\frac{1}{\delta_h^l}-1)u_h(u-\varphi_h)\\
&\quad +ch^2\int_{\Gamma(t)}|u_h|\,|\dot u-\dot u_h|+ch^2\int_{\Gamma(t)}|u_h|\,|\dot u-\dot\varphi_h|.
\end{aligned}$$

Standard applications of the Cauchy-Schwarz and Young inequalities then lead to the estimate

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Gamma(t)}(u-u_h)^2+\frac{d_0}{2}\int_{\Gamma(t)}|\nabla_\Gamma(u-u_h)|^2 \\
&\quad\leqslant c\int_{\Gamma(t)}|\nabla_\Gamma(u-\varphi_h)|^2-\frac{d}{dt}\int_{\Gamma(t)}(\frac{1}{\delta_h^l}-1)u_h(u-\varphi_h)+c\int_{\Gamma(t)}(u-u_h)^2 \\
&\quad+\frac{d}{dt}\int_{\Gamma(t)}(\frac{1}{\delta_h^l}-1)u_h(u-u_h)+ch^2\|u_h\|_{L^2(\Gamma(t))}\|\dot u-\dot u_h\|_{L^2(\Gamma(t))}+c\int_{\Gamma(t)}(u-\varphi_h)^2 \\
&\quad+\|\dot u-\dot u_h\|_{L^2(\Gamma(t))}\|u-\varphi_h\|_{L^2(\Gamma(t))}+ch^2\|u_h\|_{L^2(\Gamma(t))}\|\dot u-\dot\varphi_h\|_{L^2(\Gamma(t))}+ch^4\|u_h\|^2_{H^1(\Gamma(t))}.
\end{aligned}
$$

Integrating with respect to time, we arrive at the following estimate, with $u_{h0}=u_h(\cdot,0)\in S_h^l(0)$, using the estimate (5.5) at several places:

$$
\begin{aligned}
&\|u-u_h\|^2_{L^2(\Gamma)}+d_0\int_0^t\|\nabla_\Gamma(u-u_h)\|^2_{L^2(\Gamma)}dt \\
&\quad\leqslant\|u_0-u_{h0}\|^2_{L^2(\Gamma_0)}+c\int_0^t\|\nabla_\Gamma(u-\varphi_h)\|^2_{L^2(\Gamma)}dt+c\int_0^t\|u-\varphi_h\|^2_{L^2(\Gamma)}dt \\
&\quad+ch^2\|u_h\|_{L^2(\Gamma)}\|u-\varphi_h\|_{L^2(\Gamma)}+ch^2\|u_{h0}\|_{L^2(\Gamma_0)}\|u_0-\varphi_h(\cdot,0)\|_{L^2(\Gamma_0)} \\
&\quad+c\int_0^t\|u-u_h\|^2_{L^2(\Gamma)}dt+ch^2\|u_h\|_{L^2(\Gamma)}\|u-u_h\|_{L^2(\Gamma)} \\
&\quad+ch^2\|u_{h0}\|_{L^2(\Gamma_0)}\|u_0-u_{h0}\|_{L^2(\Gamma_0)}+ch^2\int_0^t\|u_h\|_{L^2(\Gamma)}\|\dot u-\dot u_h\|_{L^2(\Gamma)}dt \\
&\quad+c\int_0^t\|\dot u-\dot u_h\|_{L^2(\Gamma)}\|u-\varphi_h\|_{L^2(\Gamma)}dt+ch^2\int_0^t\|\dot u-\dot\varphi_h\|^2_{L^2(\Gamma)}dt+ch^4\int_0^t\|u_h\|^2_{H^1(\Gamma)}dt.
\end{aligned}
$$

We use the stability estimates from Lemma 6.1 for $u_h=U^l$ and use the interpolation estimates from Lemma 5.3 for $u_{h0}=I_hu_0$, $\varphi_h=I_hu$. This gives us the estimate

$$
\begin{aligned}
&\|u-u_h\|^2_{L^2(\Gamma)}+\int_0^t\|\nabla_\Gamma(u-u_h)\|^2_{L^2(\Gamma)}dt \\
&\quad\leqslant c\int_0^t\|u-u_h\|^2_{L^2(\Gamma)}dt+ch^2\|u_0\|^2_{H^2(\Gamma_0)}+ch^4\|u\|^2_{H^2(\Gamma)} \\
&\quad+ch^2\int_0^t\|\dot u-\dot u_h\|^2_{L^2(\Gamma)}dt+ch^2\int_0^t\|\dot u-\big(I_hu\big)^{\cdot}\|^2_{L^2(\Gamma)}dt
\end{aligned}
\tag{6.11}
$$

For the last two terms on the right hand side of this inequality we observe that from Theorem 4.2 and Lemma 6.1

$$
\int_0^t\|\dot u-\dot u_h\|^2_{L^2(\Gamma(s))}ds\leqslant c\|u_0\|^2_{H^1(\Gamma_0)}+c\|u_{h0}\|^2_{H^1(\Gamma_0)}\leqslant c\|u_0\|^2_{H^2(\Gamma_0)}
$$

and with (6.6) we get

$$
\int_0^t\|\dot u-\big(I_hu\big)^{\cdot}\|^2_{L^2(\Gamma)}dt\leqslant ch^4\int_0^t\|\dot u\|^2_{H^2(\Gamma)}+\|u\|^2_{H^2(\Gamma)}dt.
$$

A Gronwall argument then leads to the final estimate

$$\|u(\cdot,t)-u_h(\cdot,t)\|^2_{L^2(\Gamma(t))}+\int_0^t \|\nabla_\Gamma(u(\cdot,s)-u_h(\cdot,s))\|^2_{L^2(\Gamma(s))}ds$$
$$\leqslant ch^2\|u_0\|^2_{H^2(\Gamma_0)}+ch^4\sup_{s\in(0,T_0)}\|u(\cdot,s)\|^2_{H^2(\Gamma(s))}+ch^6\int_0^{T_0}\|\dot{u}(\cdot,s)\|^2_{H^2(\Gamma(s))}ds$$

for $t\in(0,T_0)$ and the Theorem is proved. □

7. Implementation and numerical results

7.1 *Implicit Euler scheme*

The time discretization in our computations is done by an implicit method. We discretize the variational form (1.2) in time. The spatially discrete problem is

$$\frac{d}{dt}\int_{\Gamma_h(t)} U\phi+\int_{\Gamma_h(t)}\mathscr{D}_0^{-l}\nabla_{\Gamma_h}U\cdot\nabla_\Gamma\phi=\int_{\Gamma_h(t)}U\dot{\phi}\quad\forall\phi\in S_h(t). \tag{7.1}$$

We introduce a time step size $\tau>0$ and use upper indices for the time levels. Thus U^m represents $U(\cdot,m\tau)$ and $\Gamma^m=\Gamma(m\tau)$. With these notations we propose the following Algorithm.

ALGORITHM 7.1 (FULLY DISCRETE SCHEME) Let $U^0\in S_h(0)$ be given. For $m=0,\dots,m_{T_0}$ solve the linear system

$$\frac{1}{\tau}\int_{\Gamma_h^{m+1}}U^{m+1}\varphi_j^{m+1}+\int_{\Gamma_h^{m+1}}\mathscr{D}_0^{-l,m+1}\nabla_{\Gamma_h^{m+1}}U^{m+1}\cdot\nabla_{\Gamma_h^{m+1}}\varphi_j^{m+1}=\frac{1}{\tau}\int_{\Gamma_h^m}U^m\varphi_j^m \tag{7.2}$$
$$j=1,\dots,N.$$

7.2 *Implementation*

A typical finite element program sets up stiffness matrix, mass matrix and right hand side of the linear system (7.2) within a loop over all triangles (elements). Let us describe how in our algorithm the stiffness matrix setup is implemented. On each triangle $e=\mathrm{conv}\{X_1,X_2,X_3\}$ with vertices $X_k\in\mathbb{R}^3$ the element stiffness matrix

$$\mathscr{S}^e_{ij}=\int_e\nabla_e\phi^e_i\cdot\nabla_e\phi^e_j,\quad i,j=1,2,3$$

with local basis functions ϕ^e_j, $j=1,2,3$, is computed and then is summed to the correct globally numbered places of the matrix $\mathscr{S}$. Here $\nabla_e=\nabla_{\Gamma_h}$ is the tangential gradient on the triangle $e\subset\Gamma_h$. Obviously the tangential gradient on a plane is a cartesian gradient. The triangle e can be parametrized over the unit triangle $\hat{e}\subset\mathbb{R}^2$ as in (5.12). Then the usual transformation matrices for the map between $\hat{e}$ and e is used to compute the element stiffness matrix. The area of the triangle is trivially given by elementary geometry.

A drawback of our method is the possibility of degenerating grids. The prescribed velocity may lead to the effect, that the triangulation $\Gamma_h(t)$ is distorted and that the solver for the linear system does not converge. In all our computational examples this problem did not occur. But, of course, in general situations this problem may appear. A remedy then is to retriangulate the surface by some method, preferably this is done by conformally reparametrizing the surface and mapping a nice grid onto the surface.

It is straightforward to handle both Dirichlet and Neumann boundary conditions when $\partial\Gamma(t)$ is non-empty. Some examples are included in the numerical results.

7.3 *Numerical tests*

Example 7.1 To start with we solve the heat equation on the unit sphere. Here the surface does not move. This example shows that our method also produces a finite element method for parabolic PDEs on surfaces which do not move. The function $u(x,t) = e^{-6t}x_1x_2$ is an exact solution of

$$u_t - \Delta_\Gamma u = 0$$

on $\Gamma(t) = \Gamma_0 = S^2$ with initial data $u_0(x) = x_1x_2$. We have chosen the coupling $\tau = h^2$ in order to show the higher order convergence for L^2 and L^∞ errors. The time interval is $T_0 = 2.0$. In Table 1 we show the absolute errors and the corresponding experimental orders of convergence for the norms

$$L^\infty(L^\infty) = \sup_{(0,T_0)} \|u - U^l\|_{L^2(\Gamma)}, \quad L^\infty(L^2) = \sup_{(0,T_0)} \|u - U^l\|_{L^2(\Gamma)},$$

$$L^2(H^1) = \left(\int_0^{T_0} \|\nabla_\Gamma(u - U^l)\|^2_{L^2(\Gamma)}\right)^{\frac{1}{2}}.$$

For an error $E(h_1)$ and $E(h_2)$ for the grid sizes h_1 and h_2 the experimental order of convergence is defined as $\mathrm{eoc}(h_1,h_2) = \log\frac{E(h_1)}{E(h_2)}\left(\log\frac{h_1}{h_2}\right)^{-1}$.

h	$L^\infty(L^\infty)$	eoc	$L^\infty(L^2)$	eoc	$L^2(H^1)$	eoc
1.	0.088590	-	0.12023	-	0.24265	-
0.55745	0.089525	-0.02	0.14399	-0.31	0.22904	0.10
0.28664	0.036723	1.34	0.060878	1.29	0.10258	1.21
0.14433	0.010891	1.77	0.018351	1.75	0.040083	1.37
0.072293	0.0028831	1.92	0.0048303	1.93	0.017503	1.20
0.036162	0.00073909	1.97	0.0012250	1.98	0.0083646	1.07

Table 1. Heat equation on the sphere. Errors and experimental orders of convergence for Example 7.1.

Example 7.2 The second computational example is a PDE on a moving surface with time dependent curvature. The surface is given by the level set function

$$d(x,t) = \frac{x_1^2}{a(t)} + x_2^2 + x_3^2 - 1, \tag{7.1}$$

so that the moving surface $\Gamma(t) = \{x \in \mathbb{R}^3 \,|\, d(x,t) = 0\}$ is an ellipsoid with time dependent axis. We have chosen $a(t) = 1 + 0.25\sin(t)$. As exact continuous solution we choose $u(x,t) = e^{-6t}x_1x_2$ and compute a right hand side for the PDE from the equation

$$f = u_t + v \cdot \nabla u + u\nabla_\Gamma \cdot v - \Delta_\Gamma u.$$

The time step size was taken to be the square of the initial maximal grid diameter. The time interval was $[0,4]$. In Table 2 we show the error in three norms together with the experimental orders of convergence. The grid size h in this example depends on time. We compute the eocs with the use of the grid size at the final time $T_0 = 4$.

$h(T_0)$	$L^\infty(L^\infty)$	eoc	$L^\infty(L^2)$	eoc	$L^2(H^1)$	eoc
0.82737	0.095488	-	0.15424	-	0.29287	-
0.43422	0.057944	0.77	0.097788	0.71	0.17507	0.80
0.21939	0.018764	1.65	0.033083	1.59	0.074327	1.26
0.10994	0.0050819	1.89	0.0089784	1.89	0.033367	1.16
0.055007	0.0013038	1.97	0.0022950	1.97	0.016053	1.06

Table 2. Errors and experimental orders of convergence for Example 7.2.

Example 7.3 We compute solutions on a rotating planar disk

$$\Gamma(t) = \{(\cos(100t)x_1 - \sin(100t)x_2, \sin(100t)x_1 + \cos(100t)x_2, 0) \,|\, x \in \Gamma_0\}$$

where $\Gamma_0 = \{x \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 \leqslant 1, x_3 = 0\}$. Here $\mathscr{D}_0 = 0.1\mathscr{I}$ and $f(x_1,x_2,0) = 100.0$ where $(x_1-0.5)^2 + x_2^2 \leqslant 0.01$, $f(x_1,x_2,0) = 0.0$ where $(x_1-0.5)^2 + x_2^2 > 0.02$ and f smooth elsewhere. We have used homogeneous Dirichlet boundary conditions on $\partial\Gamma$, and as initial value we have taken $u_0 = 0$. The time step size was $\tau = 0.00016$ and the triangulation had 16384 triangles. In Figure 2 we show some time steps of the computations. In order to show that large time steps (experimentally) are allowed we computed the same example with a time step $\tau = 0.01$. The results for the first three time steps are shown in Figure 3. They show the Lagrangian property of our algorithm. Note the large velocity of the rotation of the disk.

We computed the solution of (7.4) with $\mathscr{D}_0 = 0.1\mathscr{I}$ on a graph $\Gamma(t)$ above the unit disk, which vertically moves according to the parametrization

$$x(\theta,t) = \left(\theta_1, \theta_2, \frac{1}{2}\left(1 + te^{-t}\right)\sin(2\pi\theta_1)\sin(3\pi\theta_2)\right) \qquad \theta = (\theta_1,\theta_2) \in B_1(0). \tag{7.2}$$

The initial value was $u_0 = 0$ and the right hand side was a smoothened characteristic function $f = 100\chi_D$ with $D = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \,|\, (x_1-0.5)^2 + x_2^2 \leqslant 0.1^2\}$. The high curvatures and the velocity of the surface lead to transport and diffusion shown in Figure 4.

Example 7.4 In Figures 5 and 6 we show the results of a computation on a rotating sphere $\Gamma(t)$ with $\Gamma_0 = S^2$. The parametrization of $\Gamma(t)$ is given by

$$x(\theta,t) = \left(\cos(\eta t)\theta_1 - \sin(\eta t)\theta_2, \cos(\eta t)\theta_1 + \sin(\eta t)\theta_2, \theta_3\right), \quad \theta \in S^2 \tag{7.3}$$

with $\eta = 25$. For the initial data $u_0(x) = 0$ and the right hand side $f(x,t) = 100\chi_{B_R(x_0)\cap\Gamma(t)}(x)$ with $x_0 = (0,1,0)$, $R = 0.25$ we solve

$$\dot{u} + u\nabla_\Gamma \cdot v - \nabla_\Gamma \cdot (\mathscr{D}_0 \nabla_\Gamma u) = f \tag{7.4}$$

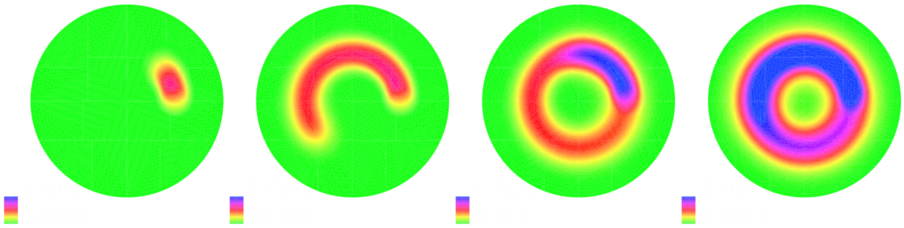

FIG. 2. Level set picture of the solution $u(\cdot,t)$ of Example 7.3 at times $t = 0.008, 0.040, 0.08$ and 0.16. Time step size $\tau = 0.00016$.

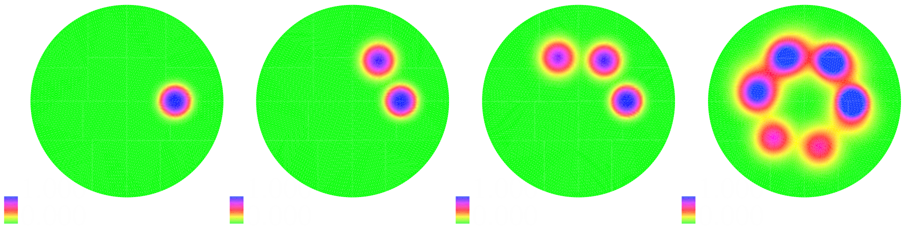

FIG. 3. Level set picture of the solution $u(\cdot,t)$ of Example 7.3 for large time step size $\tau = 0.01$ at the time steps $1, 2, 3$ and 10. Lagrangian property of the scheme.

on $\Gamma(t)$ with $\mathscr{D}_0 = \mathscr{I}$. In Figure 5 we show the levels of the solution at times $t = 0.005213, t = 0.5213$ and $t = 1.5639$ (slightly tilted). We used a grid with 8194 vertices. Note that the shading in each time step is done for an equal distribution between maximum and minimum of the discrete solution. In Figure 6 we show level lines of the stationary solution seen from the x_1 axis. Left figure shows the level lines on the front side of the sphere and the right figure shows the level lines on the back side of the sphere.

Example 7.5 Figure 8 shows computational results for a rotating cylinder and for small diffusivity. Here,

$$\Gamma_0 = \{x \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 = 1, 0 \leqslant x_3 \leqslant 1\},$$

and this cylinder is rotated according to (7.3) with $\eta = 5$. As initial function we choose $u_0(\cos\varphi, \sin\varphi) = \chi_{\{|\varphi| \leqslant 0.01\}}$. We imposed boundary conditions $u = u_0$ on $\partial\Gamma(t)$. The boundary conditions are independent of time t. As diffusivity we have chosen the relatively small number $\mathscr{D}_0 = 0.1\mathscr{I}$. We used the triangulation in Figure 7 with 3200 vertices and 6144 triangles and the time step size was $\tau = 0.1h$.

Example 7.6 Figure 9 shows the solution of (1.1) on a rotating cylindrical surface with Dirichlet boundary conditions $u(x,t) = u_0(x)$ for $x \in \partial\Gamma(t)$. Thus mass concentration is kept fixed during the evolution on the boundary of the surface. u_0 is a smoothened version of the function $100\chi_{B_{0.25}}((1,0,1))$. The surface is a deformed cylinder - not a catenoid. So, its mean curvature does not vanish identically. The surface is rotated according to (7.3) with $\eta = 10$.

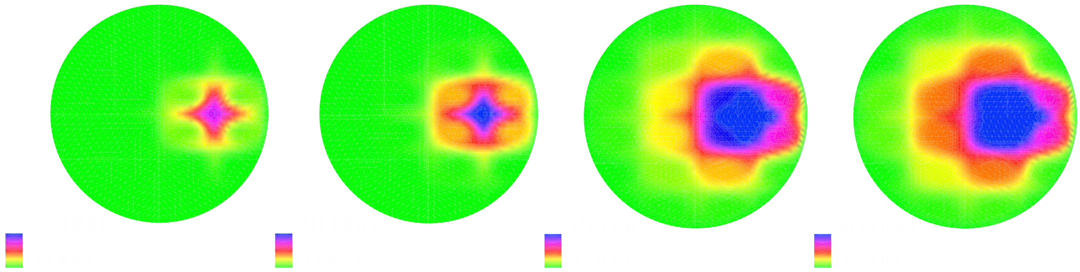

FIG. 4. Transport and diffusion on a vertically moving graph, see (7.2). Times $t = 0.795, 1.59, 6.36$ and 15.9.

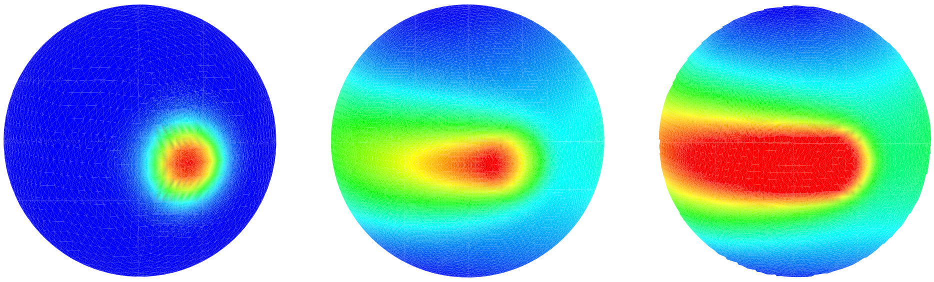

FIG. 5. Diffusion and transport on a rotating sphere. Example 7.4.

Example 7.7 Our ESFEM allows the solution on surfaces with strongly varying principal curvatures. As a test for this we have chosen the surface Figure 10. It represents a buckley initial surface which is evolved into part of a sphere of radius 4 as the time tends to infinity. Figure 11 shows some time steps of the solution for problem (7.4) with a right hand side $f = 1$ and diffusion coefficient $\mathscr{D}_0 = 0.1\mathscr{I}$. We have used Neumann boundary conditions. The initial function u_0 was taken to depend on random numbers.

8. Concluding remarks

The approach described here is directly applicable to other boundary conditions when $\partial\Gamma(t)$ is non-empty such as the non-homogeneous Dirichlet condition

$$u = g \text{ on } \partial\Gamma(t).$$

or Neumann boundary condition

$$\nabla_\Gamma u \cdot \mu = g \text{ on } \partial\Gamma(t).$$

The method is directly applicable to a system in which there is mass accumulation and deposition onto the surface from outside such as

$$\dot{u} + u\nabla_\Gamma \cdot v - \nabla_\Gamma \cdot (\mathscr{D}_0 \nabla_\Gamma u) = V u_a + f$$

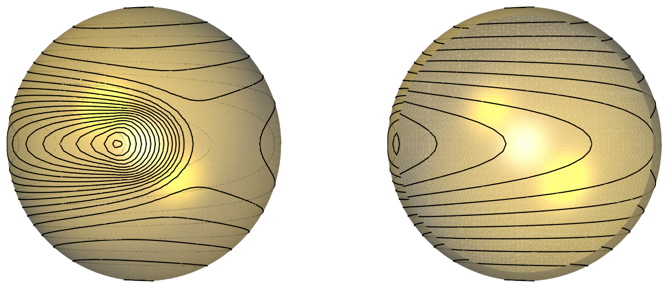

FIG. 6. Level lines of the stationary solution of Example 7.4. Front side of the sphere (left) and backside (right).

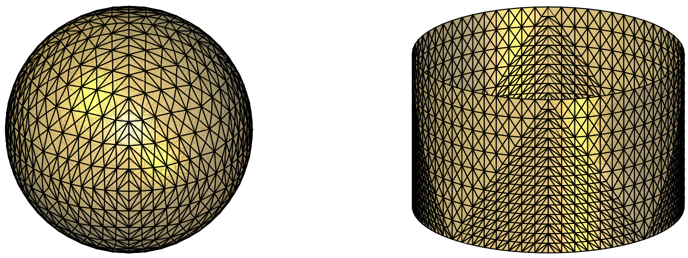

FIG. 7. Triangulation of the sphere (Example 7.4) and of the cylinder (Example 7.5).

where u_a is an ambient external concentration and f is a prescribed deposition rate.

The method could be developed to apply to a coupling with field equations away from the surface such as the Navier-Stokes equations with surfactant transport on the interface between two immiscible fluids.

The methodology is applicable to more general equations such as semi-linear reaction diffusion systems and fourth order equations such as the Cahn-Hilliard equation which can be split into two second order problems, so allowing the use of piece-wise linear finite elements.

The exposition has been concerned with an evolving discretized surface which preserves the quasi-regularity of the mesh as time evolves. In practice this may be a short time property and the issue of remeshing arises. Observe that the approximating surfaces are polyhedral. It is a challenge to extend this approach to higher order approximations of the surface and higher order finite element methods.

Although the exposition has been concerned with triangulated surfaces in $\mathbb{R}^3$, immediately applicable to curves, the methodology is also applicable to hypersurfaces in higher space dimensions.

9. Appendix: Proof of the Leibniz formula

Proof. Let $\Omega \subset \mathbb{R}^n$ be open and $X = X(\theta,t)$, $\theta \in \Omega$, $X(\cdot,t) : \Omega \to G \cap \Gamma$ be a local regular parametrization of the open portion $G \cap \Gamma$ of the surface Γ which evolves so that $X_t = v(X(\theta,t),t)$. The metric $(g_{ij})_{i,j=1,\ldots,n}$ is given by $g_{ij} = X_{\theta_i} \cdot X_{\theta_j}$ with determinant $g = det(g_{ij})$. Let $(g^{ij}) = (g_{ij})^{-1}$. Define $F(\theta,t) = f(X(\theta,t),t)$ and $\mathscr{V}(\theta,t) = v(X(\theta,t),t)$.

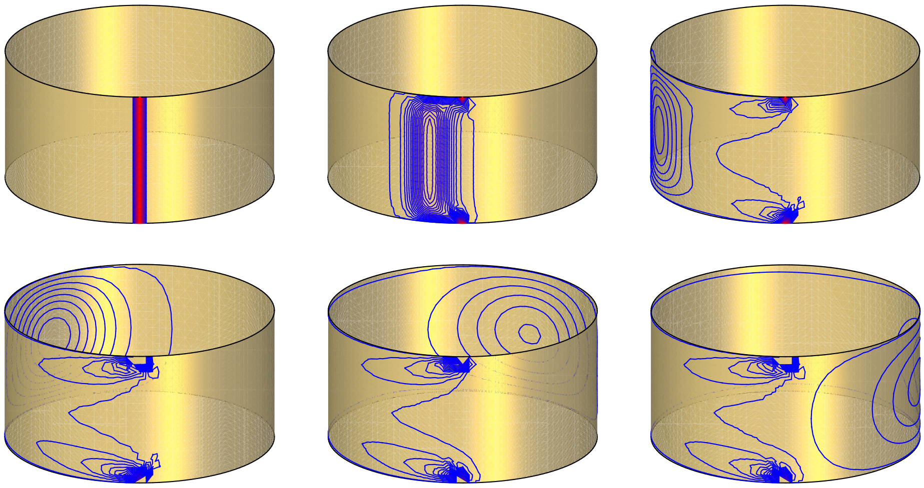

FIG. 8. Level lines of a solution of (1.1) on a rotating cylinder with Dirichlet boundary conditions and small diffusivity. Time steps 0, 10 and 50, and 100, 150 and 200 (half level spacing).

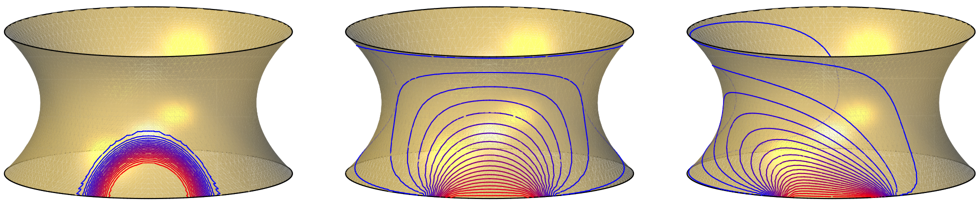

FIG. 9. Level lines of a solution of (1.1) on a cylindrical surface. Left: initial function, middle: stationary solution for stationary surface, right: stationary solution for rotating surface (Example 7.6).

Then with the Euler relation for the derivative of the determinant,

$$\frac{\partial}{\partial t}\sqrt{g} = \sqrt{g}\sum_{i,j=1}^{n} g^{ij} X_{\theta_i} \cdot \mathscr{V}_{\theta_j},$$

we have the following proof of (2.9):

$$\begin{aligned} \frac{d}{dt}\int_{\Gamma\cap G} f &= \frac{d}{dt}\int_{\Omega} F\sqrt{g} = \int_{\Omega} \frac{\partial F}{\partial t}\sqrt{g} + F\frac{\partial\sqrt{g}}{\partial t} \\ &= \int_{\Omega}\left(\frac{\partial f}{\partial t} + \nabla f(X,\cdot)\cdot X_t\right)\sqrt{g} + f(X,\cdot)\sqrt{g}\sum_{i,j=1}^{n} g^{ij} X_{\theta_i}\cdot\mathscr{V}_{\theta_j} \\ &= \int_{\Gamma\cap G} \dot{f} + f\nabla_{\Gamma}\cdot v. \end{aligned}$$

where in the last step we have used that $\mathscr{V} = X_t$ and that the tangential divergence of v is given by

$$\left(\nabla_{\Gamma}\cdot v\right)(X,\cdot) = \sum_{i,j=1}^{n} g^{ij} X_{\theta_i}\cdot\mathscr{V}_{\theta_j}.$$

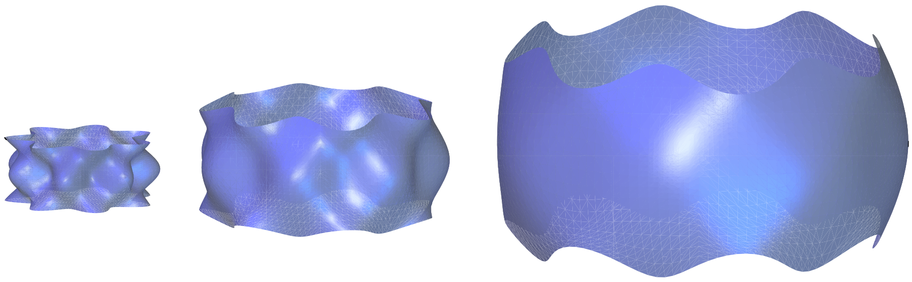

FIG. 10. Moving surface changing its curvatures strongly. From left to right: surface at time $t = 0, 0.5, 2.45$.

The formula (2.10) follows from the first equation by observing the velocity decomposition (2.6) and equation (2.8).

For the right hand side of (2.9) this leads to

$$\begin{aligned}\int_\Gamma \dot f + f\nabla_\Gamma \cdot v &= \int_\Gamma \left(f_t + v\cdot\nabla f + f\nabla_\Gamma\cdot v\right)\\ &= \int_\Gamma \left(f_t + V\frac{\partial f}{\partial \nu} - fVH + \nabla_\Gamma\cdot(fT)\right).\end{aligned}$$

Thus we have the equivalent form (2.10) for (2.9).

For the proof of (2.11) we first observe that we have

$$|\left(\nabla_\Gamma f\right)(X,\cdot)|^2 = \sum_{i,j=1}^n g^{ij}F_{\theta_i}F_{\theta_j}, \tag{9.5}$$

so that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Gamma\cap G}|\nabla_\Gamma f|^2 &= \int_\Omega \sqrt{g}\sum_{i,j=1}^n g^{ij}F_{\theta_i}F_{\theta_j,t} + \frac{1}{2}\int_\Omega\sqrt{g}\sum_{i,j=1}^n g_t^{ij}F_{\theta_i}F_{\theta_j}\\ &\quad + \frac{1}{2}\int_\Omega\sqrt{g}\sum_{i,j,k,l=1}^n g^{ij}g^{kl}X_{\theta_k}\cdot\mathscr{V}_{\theta_l}F_{\theta_i}F_{\theta_j}.\end{aligned}$$

An easy calculation shows that

$$g_t^{ij} = -\sum_{k,l=1}^n g^{ik}g^{jl}g_{kl,t} = -\sum_{k,l=1}^n g^{ik}g^{jl}\left(X_{\theta_k}\cdot X_{\theta_l}\right)_t = -\sum_{k,l=1}^n g^{ik}g^{jl}\left(\mathscr{V}_{\theta_k}\cdot X_{\theta_l} + X_{\theta_k}\cdot\mathscr{V}_{\theta_l}\right)$$

and we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_{\Gamma\cap G}|\nabla_\Gamma f|^2 = \int_\Gamma \nabla_\Gamma f\cdot\nabla_\Gamma \dot f - \int_\Gamma \sum_{i,j=1}^n \underline{D}_i v_j \underline{D}_i f\underline{D}_j f + \frac{1}{2}\int_\Gamma |\nabla_\Gamma f|^2\nabla_\Gamma\cdot v,$$

and have proved (2.11). □

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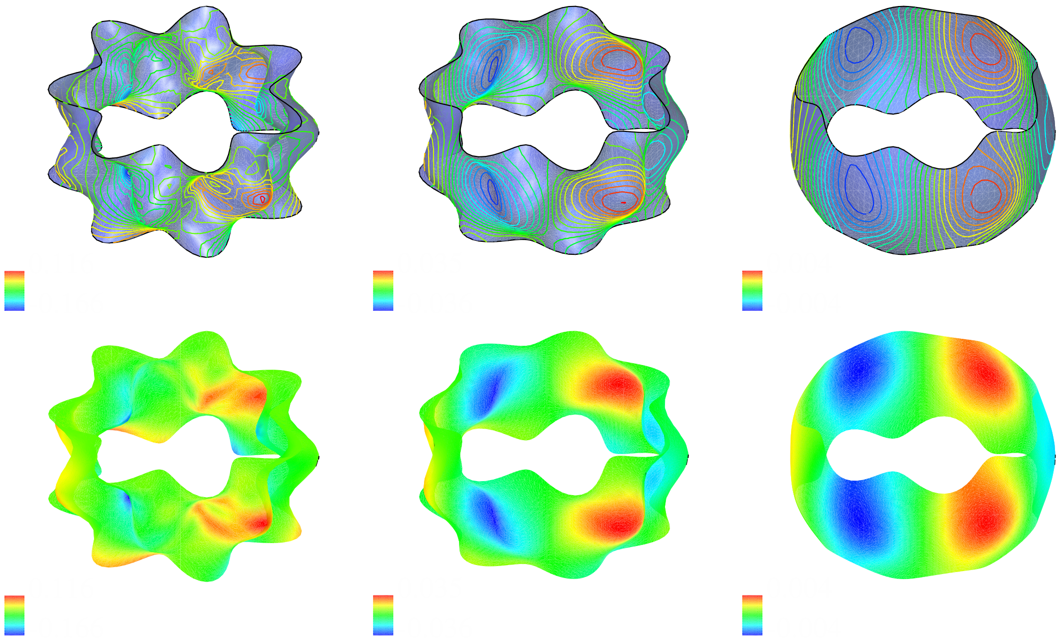

FIG. 11. Results for Example 7.7 at times $t = 0.0, 0.5$ and 2.45. The level lines are equally spaced between maximum and minimum of the solution.

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